LONG-TERM RETURNS IN STOCHASTIC INTEREST RATE MODELS: APPLICATIONS.

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Abstract

We extend the Cox-Ingersoll-Ross (1985) model of the short interest rate by assuming a stochastic reversion level, which better reflects the time dependence caused by the cyclical nature of the economy or by expectations concerning the future impact of monetary policies. In this framework, we have studied the convergence of the long-term return by using the theory of generalised Bessel-square processes. We emphasize the applications of the convergence results. A limit theorem proves evidence of the use of a Brownian motion with drift instead of the integral $\int_0^t r_u du$. For practice, however, this approximation turns out to be only appropriate when there are no explicit formulae and calculations are very timeconsuming.

Keywords

Interest rates; Cox-Ingersoll-Ross model; Stochastic reversion level; Generalised Bessel-square processes; Convergence; Bond prices; Life insurance.

1. introduction

In this paper, which has been presented at the 5th AFIR International Colloquium, we concentrate on the convergence of the long-term return $t^{-1}\int_0^t r_u du$, using a very general two-factor model, which is an extension of the Cox-Ingersoll-Ross (1985) model. Cox, Ingersoll & Ross (1985) express the short interest rate dynamics as

$$dr_t = \kappa(\gamma - r_t)dt + \sigma\sqrt{r_t}dB_t$$

with $(B_t)_{t\geq 0}$ a Brownian motion and κ, γ and σ positive constants. This model has some realistic properties. First, negative interest rates are precluded. Second, the absolute variance of the interest rate increases when the interest rate itself increases. Third, the interest rates are elastically pulled to the long-term value γ, where κ determines the speed of adjustment. Empirical studies like Chan, Karolyi, Longstaff & Sanders (1992) or Brown & Schaefer (1994), however, have shown that there is only weak evidence for the existence of a constant long run level of reversion.

We stress the long-term reversion level and the long-term interest rates since they are important in several issues in finance and insurance. For instance, for pricing an option to exchange a long bond for a short bond; or for mortgage pricing where the long rate determines when homeowners refinance their mortgages. In insurance, whole-life insurances are long-term products and the long-term interest rates play a dominant role.

We therefore follow the idea of Brennan & Schwartz (1982), who introduced a two-factor model by using short-term interest rates and consol rates (see Hogan (1993) for comments on this model).

In this paper, we assume that the short interest rate X is governed by the stochastic differential equation

$$dX_s = (2\beta X_s + \delta_s)ds + v\sqrt{X_s}dB_s$$

with the drift rate parameter $\beta < 0$, v a constant and δ a nonnegative predictable stochastic process such that $\int_0^t \delta_u du < \infty$ a.e. for all $t \in I\!R^+$. This stochastic differential equation has a unique (non-negative) strong solution.

It should be noted that the stochastic process $(\delta_s)_{s\geq 0}$ determines a reversion level. If it is chosen to be a constant and if $v = 2$, the process $(X_s)_{s\geq 0}$ is a Bessel-square process with drift, a process which is studied in great detail by for example Pitman & Yor (1982) and Revuz & Yor (1991). As the model is a generalisation of Bessel-square processes with drift, it is fairly easy to treat.

In Section 2, we concentrate on the convergence almost everywhere of the long-term return $t^{-1}\int_0^t r_u du$. We are interested in this limit as $\left(\exp(\int_0^t r_u du)\right)^{1/t}$ is the average of the accumulating factor (also called return) which can be useful in the determination of models of participation in the benefit or of saving products with a guaranteed minimum return. Using the results of Deelstra & Delbaen (1995a), we found that in most existing interest rate models, $\left(\exp(\int_0^t r_u du)\right)^{1/t}$ converges almost everywhere to a constant independent of the current market, as the observing period tends to infinity. We then say that the model has the "strong convergence property" (SCP), whereas we refer to models with the "weak convergence property" when the returns converge to a constant, that will generally depend upon the current economic environment and that may change in a stochastic fashion over time. This terminology appeared in a preliminary version entitled "Do interest rates converge" (1986) of Dybvig, Ingersoll & Ross (1996).

Dybvig, Ingersoll & Ross (1996) proved that the assumption of no-arbitrage implies that the long forward rate and the asymptotic zero-coupon rate never fall and moreover, they show that nearly all models have the surprising implication that long run forward rates and zero coupon rates converge to a constant, which is independent of the current state of the economy. El Karoui, Frachot & Geman (1998) discuss the theoretical and practical consequences of this observation for existing models. They also focus on some issues encountered in empirical work which can be related to the behavior of the long-term yield structure of interest rates.

As noted by El Karoui, Frachot & Geman (1998) and Pearson & Sun (1994), parameter estimates are generally very unstable over time and this fact can be interpreted as an indicator of misspecification : the parameters have to capture the remaining uncertainty due to the stochastic long-term rates. As illustrated by Pearson & Sun (1994) and Chen & Scott (1992), the estimation of multifactor versions with no stochastic long-term reversion level, show low mean-reversion for one of the state variables. El Karoui, Frachot & Geman (1998) argue that this low mean-reversion reflects the fact that the long-term yield is not constant over time.

Using the almost everywhere convergence theorem of Deelstra & Delbaen (1995a), we show that it is possible to build a model with the WCP in which the long-term return converges almost surely to a reversion level which is random itself. As an example we adapt the model of Tice & Webber (1997).

In Deelstra & Delbaen (1995b), we found conditions necessary to prove the convergence in law of a sequence of transformations of the long-term return to a Brownian motion. In Section 3, we propose a generalised theorem with measure-invariant hypotheses and we recall the idea of approximating $\int_0^t r_u du$ for t large enough. If the objective is to approximate the distribution of the long-term return of an investment made at time 0, it is appropriate to approximate $\int_0^t r_u du$ by a scaled Brownian motion with drift for t going to infinity. In the past, many authors have proposed Wiener models since in the long term, the Central Limit Theorems are applicable. In insurance, e.g. Beekman & Fuelling (1991), Dufresne (1990), Giacotto (1986), Goovaerts et al. (1994, 1995) and Milevsky (1997) modeled the accumulating factor $exp\left(\int_0^t r_u du\right)$ by the exponential of a Brownian motion with drift for the derivation of prices of different insurance products like annuities and perpetuities.

For practical reasons, we are interested in an approximation of $\int_0^t r_u du$ for all values of t. Therefore we suggest an improved approximation, which is discussed and evaluated by looking at bond prices. The results show that one should be very careful by replacing the integral $\int_0^t r_u du$ by a Brownian motion with drift. This approximation should only be used if no exact formulae are available and the exact computations are very time consuming like could be the case in the derivation of annuities.

In Section 4, we turn to the pricing of n-year temporary life assurances, whole-life assurances and endowment assurances. We calculate the present value and the variance and skewness of this present value of the benefit under these contracts by using on one hand the Cox-Ingersoll-Ross (1985) model and on the other hand a Brownian motion with drift which is suggested by the Central Limit Theorem. The results show that in general, it is inappropriate to use the Brownian motion with drift instead of the Cox-Ingersoll-Ross (1985) model or its extensions.

Without further notice we assume that a probability space $\left(\Omega, (\mathcal{F}_t)_{0\leq t}, I\!P\right)$ is given and that the filtration $(\mathcal{F}_t)_{0\leq t}$ satisfies the usual assumptions with respect to $I\!P$, a fixed probability on the sigma-algebra $\mathcal{F}_\infty = \bigvee_{t\geq 0} \mathcal{F}_t$. Also B is a continuous process which is a Brownian motion with respect to $(\mathcal{F}_t)_{0\leq t}$.

2. FACTOR MODELS WITH SCP AND WCP

In this section, we show by using a theorem obtained in Deelstra & Delbaen (1995a) that it is easy to verify that existing generalisations of the Cox-Ingersoll-Ross model have the strong convergence property, which means that the long-term return converges to a constant, which is independent of the earlier shape of the term structure and of the current state of the economic environment. By looking at anologous convergence theorems in e.g. a Gaussian setting, we could as a matter of fact prove that most existing interest rate models have the SCP, but this will not be done within this paper.

Afterwards, we use the model of Tice and Webber (1997) to show that multi-factor models do not necessarily imply that the strong convergence property holds.

It should be noted that the almost everywhere convergence limit of $t^{-1}\int_0^t r_u du$ is interesting to study since economists and actuaries work with the multiplicative accumulating factor (return) over t years, namely $\exp(\int_0^t r_u du)$. The average return in one year, where the average is taken over t years, is denoted by $\left(\exp(\int_0^t r_u du)\right)^{1/t}$. If the observing period goes to infinity, it converges to the exponent of the almost everywhere limit of $t^{-1}\int_0^t r_u du$.

We recall from Deelstra & Delbaen (1995a) that if X is defined by

$$dX_s = (2\beta X_s + \delta_s)ds + v\sqrt{X_s}dB_s$$

with $(B_s)_{s\geq 0}$ a Brownian motion, $\beta < 0$, v a constant and δ a positive, predictable stochastic process such that $s^{-1}\int_0^s \delta_u du \xrightarrow{a.e.} \overline{\delta}$ with $\overline{\delta} : \Omega \longrightarrow I\!R^+$, then the following convergence almost everywhere holds:

$$\frac{1}{s}\int_0^s X_u du \xrightarrow{a.e.} \frac{-\overline{\delta}}{2\beta}.$$

It is easy to show that for $r_t = \sigma^2 X_t/4$, $v = 2$, $\beta = -\kappa/2$ and $\delta_t = 4\kappa\gamma_t/\sigma^2$, we obtain a generalised two-factor Cox-Ingersoll-Ross (1985) model

$$dr_t = \kappa(\gamma_t - r_t)dt + \sigma\sqrt{r_t}dB_t$$

with $(\gamma_s)_{s\geq 0}$ a positive stochastic reversion level process. To ensure that the interest rate process $(r_t)_t$ remains a.s. strictly positive, we should add some hypotheses. Comparison theorems for Bessel-square processes with stochastic reversion level (see Deelstra (1995)) can be used to obtain some. Indeed, if $X^{(1)}$ and $X^{(}2)$ are two Bessel-square processes with respectively stochastic reversion level $\delta^{(1)}$, $\delta^{(2)}$ and issued from $x^{(1)}$, $x^{(2)}$ with $x^{(2)} \geq x^{(1)}$ and $\delta^{(2)} \geq \delta^{(1)}$ a.s. for all $t \in \mathbb{R}^+$, then

$$\mathbb{P}\left[X_t^{(2)} \geq X_t^{(1)} \text{ for all } t \geq 0\right] = 1.$$

Now, it is well-known that if $X^{(1)}$ is a Bessel-square process with constant dimension $\delta^{(1)} \geq 2$, then $X_t^{(1)} > 0$ a.s.. Therefore, hypotheses like $4\kappa\gamma_t/\sigma^2 \geq 2$ a.s. for all $t \in \mathbb{R}^+$, imply the strict positivity of $(r_t)_t$ a.s.. Remark that this is the generalisation of the constraint in case of the Cox-Ingersoll-Ross model.
In this paper, we further choose the process $(\gamma_s)_{s\geq 0}$ such that $t^{-1}\int_0^t \gamma_s ds$ converges almost everywhere to a random variable $\gamma^* = \sigma^2\overline{\delta}/4\kappa : \Omega \longrightarrow \mathbb{R}^+$. The central tendency process $(\gamma_s)_{s\geq 0}$ may be dependent or independent of the short interest rate process.

We stress this fact since if the reversion level process $(\gamma_s)_{s\geq 0}$ is independent of the short-term interest rates it is possible to derive (quasi-)explicit formulae for bond prices by using scaling properties of Bessel-square processes. This approach has been used in the papers by e.g. Maghsoodi (1996), Delbaen & Shirakawa (1996) and Deelstra (2000), who consider time-dependent but deterministic $(\gamma_s)_{s\geq 0}$. However if the reversion level process $(\gamma_s)_{s\geq 0}$ is dependent on the short interest rate process, no such formulae can be obtained.

As an example, let us describe the stochastic reversion level process $(\gamma_s)_{s\geq 0}$ by a Cox-Ingersoll-Ross (1985) square root process

$$d\gamma_t = \tilde{\kappa}(\gamma^* - \gamma_t)dt + \tilde{\sigma}\sqrt{\gamma_t}d\tilde{B}_t$$

or by a Courtadon (1982) process

$$d\gamma_t = \tilde{\kappa}(\gamma^* - \gamma_t)dt + \tilde{\sigma}\gamma_t d\tilde{B}_t \qquad \text{with } \tilde{\sigma}^2 \leq 2\tilde{\kappa}$$

with $(\tilde{B}_s)_{s\geq 0}$ a Brownian motion and with $\tilde{\kappa}, \gamma^*$ and $\tilde{\sigma}$ positive constants. The Brownian motion $(\tilde{B}_s)_{s\geq 0}$ may be correlated with the Brownian motion $(B_s)_{s\geq 0}$ of the short rate process and this correlation may be in a random way. As mentioned above, we do not need the technical assumption of fixed correlation or independence between the two factors of the model: for example as in Brennan & Schwartz (1982).

The two proposed reversion level processes are from the same family. They both remain positive for $\tilde{\kappa}, \gamma^* \geq 0$, a property which is necessary if one wants to work with nominal interest rates. For $\tilde{\kappa}, \gamma^* > 0$, these processes are mean-reverting to the long-term constant value γ^*, where $\tilde{\kappa}$ represents the speed of adjustment. The volatility increases in both cases with the reversion level.

For this class of stochastic reversion levels, $t^{-1}\int_0^t \gamma_s ds \xrightarrow{a.e.} \gamma^*$ and since $\delta_s = 4\kappa\gamma_s/\sigma^2$, $t^{-1}\int_0^t \delta_s ds \xrightarrow{a.e.} 4\kappa\gamma^*/\sigma^2$. By the theorem mentioned above (see Deelstra & Delbaen (1995a)), the long-term return is shown to converge almost everywhere to a constant:

$$\frac{1}{t}\int_0^t r_s ds = \frac{1}{t}\int_0^t \frac{\sigma^2}{4} X_s ds \xrightarrow{a.e.} \gamma^*.$$

We conclude that the long-term return in these two-factors model of short interest rates satisfies the strong convergence property. The average accumulating factor, where the average is taken over a period t, is found to converge almost everywhere to a constant as the period t tends to infinity, and this constant is independent of the current state of the economy:

$$\left(e^{\int_0^t r_u du}\right)^{1/t} \xrightarrow{a.e.} e^{\gamma^*}.$$

As another example, we treat the two-factor model proposed by Cox, Ingersoll & Ross (1985). They assumed a stochastic reversion level process depending on Y, the state variable which describes the change in the production opportunities, namely

$$\begin{aligned} dr_t &= \kappa(\gamma_t - r_t)dt + \sigma\sqrt{r_t}dB_t \\ d\gamma_t &= \tilde{\kappa}(Y_t - \gamma_t)dt \\ dY_t &= -\xi(\frac{-\zeta}{\xi} - Y_t)dt + \tilde{\sigma}\sqrt{Y_t}dB_t', \end{aligned}$$

with $\kappa, \tilde{\kappa}, \sigma, \zeta$ and $\tilde{\sigma}$ strictly positive constants. We assume that ξ is a strictly negative constant. We here only theoretically show that this model also has the SCP for the long-term return: since $t^{-1}\int_0^t Y_s ds \xrightarrow{a.e.} -\zeta/\xi$, we have that $t^{-1}\int_0^t \gamma_s ds \xrightarrow{a.e.} -\zeta/\xi$ and by the same reasoning as above, we obtain

$$\frac{1}{t}\int_0^t r_s ds \xrightarrow{a.e.} \frac{-\zeta}{\xi}.$$

As a consequence of the convergence of the long-term return to a constant, we can conclude that the long-term yield $\lim_{T\to\infty} Y(t,T)$ is uniformly bounded above as by Jensens's inequality (see also Yao (1998))

$$Y(t,T) = \mathbb{E}\left[\exp\left(\frac{1}{T-t}\int_t^T r_u du\right)\right] \leq \exp\left(\frac{1}{T-t}\int_t^T \mathbb{E}[r_u] du\right).$$

It is not surprising that the previous examples satisfy the SCP since in each model, the reversion level process itself is elastically pulled to a constant independent of the economic state. We recall that the convergence theorem from Deelstra & Delbaen (1995a) has no such strong hypothesis; on the contrary, the assumptions are very general. For example, the reversion level process does not have to be continuous. The convergence theorem only assumes a positive, predictable reversion $(\delta_u)_{u\geq 0}$ such that $s^{-1}\int_0^s \delta_u du \xrightarrow{a.e.} \overline{\delta}$, where $\overline{\delta}$ may be a random variable. Models in which this $\overline{\delta}$ really is a random variable, would imply that the long-term return converges to a random variable which will generally depend on the economic environment.

As an example, let us look at the general dynamic mean interest rate model in Tice & Webber (1997)

$$dr = a(\gamma - r)dt + \sigma_r dz_r$$
$$d\gamma = b(\mu_\gamma(t,r,Y) - \gamma)dt + \sigma_\gamma dz_\gamma$$
$$dY = c(\mu_Y(t,r,\gamma,Y) - Y)dt + \sigma_Y dz_Y$$

where z_r, z_γ and z_Y denote Brownian motions, r is the short rate and γ the level to which the short rates revert. Y is assumed to be a vector process summarizing the remainder of the dynamics in the model. Tice & Webber (1997) have interpreted this model within the IS-LM framework, which is a standard model in macroeconomics (see e.g. Hicks (1937)). As a particular case, Tice & Webber (1997) study a three factor model with the third factor related to the availability of transactions credit within the economy. To simplify the notations, Tice & Webber (1997) restrict themselves to σ_r, σ_γ and σ_Y being constant but it is possible to consider e.g. $\sigma_r = \sigma\sqrt{r}$.

In that case, it is clear that we are dealing with an extension of the Cox-Ingersoll-Ross model with a stochastic reversion level. This model has the weak convergence property if the process is not recurrent.

3. APPROXIMATION OF THE LONG-TERM RETURN AND OF BOND PRICES

In this section, we give a generalised version of the Central Limit Theorem from Deelstra & Delbaen (1995b). We study the convergence in law since it is always useful to know how the long-term return is distributed in the limit so that approximations can be deduced. We are particularly interested in an approximation of $\int_0^t r_u du$ since this term appears in discounting factors, bond prices, annuities, perpetuities, etc. As a natural candidate appears a Brownian motion with drift. This process has been used in insurance before for modeling the integral $\int_0^t r_u du$, e.g. in Beekman & Fuelling (1991), Dufresne (1990), Giacotto (1986), Goovaerts et al. (1994, 1995) and Milevsky (1997). In order to evaluate this approximation, we compare in the settings of the Cox-Ingersoll-Ross model bond prices calculated by using the approximating Brownian motion with exact values.

In order to obtain convergence in law, we have to make some more assumptions about our family of process:

Theorem: Suppose that a probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, I\!\!P)$ is given and that a stochastic process $X : \Omega \times I\!\!R^+ \to I\!\!R^+$ is defined by the stochastic differential equation

$$dX_s = (2\beta X_s + \delta_s)ds + v\sqrt{X_s}dB_s \quad \forall s \in I\!\!R^+$$

with $(B_s)_{s\geq 0}$ a Brownian motion with respect to $(\mathcal{F}_t)_{t\geq 0}$, v a constant and $\beta < 0$.
Let us make the following assumptions about the adapted and measurable process $\delta : \Omega \times I\!\!R^+ \to I\!\!R^+$:

- $\frac{1}{s}\int_0^s \delta_u du \xrightarrow{a.e.} \overline{\delta}$ where $\overline{\delta}$ is a strictly positive real number;
- $\sup_{t\geq 1} \frac{1}{t}\int_0^t \delta_u^2 du < \infty$ a.e.;
- For all $a \in I\!\!R^+$ $\quad \frac{1}{t}\int_{t-a}^t \delta_u^2 du \xrightarrow{I\!\!P} 0$.

Under these conditions, the following convergence in distribution holds:

$$\left(\sqrt{\frac{-8\beta^3}{v^2\overline{\delta}n}}\int_0^{nt}\left(X_u + \frac{\delta_u}{2\beta}\right)du\right)_{t\geq 0} \xrightarrow{\mathcal{L}} (B_t)_{t\geq 0}$$

where $(B_t)_{t\geq 0}$ denotes a Brownian motion and where '$\xrightarrow{\mathcal{L}}$' denotes convergence in law.

Since the proof of this theorem follows more or less the lines of the result in Deelstra & Delbaen (1995b), the proof is omitted and we immediately turn to the applications.
Inspired by this theorem, we estimate $\int_0^t X_u du$ with X as in the settings of the theorem by

$$\int_0^t \frac{-\delta_u}{2\beta}du + \sqrt{\frac{-v^2\overline{\delta}}{8\beta^3}}B_t$$

for t large enough. In Deelstra & Delbaen (1995b), we used the hypothesis $t^{-1}\int_0^t \delta_u du \xrightarrow{a.e.} \overline{\delta}$, to approximate $\int_0^t X_u du$ by the sum of the long-term constant $-\overline{\delta}/2\beta$, to which the long-term return a.e. converges, multiplied by t and a scaled Brownian motion:

$$\int_0^t X_u du \qquad \text{by} \qquad \frac{-\overline{\delta}}{2\beta}t + \sqrt{\frac{-v^2\overline{\delta}}{8\beta^3}}B_t. \tag{1}$$

It should be noted that in the case that $(\delta_u)_u$ is a stochastic process we replace the stochastic term $(-2\beta)^{-1} \int_0^t \delta_u du$ by a constant times t.

Another drawback of this estimator is that the moments of $\int_0^t X_u du$ do not equal those of the estimator, although they are the same asymptotically. If the period observed is large enough, this is satisfactory. If the objective is to approximate the distribution of the long-term return of an investment made at time 0, it seems to be appropriate to approximate $\int_0^t X_u du$ by a scaled Brownian motion with drift since the Central Limit Theorems are applicable on long-term.

However, one of our objectives is to look at the approximation

$$\int_0^t X_u du \qquad \text{by} \qquad \frac{-\overline{\delta}}{2\beta} t + \sqrt{\frac{-v^2 \overline{\delta}}{8\beta^3}} B_t$$

to find estimations of bond prices for all maturities. Therefore, the moments of $\int_0^t X_u du$ and of the estimator should be equal for all t. A second drawback of the approximation immediately appears in the bond price, namely

$$P(0,t) = I\!\!E_{X_0} \left[e^{-\int_0^t X_u du} \right] \sim \exp\left(\frac{\overline{\delta}}{2\beta} t - \frac{v^2 \overline{\delta}}{16\beta^3} t \right).$$

It is not realistic that the estimating bond price is independent of the current short interest rate X_0. Remark that we work with the default-free bond prices. In the sequel, we omit without notice the adjective "default-free". We further assume that there is no market price of risk, since we only want to compare different approximations theoretically.

In case of the Cox-Ingersoll-Ross (1985) square root process, the approximating bond price equals:

$$I\!\!E_{r_0} \left[e^{-\int_0^t r_u du} \right] \sim \exp\left(\gamma t (\frac{\sigma^2}{2\kappa^2} - 1) \right).$$

This estimating bond price is a decreasing function of the speed of adjustment parameter κ, where in case of the Cox-Ingersoll-Ross (1985) model, two cases are distinguished: for $r_0 < \gamma$, the bond price is a decreasing function of the parameter κ, and for $r_0 > \gamma$, it is an increasing function of κ. In Deelstra & Delbaen (1995b), we compared these approximating bond prices with values obtained in the Cox-Ingersoll-Ross setting and found that there is an underestimation of bond prices if $r_0 < \gamma$ and an overestimation if $r_0 > \gamma$.

Trying to motivate the approximation of the integral of the short-term interest rates by a Brownian motion with drift, we searched for an improved approximation. It seems logical to propose the approximation

$$\int_0^t X_u du \quad \sim \quad \int_0^t I\!\!E[X_u] du + \sqrt{\frac{-v^2 \overline{\delta}}{8\beta^3}} B_t.$$

Then the expectation is equal for all t and the variance is still asymptotically equal.

Since $(X_u)_{u\geq 0}$ is defined by the stochastic differential equation

$$dX_s = (2\beta X_s + \delta_s)ds + v\sqrt{X_s}dB_s,$$

the expectation value of X_s equals:

$$\mathbb{E}[X_s] = e^{2\beta s}X_0 + e^{2\beta s}\int_0^s e^{-2\beta u}\mathbb{E}[\delta_u]du,$$

which can only be calculated if $\mathbb{E}[\delta_u]$ is known and $\int_0^s \mathbb{E}[\delta_u]du < \infty$. As above, it should be noted that in the case of $(\delta_u)_u$ being a stochastic process, we replace the stochastic term $(2\beta)^{-1}\int_0^t \delta_u du$ by a deterministic time-dependent term. But at least in this way, the current state X_0 is introduced in the approximation.

As an example of the approximation, let us look again at the Cox-Ingersoll-Ross (1985) two-factor model:

$$dr_t = \kappa(\gamma_t - r_t)dt + \sigma\sqrt{r_t}dB_t,$$

$$d\gamma_t = \tilde{\kappa}(\gamma^* - \gamma_t)dt + \tilde{\sigma}\sqrt{\gamma_t}d\tilde{B}_t.$$

The approximation becomes

$$\begin{aligned}\int_0^t r_u du &\sim \int_0^t \mathbb{E}[r_u]du + \sqrt{\frac{\sigma^2\gamma^*}{\kappa^2}}B_t \\ &\sim \gamma^* t + \frac{1-e^{-\kappa t}}{\kappa}\left(r_0 - \gamma^* - \frac{\gamma_0 - \gamma^*}{\kappa - \tilde{\kappa}}\kappa\right) \\ &\quad + \frac{1-e^{-\tilde{\kappa} t}}{\tilde{\kappa}}\left(\frac{\gamma_0 - \gamma^*}{\kappa - \tilde{\kappa}}\right)\kappa + \sqrt{\frac{\sigma^2\gamma^*}{\kappa^2}}B_t.\end{aligned}$$

The bond price is estimated by:

$$\begin{aligned}\mathbb{E}_{r_0}\left[e^{-\int_0^t r_u du}\right] &\sim \exp\left(\gamma^* t(\frac{\sigma^2}{2\kappa^2} - 1)\right). \\ &\exp\left(-\frac{1-e^{-\kappa t}}{\kappa}\left(r_0 - \gamma^* - \frac{\gamma_0 - \gamma^*}{\kappa - \tilde{\kappa}}\kappa\right) - \frac{1-e^{-\tilde{\kappa} t}}{\tilde{\kappa}}\left(\frac{\gamma_0 - \gamma^*}{\kappa - \tilde{\kappa}}\right)\kappa\right).\end{aligned}$$

Let us evaluate the approximation in case of the Cox-Ingersoll-Ross (1985) single-factor model:

$$dr_t = \kappa(\gamma - r_t)dt + \sigma\sqrt{r_t}dB_t.$$

An anonymous referee remarked (see Deelstra & Delbaen (1995b)) that in this case, the moments of the first proposal (1) are equal for all t, as soon as the

Table 1: Bond prices: Exact values and approximations.

t	$r_0 = 0.04$			$r_0 = 0.1$		
	Exact	Approx	Approx-Exact	Exact	Approx	Approx-Exact
1	.9565	.9617	.0051	.9068	.9116	.0048
6	.7061	.7254	.0192	.5843	.5978	.0134
7	.6587	.6788	.0200	.5386	.5521	.0134
8	.6135	.6339	.0204	.4970	.5102	.0131
9	.5708	.5912	.0204	.4591	.4719	.0127
10	.5305	.5507	.0201	.4244	.4367	.0122
20	.2503	.2630	.0127	.1968	.2040	.0071
30	.1171	.1239	.0068	.0919	.0959	.0039
40	.0547	.0582	.0035	.0430	.0451	.0021

current short interest rate r_0 is distributed according to the steady state distribution of the square root process, namely the gamma-function with parameters $\alpha = 2\kappa\gamma/\sigma^2$ and $\beta = 2\kappa/\sigma^2$:

$$\mathbb{E}_{r_0}\left[\int_0^t r_u du\right] = \gamma t = \mathbb{E}_0\left[\gamma t + \sqrt{\frac{\sigma^2}{2\kappa^2}} B_t\right].$$

In reality, r_0 is not distributed this way, so an improvement is also necessary here to obtain good estimations of bond prices:

$$\int_0^t r_u du \sim \int_0^t \mathbb{E}[r_u] du + \sqrt{\frac{\sigma^2\gamma}{\kappa^2}} B_t.$$

Substituting the mean of the short interest rate, gives the expression

$$\int_0^t r_u du \sim \gamma t + \frac{1 - e^{-\kappa t}}{\kappa}(r_0 - \gamma) + \sqrt{\frac{\sigma^2\gamma}{\kappa^2}} B_t$$

and the estimating bond price is found to be

$$\mathbb{E}_{r_0}\left[e^{-\int_0^t r_u du}\right] \sim \exp\left(\gamma t(\frac{\sigma^2}{2\kappa^2} - 1) - \frac{1 - e^{-\kappa t}}{\kappa}(r_0 - \gamma)\right).$$

In case of the previous approximation (1), we found for $r_0 < \gamma$ an underestimation of the bond prices. The approximation in this paper is larger since for $r_0 < \gamma$, a positive term is added to the exponent, namely $-\frac{1-e^{-\kappa t}}{\kappa}(r_0 - \gamma)$. In the same way, the underestimation in case of $r_0 > \gamma$ is reduced.

For the Cox-Ingersoll-Ross (1985) square root process, an explicit formula for the bond price is given by Pitman & Yor (1982) and Cox, Ingersoll & Ross (1985). We recall the bond price from Pitman & Yor (1982):

$$\begin{aligned} & \mathbb{E}_{r_0}\left[\exp\left(-\int_0^t r_u du\right)\right] \\ = & \frac{\exp\left\{-\frac{r_0}{\sigma^2} w \frac{1+\kappa/w \coth(wt/2)}{\coth(wt/2)+\kappa/w}\right\} e^{\kappa x/\sigma^2} e^{\kappa^2\gamma^* t/\sigma^2}}{(\cosh(wt/2)+\kappa/w \sinh(wt/2))^{\frac{2\kappa\gamma^*}{\sigma^2}}} \end{aligned}$$

with $w = \sqrt{\kappa^2 + 2\sigma^2}$.

Using various values for the parameters, we have calculated this exact bond price and the improved approximation, for a large range of maturities. The deviations are always very small. The largest absolute deviations appear when the bond price has a value about 0.5. The reason therefore is that the bond price is a decreasing convex function of maturity and that the endpoints are fixed, namely for $t = 0$, the bond price equals 1, and for $t = \infty$, the bond price converges to 0. Consequently, the largest deviations are to be expected around one half.

In Table 1, the exact bond prices and the estimating bond prices are calculated with the parameters estimated by Chan, Karolyi, Longstaff & Sanders (1992), namely $\kappa = 0.23394$, $\gamma = 0.0808$ and $\sigma = 0.0854$. The results are given for $r_0 = 0.04$ and for $r_0 = 0.1$. We present the maturities between 6 and 10 since then, the bond price is approximately 0.5 and the largest absolute deviations appear. Although the absolute error as presented in Table 1 is not a monotonic function, one should note that the error in the rate $-\ln P(0,t)/t$ does reduce for large values of t.

In comparison with the first approximation (1), the underestimation and overestimation are reduced but the difference between the exact result and the approximation remains too large to be useful in practice. This approximation should only be used if no exact formulae are available and the exact computations are very timeconsuming like could be the case in the derivation of annuities.

4. APPLICATIONS IN LIFE ASSURANCE

In this section, we follow the lines of Parker (1993, 1994) for deriving the net single premium and the variance and the skewness of the present value of the benefit payable under some insurance contracts. If the short-term interest rates are determined by a Cox-Ingersoll-Ross model, the exact formulae follow from the result of Pitman & Yor (1982). We compare these values with the approximation derived in Section 3.

Following the notation of Parker (1992), we denote by K the integer-valued discrete random variable which represents the number of completed years to be lived by a life assured, whose age is exactly x year at the issue of the contract. We let $\mathcal{Z}$ be the present value of the benefit payable under a given assurance contract. As the precise definition of $\mathcal{Z}$ depends on the specific assurance under consideration, we look at some examples: the n-year temporary assurance, the whole-life assurance and the endowment assurance (see e.g. Bowers et al. (1986)).

Under the n-year temporary assurance, the benefit of 1 is payable at the end of the year of death of a life assured, if the death occurs within n years from the date of issue. Thus $\mathcal{Z}$ is defined to be:

$$\mathcal{Z} = \begin{cases} \exp\left(-\int_0^{K+1} X_u du\right) & K = 0, 1, \ldots, n-1 \\ 0 & K = n, n+1, \ldots \end{cases}$$

where $(X_u)_{u \geq 0}$ denotes as before the short interest rate, defined by the stochastic differential equation

$$dX_t = (2\beta X_t + \delta_t)dt + v\sqrt{X_t}d\tilde{B}_t.$$

The m-th non-centered moment of $\mathcal{Z}$ is given by:

$$I\!E\left[\mathcal{Z}^m\right] = \sum_{k=0}^{n-1} I\!E\left[\exp\left(-m\int_0^{k+1} X_u du\right)\right] {}_{k|}q_x,$$

where ${}_{k|}q_x$ denotes the probability that the life assured dies between his $(x+k)$-th and his $(x+k+1)$-th birthday.

Remark that for a whole-life assurance, the benefit certainly will be paid once, namely at the end of the year of death. Consequently,

$$\mathcal{Z} = \exp\left(-\int_0^{K+1} X_u du\right) \qquad K = 0, 1, \cdots, \omega - x - 1,$$

where ω is the least age so that $l_x = 0$. The m-th non-centered moment is given by:

$$I\!E\left[\mathcal{Z}^m\right] = \sum_{k=0}^{\omega-x-1} I\!E\left[\exp\left(-m\int_0^{k+1} X_u du\right)\right] {}_{k|}q_x.$$

Under the endowment assurance contract, the benefit is payable at the end of the year of death if death occurs within n years of the issue date or, if the insured person survives n years, the benefit is payable at time n. Consequently, the present value $\mathcal{Z}$ of an endowment assurance is defined as:

$$\mathcal{Z} = \begin{cases} \exp\left(-\int_0^{K+1} X_u du\right) & K = 0, 1, \ldots, n-1 \\ \exp\left(-\int_0^{n} X_u du\right) & K = n, n+1, \ldots \end{cases}$$

Table 2: Net single premiums: Exact values and Approximations.

n	life assurance			endowment assurance		
	Exact	Approx	Approx-Exact	Exact	Approx	Approx-Exact
1	.00154	.00155	.000008	.9313	.9363	.0049
10	.01453	.01484	.000314	.4785	.4944	.0158
20	.02896	.02985	.000887	.2354	.2453	.0098
40	.06222	.06479	.002572	.0894	.0935	.0041
60	.07635	.07979	.003439	.0767	.0801	.0034
80	.07664	.08010	.003459	.0766	.0801	.0034

Table 3: The variances: Exact values and Approximations.

n	life assurance			endowment assurance		
	Exact	Approx	Exact-Approx	Exact	Approx	Exact-Approx
1	.00153	.00147	.00006	.00949	.05587	.04638
10	.01182	.01071	.00111	.02844	.07711	.04867
20	.01763	.01587	.00175	.01849	.02994	.01148
40	.02021	.01796	.00225	.01567	.01833	.00266
60	.01902	.01658	.00243	.01653	.01897	.00244
80	.01898	.01654	.00244	.01654	.01898	.00244

The m-th non-centered moment of the present value is given by:

$$\mathbb{E}\left[\mathcal{Z}^m\right] = \sum_{k=0}^{n-1} \mathbb{E}\left[\exp\left(-m\int_0^{k+1} X_u du\right)\right] {}_{k|}q_x + \mathbb{E}\left[\exp\left(-m\int_0^{n} X_u du\right)\right] {}_{n}p_x.$$

Approximations of the net single premium of each contract are easily calculated. Indeed, approximations of the expected value of $\mathcal{Z}$ are obtained by taking $m = 1$ and by substituting the estimating bond price, proposed in the previous section.

We have evaluated this approximation in case of the Cox-Ingersoll-Ross single factor model, with the parameters estimated within Chan, Karolyi, Longstaff & Sanders (1992) and with $r_0 = 0.07$. We used the mortality table HD (1968-72), which is commonly used in Belgium and which is based on Makeham's formula $l_x = k\, s^x\, g^{c^x}$ with for the ages between 0 and 69: $k = 1,000,268$, $s = 0.999147835528$, $g = 0.999731696667$ and $c = 1.115094352734$; and otherwise $k = 1,292,726$, $g = 0.995564574228$, $c = 1.077130677635$ and the same value of s.

Table 4: The skewness: Exact values and Approximations.

n	life assurance			endowment assurance		
	Exact	Approx	Exact-Approx	Exact	Approx	Exact-Approx
1	25.245	24.866	0.379	-3.685	0.313	-3.998
10	7.540	7.421	0.119	-0.957	1.048	-2.005
20	5.019	5.002	0.017	0.443	2.046	-1.602
40	3.655	3.719	-0.064	3.668	3.961	-0.293
60	3.724	3.891	-0.166	3.733	3.904	-0.171
80	3.731	3.902	-0.170	3.731	3.902	-0.170

In Table 2, the exact values and the approximations are given for the net single premiums of n-year temporary life assurances and endowment contracts. Remark that for n larger than 60 years, both assurances become whole-life assurances since the life assured is aged $x = 30$ at the date of issue. We conclude that the approximations of the single net premiums are not encouraging.

The variance and the skewness of $\mathcal{Z}$ also are easy to find since the variance is defined as

$$\mathrm{var}\,[\mathcal{Z}] = \mathbb{E}\left[\mathcal{Z}^2\right] - \mathbb{E}[\mathcal{Z}]^2,$$

and the skewness is defined as

$$\begin{aligned}\mathrm{sk}[\mathcal{Z}] &= \frac{\mathbb{E}\left[(\mathcal{Z} - \mathbb{E}[\mathcal{Z}])^3\right]}{\mathrm{var}[\mathcal{Z}]^{3/2}} \\ &= \frac{\mathbb{E}\left[\mathcal{Z}^3\right] - 3\mathbb{E}\left[\mathcal{Z}^2\right]\,\mathbb{E}[\mathcal{Z}] + 2\mathbb{E}[\mathcal{Z}]^3}{\mathrm{var}[\mathcal{Z}]^{3/2}}.\end{aligned}$$

Each of these terms can be calculated by substituting m =1, 2 or 3 in $\mathbb{E}\left[\mathcal{Z}^m\right]$ and by using the approximation of the m-th non-centered moment of the discounting factor, namely

$$\mathbb{E}\left[\exp\left(-m\int_0^t X_u du\right)\right] \sim \exp\left(-m\int_0^t \mathbb{E}[X_u]du - \frac{m^2\overline{\delta}v}{16\beta^3}t\right).$$

In Tables 3 and 4, the variance and the skewness of $\mathcal{Z}$ are calculated, for $\mathcal{Z}$ being the present value of the benefit under an n-year temporary life-assurance, an endowment assurance and a whole-life assurance (if n is very large). Again, we used the formula of Makeham and the Cox-Ingersoll-Ross (1985) model with the same parameters as above. These results seem to be an indicator that the approximation by a Brownian motion with drift can only be used in practice when there are no explicit formulae or when the calculation is very timeconsuming.

We further admit that the major problem of taking into account stochastic interest rates in long-term life insurance products, is that the policies become dependent. With regard to the problems of setting contingency reserves and assessing the solvency of life assurance companies, it is therefore interesting to study portfolios of assurance policies (see e.g. Parker (1992, 1997)).

ACKNOWLEDGMENTS

The author wants to thank G. Parker for his suggestions and for giving a copy of his Ph.D. thesis and his articles. The author is also grateful to F. Delbaen for his interesting comments. P. Dybvig is acknowledged for sending his working papers. An anonymous referee of Deelstra (2000) is acknowledged for pointing out the use of comparison theorems to ensure the strict positivity of the generalised Cox-Ingersoll-Ross model.

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