Bounds for the price of discrete arithmetic Asian options

M. Vanmaele[*], G. Deelstra[†], J. Liinev[*], J. Dhaene[‡], M.J. Goovaerts[‡]

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[*]Department of Applied Mathematics and Computer Science, Ghent University, Krijgslaan 281, S9, 9000 Gent, Belgium, Michele.Vanmaele@ugent.be, Jan.Liinev@ugent.be, Tel.: +32 9 2644895.

[†]Department of Mathematics, ISRO and ECARES, Université Libre de Bruxelles, CP 210, 1050 Brussels, Belgium Griselda.Deelstra@ulb.ac.be, Tel. + 32 2 6505046. This research was carried out while the author was employed at the Ghent University.

[‡]Department of Applied Economics, Catholic University Leuven, Naamsestraat 69, 3000 Leuven, Belgium, Jan.Dhaene@econ.kuleuven.ac.be, Marc.Goovaerts@econ.kuleuven.ac.be, Tel. +32 16 326750.

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Abstract

In this paper the pricing of European-style discrete arithmetic Asian options with fixed and floating strike is studied by deriving analytical lower and upper bounds. In our approach we use a general technique for deriving upper (and lower) bounds for stop-loss premiums of sums of dependent random variables, as explained in Kaas, Dhaene and Goovaerts (2000), and additionally, the ideas of Rogers and Shi (1995) and of Nielsen and Sandmann (2003). We are able to create a unifying framework for European-style discrete arithmetic Asian options through these bounds, that generalizes several approaches in the literature as well as improves the existing results. We obtain analytical and easily computable bounds. The aim of the paper is to formulate an advice of the appropriate choice of the bounds given the parameters, investigate the effect of different conditioning variables and compare their efficiency numerically. Several sets of numerical results are included. We also discuss hedging using these bounds. Moreover, our methods are applicable to a wide range of (pricing) problems involving a sum of dependent random variables.

1 Introduction

In this paper the pricing of European-style discrete arithmetic Asian options with fixed and floating strike is studied.

A European-style discrete arithmetic Asian call option is a financial derivative instrument with exercise date T, n averaging dates and fixed strike price K, which generates at T a pay-off

$$\left(\frac{1}{n}\sum_{i=0}^{n-1} S(T-i) - K\right)_+ ,$$

where $x_+ = \max\{x, 0\}$ and $S(T-i)$ is the price of a risky asset at time $T-i$, $i = 0, \ldots, n-1$. The risk neutral price of this call option at current time $t = 0$ is given by

$$AC(n, K, T) = \frac{e^{-rT}}{n} E^Q\left[\left(\sum_{i=0}^{n-1} S(T-i) - nK\right)_+\right] \tag{1}$$

under a martingale measure Q and with some risk-neutral interest rate r.

A European-style discrete arithmetic Asian put option with exercise date T, n averaging dates ($n \leq T+1$) and floating strike price with percentage β, generates at T a pay-off

$$\left(\frac{1}{n}\sum_{i=0}^{n-1} S(T-i) - \beta S(T)\right)_+ .$$

A European-style arithmetic Asian call option with continuous averaging is based on a similar pay-off as in (1) but by replacing the discrete average by an integral divided by the length of the averaging period. We focus on discrete averaging which is the normal specification in real contracts. Discrete arithmetic Asian options are path-dependent contingent claims with pay-offs that depend on the average of the underlying asset price over some prespecified period of time, often a low number of trading days in the discrete averaging case. Such contracts form an attractive specification for thinly traded asset markets where price manipulation on or near a maturity date is possible. In markets where prices are prone to periods of extreme volatility the averaging performs a smoothing operation. For buyers as well as for writers, an Asian option is a useful hedging instrument. These Asian options provide for the buyer a cost efficient way of hedging cash or asset flows over extended periods, e.g., for foreign exchange, interest rate, or commodities like oil or gold. For the writer of an Asian option, the advantages include more manageable hedge ratios and the ability to unwind his position more gracefully at the end.

Asian options can also be part of complex financial contracts and strategies, like retirement plans or catastrophe insurance derivatives. Indeed, as explained in Nielsen and Sandmann (2003), a typical investment plan of a retirement scheme could include fixed periodic payments invested in a specified risky asset. An Asian option on the average return can be used to guarantee a minimum rate of return on the periodic payments. On the other hand, Cat-calls are catastrophic risk options which include Asian options on the average of an underlying index (see Geman (1994)).

Within the Black & Scholes (1973) model, no closed form solutions are available for Asian options involving the discrete arithmetic average. As opposed to options on geometric average, the density function for the arithmetic average is not lognormal and has no explicit representation.

A variety of methods for the European case and especially continuously averaged fixed strike options have been developed while only a few papers deal with the more practical case of discrete arithmetic averaging. A partial list of methods includes (for references see for example Klassen (2001) and Večeř (2001)): Monte Carlo or quasi-Monte Carlo methods, exact expressions involving Laplace transforms or an infinite sum over recursively defined integrals, convolution methods using the fast Fourier transform, analytic approximations based on moment matching or conditioning on some average, a number of PDE methods, tree methods.

We focus on analytic methods, based on bounds through conditioning on some random variable. We aim to create a unifying framework for European-style discrete arithmetic Asian options through these bounds, that generalizes several approaches in the literature as well as improves the existing results.

Throughout the paper we mainly consider 'forward starting' Asian options which means that at the current time 0, the averaging has not yet started and that the n variables $S(T-n+1), \ldots, S(T)$ are random. This case states in contrast with the case that $T - n + 1 \leq 0$ where only the prices $S(1), \ldots, S(T)$ remain random. In literature, this Asian option is called 'in progress'. Note that our results for forward starting Asian options can immediately be translated to results for Asian options in progress. Most papers considering analytical approximations treat only standard Asian options which is the case of $T = n - 1$ but in a non-analytical way the PDE approach also treats easily different types of Asian options.

An analytical lower and upper bound in the case of continuous averaging was obtained by the method of conditioning in Rogers and Shi (1995). Simon, Goovaerts and Dhaene (2000) derived and computed in a general framework an analytical expression for the so-called 'comonotonic upper bound', which is in fact the smallest linear combination of prices of European call options that bounds the price of an European-style Asian option from above. Nielsen and Sandmann (2003) studied both upper and lower bounds for an European-style arithmetic Asian option in the Black & Scholes setting. In particular, they derive a special case of the Simon, Goovaerts and Dhaene upper bound using Lagrange optimization. Nielsen and Sandmann (2003) also apply

the Rogers & Shi reasoning in the arithmetic averaging case by using one specific standardized normally distributed conditioning variable.

The paper is organized as follows. Section 2 provides bounds for the European-style discrete arithmetic Asian options with fixed strike in the Black & Scholes setting. We first present in Section 2.1 lower and upper bounds based on a general technique for deriving the bounds for stop-loss premiums of sums of dependent random variables, as explained in Kaas, Dhaene and Goovaerts (2000) and Dhaene et al. (2002a). For clarity we have included a short overview of their methods in Appendix A. In Section 2.2 we show how to improve the upper bound that is based on the ideas of Rogers and Shi (1995), and generalize the approach of Nielsen and Sandmann (2003) to a general class of normally distributed conditioning variables. We also show in Section 2.3 how to sharpen the improved comonotonic upper bound of Kaas et al. (2000) and Dhaene et al. (2002a) by obtaining another so-called partially exact/comonotonic upper bound which consists of an exact part of the option price and some improved comonotonic upper bound for the remaining part. This idea of decomposing the calculations in an exact part and an approximating part goes at least back to Curran (1994). The procedures we present can also be used to price the European-style discrete arithmetic Asian put options with fixed strike (either directly or through the put-call parity), see Section 2.4. In Section 2.5 we compare and discuss all approaches and, in addition, compare our results to those of Jacques (1996), who approximates the distribution of the arithmetic average by a more tractable one. We measure the closeness of the bounds in distributional sense. Several sets of numerical results are given. We also consider hedging based on the lower and upper bounds in Section 2.6.

Section 3 treats the European-style discrete arithmetic Asian options with floating strike in the Black & Scholes setting. In independent work, Henderson and Wojakowski (2002) use the change of numeraire technique to obtain symmetry results between forward starting European-style Asian options with floating and fixed strike in case of continuous averaging. We show that their results can be extended to discrete averaging and we give also bounds for the European-style Asian floating strike options in progress.

We conclude the paper with main results and recent developments in Section 4.

One of the aims of this paper is to identify the currently best lower and upper bounds. We will show that the lower bounds are very close to the Monte Carlo values and that one of our techniques leads to very satisfying upper bounds, see Theorem 6.

2 Fixed strike Asian options in a Black & Scholes setting

In the Black & Scholes model, the price of a risky asset $\{S(t),\ t \geq 0\}$ under the risk-neutral measure Q follows a geometric Brownian motion process, with volatility σ and with drift equal to the risk-free force of interest r:

$$\frac{dS(t)}{S(t)} = rdt + \sigma dB(t), \qquad t \geq 0,$$

where $\{B(t),\ t \geq 0\}$ is a standard Brownian motion process under Q. Hence, the random variables $\frac{S(t)}{S(0)}$ are lognormally distributed with parameters $(r - \frac{\sigma^2}{2})t$ and $t\sigma^2$.

Therefore we do not have an explicit analytical expression for the distribution of the average $\frac{1}{n}\sum_{i=0}^{n-1} S(T-i)$ in (1) and determining the price of the Asian option is a complicated task. From (1) it is seen that the problem of pricing arithmetic Asian options turns out to be equivalent to calculating stop-loss premiums of a sum of dependent risks. Hence we can apply the results on comonotonic upper and lower bounds for stop-loss premiums, which have been summarized in Section 2.1 and in Appendix A.

We now shall concentrate on bounds for the European-style discrete arithmetic Asian option with fixed strike by comonotonicity reasoning and by using the approach of Rogers & Shi which has been generalized by Nielsen and Sandmann (2003). We only write down the formulae of the forward starting Asian call options in progress and corresponding Asian put options can be treated in a similar way.

2.1 Bounds based on comonotonicity reasoning

In both financial and actuarial context one encounters quite often random variables of the type $\mathbb{S} = \sum_{i=1}^{n} X_i$ where the terms X_i are not mutually independent, but the multivariate distribution function of the random vector $(X_1, X_2, \ldots, X_n)$ is not completely specified because one only knows the marginal distribution functions of the random variables X_i. In such cases, one would like to find lower bounds of the form $\underline{\mathbb{S}} = \sum_{i=1}^{n} \underline{X_i}$ and upper bounds of the form $\overline{\mathbb{S}} = \sum_{i=1}^{n} \overline{X_i}$ for the sum $\mathbb{S} = \sum_{i=1}^{n} X_i$ such that (i) the marginal distribution functions of X_i, $\underline{X_i}$ and $\overline{X_i}$ ($i = 1, \ldots, n$) are equal, and (ii) $\underline{\mathbb{S}} \preceq_{\text{cx}} \mathbb{S} \preceq_{\text{cx}} \overline{\mathbb{S}}$, where $\preceq_{\text{cx}}$ denotes the convex order, which means that $E[\underline{\mathbb{S}}] = E[\mathbb{S}] = E[\overline{\mathbb{S}}]$ and $E[(\underline{\mathbb{S}} - d)_+] \leq E[(\mathbb{S} - d)_+] \leq E[(\overline{\mathbb{S}} - d)_+]$ for all $d \in \mathbb{R}$.

Referring to Dhaene et al. (2002a), one possible choice for an upper bound $\overline{\mathbb{S}}$ is given by $\overline{\mathbb{S}} := \mathbb{S}^c$ with

$$\mathbb{S}^c \stackrel{d}{=} \sum_{i=1}^{n} F_{X_i}^{-1}(U). \tag{2}$$

In other words, we choose the components of the random vector $\left(\overline{X_1}, \overline{X_2}, \ldots, \overline{X_n}\right)$ such that $\overline{X_i} := X_i^c := F_{X_i}^{-1}(U)$, where (a) $F_X^{-1}(U)$ is the usual inverse of a distribution function, which is the non-decreasing and left-continuous function defined by

$$F_X^{-1}(U) = \inf\left\{x \in \mathbb{R} \mid F_X(x) \geq U\right\}, \qquad U \in [0,1],$$

with $\inf \emptyset = +\infty$ by convention, and (b) the corresponding random vector $(X_1^c, \ldots, X_n^c)$ is *comonotonic*, which means that each two possible outcomes $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ of $(X_1^c, \ldots, X_n^c)$ are ordered componentwise.

Another choice for the upper bound $\overline{\mathbb{S}}$ is based on the assumption that there is some additional information available concerning the stochastic nature of $(X_1, \ldots, X_n)$, represented by some random variable Λ with a given distribution function. Based on Kaas et al. (2000), we choose $\overline{\mathbb{S}} := \mathbb{S}^u$, with

$$\mathbb{S}^u = F_{X_1|\Lambda}^{-1}(U) + F_{X_2|\Lambda}^{-1}(U) + \cdots + F_{X_n|\Lambda}^{-1}(U). \tag{3}$$

Put another way, we choose the components of the random vector $\left(\overline{X_1}, \overline{X_2}, \ldots, \overline{X_n}\right)$ such that

$\overline{X_i} := F^{-1}_{X_i|\Lambda}(U)$, where $F^{-1}_{X_i|\Lambda}(U)$ is the notation for the random variable $f_i(U,\Lambda)$, with the function f_i defined by $f_i(u,\lambda) = F^{-1}_{X_i|\Lambda=\lambda}(u)$, and with U being a $(0,1)$-uniform random variable independent of Λ. The upper bound $\mathbb{S}^u$ is an improvement over the upper bound $\mathbb{S}^c$, see e.g. Dhaene et al. (2002a) for details. As a lower bound we choose $\underline{\mathbb{S}} := \mathbb{S}^\ell$, following Kaas et al. (2000), where $\mathbb{S}^\ell$ is a conditional expectation of $\mathbb{S}$ given some random variable Λ, not necessarily equal to that entering (3):

$$\mathbb{S}^\ell = E\left[\mathbb{S} \mid \Lambda\right]. \tag{4}$$

In other words, we choose the components of the random vector $\left(\underline{X_1}, \underline{X_2}, \ldots, \underline{X_n}\right)$ such that $\underline{X_i} := E\left[X_i \mid \Lambda\right]$. We remark that this idea was also suggested by Rogers and Shi (1995) for the continuous averaging case.

Summarizing, the sum $\mathbb{S}$ is bounded below and above in convex order by the sums given by (4), (3) and (2):

$$\mathbb{S}^\ell \preceq_{\text{cx}} \mathbb{S} \preceq_{\text{cx}} \mathbb{S}^u \preceq_{\text{cx}} \mathbb{S}^c,$$

which implies by definition of convex order that

$$E[(\mathbb{S}^\ell - d)_+] \leq E[(\mathbb{S} - d)_+] \leq E[(\mathbb{S}^u - d)_+] \leq E[(\mathbb{S}^c - d)_+]$$

for all d in $\mathbb{R}$, while $E[\mathbb{S}^\ell] = E[\mathbb{S}] = E[\mathbb{S}^u] = E[\mathbb{S}^c]$.

A more detailed overview of the construction of these sums and the corresponding bounds, based on the literature, is given in Appendix A. Notice that throughout the paper, especially in the proofs of theorems, we make use of the results summarized in that appendix.

We remark that the Asian option pricing in the Black & Scholes setting is in fact a particular case of sums of lognormal variables in Appendix A. Indeed, let us look at the price of the European-style discrete arithmetic Asian call option with strike price K, maturity date T and averaging over n prices of the underlying with $T - n + 1 \geq 0$:

$$AC(n,K,T) = \frac{e^{-rT}}{n} E^Q\left[(\mathbb{S} - nK)_+\right] \tag{5}$$

with

$$\mathbb{S} = \sum_{i=0}^{n-1} S(T-i) = \sum_{i=0}^{n-1} S(0)e^{(r-\frac{\sigma^2}{2})(T-i)+\sigma B(T-i)}. \tag{6}$$

This can be rewritten as a sum of lognormal random variables:

$$\mathbb{S} = \sum_{i=0}^{n-1} X_i = \sum_{i=0}^{n-1} \alpha_i e^{Y_i} \tag{7}$$

with

$$\begin{aligned} Y_i &= \sigma B(T-i) \sim N(0, \sigma^2(T-i)) \\ \alpha_i &= S(0)e^{(r-\frac{\sigma^2}{2})(T-i)} \end{aligned} \tag{8}$$

and

$$\mathrm{cov}(Y_i, Y_j) = \sigma^2 \min(T-i, T-j)$$

leading to

$$\mathrm{cov}(X_i, X_j) = \alpha_i \alpha_j e^{\frac{1}{2}\sigma^2((T-i)+(T-j))} \left[e^{\sigma^2 \min(T-i,T-j)} - 1 \right].$$

2.1.1 Lower bound

A lower bound for the Asian option price $AC(n, K, T)$ is obtained by using a normally distributed conditioning variable Λ and by substituting $\mathbb{S}^\ell$ for $\mathbb{S}$ in the right hand side of (5), where according to (4)

$$\mathbb{S}^\ell = \sum_{i=0}^{n-1} E^Q[X_i|\Lambda] = \sum_{i=0}^{n-1} \alpha_i E^Q[e^{Y_i}|\Lambda].$$

The following theorem states a lower bound for the option price $AC(n, K, T)$. The proof follows from (62), (63) and (68) in Appendix A as shown in Dhaene et al. (2002b).

Theorem 1. *Suppose the sum $\mathbb{S}$ is given by* (6)-(8) *and Λ is a normally distributed conditioning variable such that $(\sigma B(T-i), \Lambda)$ are bivariate normally distributed for all i. Then the comono-*

tonic lower bound for the option price $AC(n,K,T)$ is given by

$$\begin{aligned}\mathrm{LB}\Lambda &= \frac{e^{-rT}}{n}E^Q[(\mathbb{S}^\ell - nK)_+] \\ &= \frac{S(0)}{n}\sum_{i=0}^{n-1} e^{-ri}\,\Phi\left[\sigma\rho_{T-i}\sqrt{T-i} - \Phi^{-1}\left(F_{\mathbb{S}^\ell}(nK)\right)\right] - e^{-rT}K\left(1 - F_{\mathbb{S}^\ell}(nK)\right), \quad (9)\end{aligned}$$

where $\rho_{T-i} = \mathrm{corr}(\sigma B(T-i),\Lambda) \geq 0$ and $F_{\mathbb{S}^\ell}(nK)$ is a solution to

$$S(0)\sum_{i=0}^{n-1}\exp\left[\left(r - \frac{\sigma^2}{2}\rho_{T-i}^2\right)(T-i) + \sigma\,\rho_{T-i}\sqrt{T-i}\,\Phi^{-1}\left(F_{\mathbb{S}^\ell}(nK)\right)\right] = nK, \quad (10)$$

where $\Phi(\cdot)$ is the cumulative distribution function (cdf) of a standard normal variable and $F_{\mathbb{S}^\ell}(\cdot)$ represents the cdf of $\mathbb{S}^\ell$.

Note that the conditioning variable Λ only enters through the correlations ρ_{T-i}. We now focus on choosing the appropriate conditioning variable Λ. Taking into account that we aim to derive a closed-form expression for the lower bound, we define Λ as a normal random variable given by

$$\Lambda = \sum_{i=0}^{n-1}\beta_i B(T-i), \qquad \beta_i \in \mathbb{R}^+. \quad (11)$$

For general positive β_i, the variance of Λ is given by

$$\sigma_\Lambda^2 = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\beta_i\beta_j\min\left(T-i, T-j\right)$$

and

$$\rho_{T-i} = \mathrm{corr}(\sigma B(T-i),\Lambda) = \frac{\mathrm{cov}\left(B(T-i),\Lambda\right)}{\sqrt{T-i}\,\sigma_\Lambda} = \frac{\sum_{j=0}^{n-1}\beta_j\min\left(T-i,T-j\right)}{\sqrt{T-i}\,\sigma_\Lambda} \geq 0. \quad (12)$$

Remark that we take positive coefficients β_i implying that the correlations ρ_{T-i} are positive. This is to ensure that $\mathbb{S}^\ell$ is a sum of n comonotonic random variables.

We investigate different choices of weights β_i in expression (11) for the conditioning random variable Λ. The choice is motivated by the reasoning that the quality of the stochastic lower bound $E^Q[\mathbb{S} \mid \Lambda]$ can be judged by its variance. To maximize the quality, this variance should be made as close as possible to $\mathrm{var}^Q[\mathbb{S}]$. In other words, the average value

$$E^Q\left[\mathrm{var}^Q[\mathbb{S} \mid \Lambda]\right] = \mathrm{var}^Q[\mathbb{S}] - \mathrm{var}^Q\left[E^Q[\mathbb{S} \mid \Lambda]\right]$$

should be small. This however does not imply that the above expression should be minimized over the conditioning variable Λ. Notice that

$$\mathrm{var}^Q[\mathbb{S}] - \mathrm{var}^Q\left[\mathbb{S}^\ell\right] = 2\int_{-\infty}^{+\infty} \{\mathrm{E}^Q[(\mathbb{S}-k)_+] - (\mathrm{E}^Q[(\mathbb{S}^\ell - k)_+]\}\, dk.$$

From this relation it is seen that minimizing the difference in variance over Λ is no guarantee that the difference between the corresponding stop-loss premia for one particular k will be minimized. Intuitively, to get the best lower bound for $AC(n, K, T)$, Λ and $\mathbb{S}$ should be as alike as possible. Therefore, we have selected the following two candidates for Λ which turn out to give very good results:

1. a linear transformation of a first order approximation to $\sum_{i=0}^{n-1} S(T-i)$ in (6), as proposed in a general setting by Kaas, Dhaene and Goovaerts (2000) and used in Dhaene et al. (2002b):

$$\Lambda = \sum_{i=0}^{n-1} e^{(r-\frac{\sigma^2}{2})(T-i)} B(T-i), \tag{13}$$

2. the standardized logarithm of the geometric average $\mathbb{G} = \sqrt[n]{\prod_{i=0}^{n-1} S(T-i)}$ as in Nielsen and Sandmann (2003):

$$\Lambda = \frac{\ln \mathbb{G} - E^Q[\ln \mathbb{G}]}{\sqrt{\mathrm{var}^Q[\ln \mathbb{G}]}} = \frac{1}{\sqrt{\mathrm{var}^Q[\sum_{i=0}^{n-1} B(T-i)]}} \sum_{i=0}^{n-1} B(T-i), \tag{14}$$

where

$$\mathrm{var}^Q[\sum_{i=0}^{n-1} B(T-i)] = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \min(T-i, T-j) = n^2 T - \frac{n}{6}(n-1)(4n+1).$$

The lower bound (9)-(10) differs for the two choices (13) and (14) of Λ, only by the expression (12) for the correlation coefficient ρ_{T-i}:

1. $\rho_{T-i} = \dfrac{\sum_{j=0}^{n-1} e^{(r-\frac{\sigma^2}{2})(T-j)} \min(T-i, T-j)}{\sqrt{T-i}\,\sigma_\Lambda}$

   with

   $\sigma_\Lambda^2 = \displaystyle\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} e^{(r-\frac{\sigma^2}{2})(2T-i-j)} \min(T-i, T-j)\,,$

2. $\rho_{T-i} = \dfrac{\sum_{j=0}^{n-1} \min\left(T-i, T-j\right)}{\sqrt{n^2 T - \frac{n}{6}(n-1)(4n+1)}\sqrt{T-i}} = \dfrac{n\left(T-i\right) - \frac{(n-i-1)(n-i)}{2}}{\sqrt{n^2 T - \frac{n}{6}(n-1)(4n+1)}\sqrt{T-i}}$
since $\sigma_\Lambda = 1$.

We note that the closed-form solution of the lower bound in Nielsen and Sandmann (2003) is a special case of (9) and (10) with (14) as the conditioning variable. We also noticed that the lower bound when conditioning on the geometric average coincides with the so-called "naive" approximation of Curran (1994). In fact, formulae (9)-(10) for the lower bound are general in the sense that they hold for any normally distributed conditioning variable Λ by substituting the right ρ_{T-i}. Moreover, the lower bound can be expressed as a combination of Black & Scholes type formulae.

Theorem 2. *For a general normally distributed conditioning variable Λ, satisfying the assumptions of Theorem 1, the lower bound LBΛ of $AC(n, K, T)$ can be written as an average of Black & Scholes formulae for an artificial underlying asset of which the price process $\tilde{S}(t)$ is a geometric Brownian motion with $\tilde{S}(0) = S(0)$ and with a non-constant volatility $\tilde{\sigma}_i = \sigma\rho_{T-i}$ at time instance $T-i$:*

$$\mathrm{LB}\Lambda = \frac{e^{-rT}}{n}\sum_{i=0}^{n-1} E^Q[(\tilde{S}(T-i) - \tilde{K}_i)_+] = \frac{1}{n}\sum_{i=0}^{n-1}\left(e^{-ri}\tilde{S}(0)\Phi\left(d_{1,i}\right) - e^{-rT}\tilde{K}_i\Phi\left(d_{2,i}\right)\right)$$

with

$$\tilde{S}(T-i) = \tilde{S}(0)e^{(r-\frac{\tilde{\sigma}_i^2}{2})(T-i)+\tilde{\sigma}_i B(T-i)}$$

and strike prices

$$\tilde{K}_i = F^{-1}_{E[S(T-i)|\Lambda]}\left(F_{\mathbb{S}^\ell}(nK)\right) = S(0)e^{(r-\frac{\tilde{\sigma}_i^2}{2})(T-i)+\tilde{\sigma}_i\sqrt{T-i}\Phi^{-1}\left(F_{\mathbb{S}^\ell}(nK)\right)}$$

and where

$$\begin{aligned} d_{1,i} &= \frac{\left(r+\frac{\tilde{\sigma}_i^2}{2}\right)(T-i) - \ln\left(\frac{\tilde{K}_i}{\tilde{S}(0)}\right)}{\tilde{\sigma}_i\sqrt{T-i}} = \tilde{\sigma}_i\sqrt{T-i} - \Phi^{-1}\left(F_{\mathbb{S}^\ell}(nK)\right), \\ d_{2,i} &= d_{1,i} - \tilde{\sigma}_i\sqrt{T-i} = -\Phi^{-1}\left(F_{\mathbb{S}^\ell}(nK)\right) \end{aligned}$$

while $F_{\mathbb{S}^\ell}(nK)$ can be calculated from $\sum_{i=0}^{n-1}\tilde{K}_i = nK$ similarly to (10).

2.1.2 Improved comonotonic upper bound

As for the lower bound, we consider a conditioning normal random variable Λ. An improved comonotonic upper bound for the Asian option price $AC(n,K,T)$ is given by

$$AC(n,K,T) = \frac{e^{-rT}}{n} E^Q\left[(\mathbb{S}-nK)_+\right] \leq \frac{e^{-rT}}{n} E^Q\left[(\mathbb{S}^u-nK)_+\right], \tag{15}$$

where according to (3) $\mathbb{S}^u = \sum_{i=0}^{n-1} F^{-1}_{X_i|\Lambda}(U) = \sum_{i=0}^{n-1} F^{-1}_{\alpha_i e^{Y_i}|\Lambda}(U)$ for a $(0,1)$-uniform random variable U independent of Λ. More explicitly, we obtain the following analytic expression for this bound.

Theorem 3. *Suppose the sum $\mathbb{S}$ is given by* (6)-(8) *and Λ is a normally distributed conditioning variable such that $(\sigma B(T-i),\Lambda)$ are bivariate normally distributed for all i. Then the improved comonotonic upper bound for the option price $AC(n,K,T)$ is given by*

$$\begin{aligned}
\mathrm{ICUB}\Lambda &= \frac{e^{-rT}}{n} E^Q\left[(\mathbb{S}^u-nK)_+\right] \\
&= \frac{e^{-rT}}{n}\sum_{i=0}^{n-1} S(0)e^{r(T-i)}e^{-\frac{\sigma^2}{2}\rho^2_{T-i}(T-i)} \\
&\quad \times \int_0^1 e^{\rho_{T-i}\,\sigma\sqrt{T-i}\,\Phi^{-1}(v)}\Phi\left(\sqrt{1-\rho^2_{T-i}}\,\sigma\sqrt{T-i} - \Phi^{-1}\left(F_{\mathbb{S}^u|V=v}(nK)\right)\right)dv \\
&\quad -e^{-rT}nK\left(1-F_{\mathbb{S}^u}(nK)\right),
\end{aligned} \tag{16}$$

where

$$V = \Phi\left(\frac{\Lambda - E[\Lambda]}{\sigma_\Lambda}\right) \tag{17}$$

is a uniform$(0,1)$ random variable, $\rho_{T-i} = \mathrm{corr}(\sigma B(T-i),\Lambda)$, and

$$F_{\mathbb{S}^u}(nK) = \int_0^1 F_{\mathbb{S}^u|V=v}(nK)dv,$$

and the conditional distribution $F_{\mathbb{S}^u|V=v}(nK)$ follows from

$$nK = \sum_{i=0}^{n-1}\alpha_i \exp\left[\rho_{T-i}\sigma\sqrt{T-i}\,\Phi^{-1}(v) + \sqrt{1-\rho^2_{T-i}}\,\sigma\sqrt{T-i}\,\Phi^{-1}\left(F_{\mathbb{S}^u|V=v}(nK)\right)\right]. \tag{18}$$

Proof. We determine the cdf of $\mathbb{S}^u$ and the stop-loss premium $E\left[(\mathbb{S}^u-d)_+\right]$, where we condition on a normally distributed random variable Λ or equivalently on the uniform$(0,1)$ random variable

V, cfr. (17). The conditional probability $F_{\mathbb{S}^u|V=v}(x)$ also denoted by $F_{\mathbb{S}^u}(x \mid V=v)$, is the cdf of a sum of n comonotonic random variables and follows for $F^{-1}_{\mathbb{S}^u|V=v}(0) < x < F^{-1}_{\mathbb{S}^u|V=v}(1)$, according to (60) and (67), for $\alpha_i \geq 0$, $i = 0, \ldots, n-1$, implicitly from:

$$\sum_{i=0}^{n-1} \alpha_i \, e^{E[Y_i]+r_i\sigma_{Y_i}\Phi^{-1}(v)+\sqrt{1-r_i^2}\,\sigma_{Y_i}\Phi^{-1}(F_{\mathbb{S}^u}(x|V=v))} = x, \tag{19}$$

where $r_i = \mathrm{corr}(Y_i, \Lambda)$. The cdf of $\mathbb{S}^u$ is then given by

$$F_{\mathbb{S}^u}(x) = \int_0^1 F_{\mathbb{S}^u|V=v}(x)dv. \tag{20}$$

We now look for an expression for the stop-loss premium at retention d with $F^{-1}_{\mathbb{S}^u|V=v}(0) < d < F^{-1}_{\mathbb{S}^u|V=v}(1)$ for $\mathbb{S}^u$, see (61):

$$E\left[(\mathbb{S}^u - d)_+\right] = \int_0^1 E\left[(\mathbb{S}^u - d)_+ \mid V=v\right] dv = \sum_{i=0}^{n-1} \int_0^1 E\left[\left(F^{-1}_{X_i|\Lambda}(U \mid V=v) - d_i\right)_+\right] dv \tag{21}$$

with $d_i = F^{-1}_{X_i|\Lambda}\left(F_{\mathbb{S}^u}(d \mid V=v) \mid V=v\right)$ and with U a random variable which is uniformly distributed on $(0,1)$ and independent of V. Since $F^{-1}_{X_i|\Lambda}(U \mid V=v)$ follows a lognormal distribution with mean and standard deviation:

$$\mu_v(i) = \ln \alpha_i + E\left[Y_i\right] + r_i\sigma_{Y_i}\Phi^{-1}(v), \quad \sigma_v(i) = \sqrt{1-r_i^2}\,\sigma_{Y_i},$$

one obtains that

$$d_i = \alpha_i \exp\left[E\left[Y_i\right] + r_i\sigma_{Y_i}\Phi^{-1}(v) + \alpha_i\sqrt{1-r_i^2}\,\sigma_{Y_i}\Phi^{-1}\left(F_{\mathbb{S}^u|V=v}(d)\right)\right]. \tag{22}$$

The well-known formula (65) then yields

$$E\left[(\mathbb{S}^u - d)_+ \mid V=v\right] = \sum_{i=0}^{n-1}\left[\alpha_i e^{\mu_v(i)+\frac{\sigma_v^2(i)}{2}}\Phi(\alpha_i d_{i,1}) - d_i\Phi(\alpha_i d_{i,2})\right],$$

with, according to (66),

$$d_{i,1} = \frac{\mu_v(i) + \sigma_v^2(i) - \ln d_i}{\sigma_v(i)}, \qquad d_{i,2} = d_{i,1} - \sigma_v(i).$$

Substitution of the corresponding expressions and integration over the interval $[0,1]$ leads to the following result

$$E\left[(\mathbb{S}^u - d)_+\right] = \sum_{i=0}^{n-1} \alpha_i e^{E[Y_i]+\frac{1}{2}\sigma^2_{Y_i}\left(1-r_i^2\right)} \times$$
$$\times \int_0^1 e^{r_i \sigma_{Y_i} \Phi^{-1}(v)} \Phi\left(\text{sign}(\alpha_i)\sqrt{1-r_i^2}\,\sigma_{Y_i} - \Phi^{-1}\left(F_{\mathbb{S}^u|V=v}(d)\right)\right) dv$$
$$- d\left(1 - F_{\mathbb{S}^u}(d)\right). \tag{23}$$

The upper bound then follows from (19) and (23) for $d = nK$ by plugging in α_i, Y_i and its mean and variance from (8), while denoting the correlations r_i by ρ_{T-i}. □

We found that the conditioning variable

$$\Lambda = \sum_{k=1}^{T} \beta_k W_k, \quad \text{with } W_k \text{ i.i.d. } N(0,1) \text{ such that } B(T-i) \stackrel{d}{=} \sum_{k=1}^{T-i} W_k,\ i = 0, ...n-1, \tag{24}$$

with all β_k equal to a same constant (for simplicity taken equal to one) leads to a sharper upper bound than other choices for β_k or than the conditioning variables in the lower bound.

For $\Lambda = \sum_{k=1}^{T} W_k \stackrel{d}{=} B(T)$ the correlation terms have the form:

$$r_i = \rho_{T-i} = \frac{\text{cov}\left(B(T-i), \Lambda\right)}{\sqrt{T-i}\,\sigma_\Lambda} = \frac{T-i}{\sqrt{T-i}\sqrt{T}} = \frac{\sqrt{T-i}}{\sqrt{T}},\ i = 0, \ldots, n-1, \tag{25}$$

and the dependence structure of the terms in the sum $\mathbb{S}^u$ corresponds better to that of the terms in the sum $\mathbb{S}$ than for other choices of Λ. Investigating the correlations

$$\text{corr}\left[F^{-1}_{S(T-i)|\Lambda}(U), F^{-1}_{S(T-j)|\Lambda}(U)\right] = \frac{e^{[\rho_{T-i}\rho_{T-j}+\sqrt{1-\rho^2_{T-i}}\sqrt{1-\rho^2_{T-j}}]\sigma^2\sqrt{T-i}\sqrt{T-j}} - 1}{\sqrt{e^{\sigma^2(T-i)}-1}\sqrt{e^{\sigma^2(T-j)}-1}}$$
$$\text{corr}\left[S(T-i), S(T-j)\right] = \frac{e^{\sigma^2 \min(T-i,T-j)} - 1}{\sqrt{e^{\sigma^2(T-i)}-1}\sqrt{e^{\sigma^2(T-j)}-1}},$$

it can be seen that for ρ_{T-i} given by (25) these correlations not only coincide for $i = j$ but also when one of the indices i or j equals zero. Moreover, for $i \neq j$, the differences

$$\left|[\rho_{T-i}\rho_{T-j} + \sqrt{1-\rho^2_{T-i}}\sqrt{1-\rho^2_{T-j}}]\sigma^2\sqrt{T-i}\sqrt{T-j} - \sigma^2 \min(T-i, T-j)\right|$$

are small for all i and j in $\{0, \ldots, n-1\}$ in comparison to other choices of Λ.

As in the case of the lower bound, we can rewrite the upper bound as an expression of Black & Scholes formulae.

Theorem 4. *For a general normally distributed conditioning variable Λ, satisfying the assumptions of Theorem 1, the improved upper bound of $AC(n,K,T)$ can be written as a combination of Black & Scholes formulae for an artificial underlying asset $\tilde{S}(t)$ with $\tilde{S}(0)=S(0)$ and with volatilities $\tilde{\sigma}_i=\sigma\sqrt{1-\rho^2_{T-i}}$:*

$$\begin{aligned}\frac{e^{-rT}}{n}E^Q\left[(\mathbb{S}^u-nK)_+\right] &= \int_0^1 \frac{1}{n}\sum_{i=0}^{n-1} e^{\rho_{T-i}\,\sigma\sqrt{T-i}\,\Phi^{-1}(v)-\frac{\sigma^2}{2}\rho^2_{T-i}(T-i)} \\ &\quad\times\left\{e^{-ri}\tilde{S}(0)\Phi\left(d_{1,i}(v)\right)-e^{-rT}\tilde{K}_i(v)\Phi\left(d_{2,i}(v)\right)\right\}dv\end{aligned}$$

with

$$\tilde{S}(T-i)=\tilde{S}(0)e^{(r-\frac{\tilde{\sigma}_i^2}{2})(T-i)+\tilde{\sigma}_i B(T-i)}$$

and the strike prices defined by

$$\tilde{K}_i(v)=S(0)e^{(r-\frac{\tilde{\sigma}_i^2}{2})(T-i)+\tilde{\sigma}_i\sqrt{T-i}\Phi^{-1}\left(F_{\mathbb{S}^u|V=v}(nK)\right)}$$

where

$$\begin{aligned}d_{1,i}(v) &= \frac{\left(r+\frac{\tilde{\sigma}_i^2}{2}\right)(T-i)-\ln\left(\frac{\tilde{K}_i(v)}{\tilde{S}(0)}\right)}{\tilde{\sigma}_i\sqrt{T-i}}=\tilde{\sigma}_i\sqrt{T-i}-\Phi^{-1}\left(F_{\mathbb{S}^u|V=v}(nK)\right)\\ d_{2,i}(v) &= d_{1,i}(v)-\tilde{\sigma}_i\sqrt{T-i}=-\Phi^{-1}\left(F_{\mathbb{S}^u|V=v}(nK)\right)\end{aligned}$$

and $F_{\mathbb{S}^u|V=v}(nK)$ can be calculated similarly to (18) *from $\sum_{i=0}^{n-1}\tilde{K}_i(v)=nK$.*

2.2 Bounds based on the Rogers & Shi approach

As an alternative to Section 2.1.2, following the ideas of Rogers and Shi (1995), we derive an upper bound based on the lower bound. Indeed, we apply the following general inequality for any random variable Y and Z from Rogers and Shi (1995):

$$0\le E\left[E\left[Y^+\mid Z\right]-E\left[Y\mid Z\right]^+\right]\le\frac{1}{2}E\left[\sqrt{\text{var}(Y\mid Z)}\right]. \tag{26}$$

Theorem 5. *Let $\mathbb{S}$ be given by* (6)-(8) *and Λ is a normally distributed conditioning variable such that $(\sigma B(T-i),\Lambda)$ are bivariate normally distributed for all i. Then an upper bound of the option*

price $AC(n,K,T)$ is given by

$$\mathrm{UB}\Lambda = \frac{e^{-rT}}{n}\left\{E^Q\left[(\mathbb{S}^\ell - nK)^+\right] + \varepsilon\right\}, \tag{27}$$

where the error bound ε equals

$$\begin{aligned}\varepsilon =& \frac{1}{2}E^Q\left[\sqrt{\mathrm{var}^Q(\mathbb{S}\mid\Lambda)}\right]\\ =& \frac{1}{2}\int_0^1\left\{\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\alpha_i\alpha_j e^{r_{ij}\sigma\sigma_{ij}\Phi^{-1}(v)+\frac{1}{2}(1-r_{ij}^2)\sigma^2\sigma_{ij}^2}\right.\\ &\left.-\left(\sum_{i=0}^{n-1}S(0)\,e^{\left(r-\frac{1}{2}\sigma^2\rho_{T-i}^2\right)(T-i)+\rho_{T-i}\sigma\sqrt{T-i}\Phi^{-1}(v)}\right)^2\right\}^{\frac{1}{2}}dv,\end{aligned} \tag{28}$$

with

$$\alpha_i\alpha_j = S(0)^2\exp\left[(r-\frac{\sigma^2}{2})(2T-i-j)\right], \tag{29}$$

$$\sigma_{ij} = \sqrt{(T-i)+(T-j)+2\min(T-i,T-j)}, \tag{30}$$

$$r_{ij} = \frac{\sqrt{T-i}}{\sigma_{ij}}\rho_{T-i}+\frac{\sqrt{T-j}}{\sigma_{ij}}\rho_{T-j}. \tag{31}$$

Proof. By applying (26) to the case of Y being $\sum_{i=0}^{n-1}S(T-i)-nK$ and Z being a conditioning variable Λ, we obtain an error bound for the difference of the option price and its lower bound

$$0 \le E^Q\left[E^Q\left[(\mathbb{S}-nK)^+\mid\Lambda\right]-(\mathbb{S}^\ell-nK)^+\right] \le \frac{1}{2}E^Q\left[\sqrt{\mathrm{var}^Q(\mathbb{S}\mid\Lambda)}\right]. \tag{32}$$

Consequently, (27) follows after discounting as the upper bound for the option price $AC(n,K,T)$. Using properties of lognormal distributed variables, $E^Q\left[\sqrt{\mathrm{var}^Q(\mathbb{S}\mid\Lambda)}\right]$ can be written out explicitly, giving some lengthy, analytical, computable expression:

$$\begin{aligned}E^Q\left[\sqrt{\mathrm{var}^Q(\mathbb{S}\mid\Lambda)}\right] &= E^Q\left[\left(E^Q\left[\mathbb{S}^2\mid\Lambda\right]-E^Q\left[\mathbb{S}\mid\Lambda\right]^2\right)^{1/2}\right] \qquad (33)\\ &= E^Q\left[\left(\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}E^Q\left[S(T-i)S(T-j)\mid\Lambda\right]-\left(\mathbb{S}^\ell\right)^2\right)^{1/2}\right],\end{aligned}$$

where the first term in the expectation in the right hand side equals

$$\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\alpha_i\alpha_j\exp\left(r_{ij}\sigma\sigma_{ij}\Phi^{-1}(V)+\frac{1}{2}(1-r_{ij}^2)\sigma^2\sigma_{ij}^2\right), \tag{34}$$

where V is uniformly distributed on the interval $(0,1)$. The second term in the expectation in the right hand side of (33) can according to (68) in Theorem 12 be written as

$$\mathbb{S}^{\ell} \stackrel{d}{=} \sum_{i=0}^{n-1} S(0)\, e^{\left(r-\frac{1}{2}\sigma^2\rho_{T-i}^2\right)(T-i)+\rho_{T-i}\sigma\sqrt{T-i}\Phi^{-1}(V)} \tag{35}$$

by plugging in α_i, Y_i and its mean and variance from (8), while denoting the correlations r_i by ρ_{T-i}, and simplifying. □

Note that the error bound (32) and hence ε are independent of the strike price K. In the following theorem we show how to strengthen the error bound ε in Theorem 5 by making it dependent on the strike price through a suitably chosen constant d_Λ such that $\Lambda \geq d_\Lambda$ implies that $\mathbb{S} \geq nK$. The meaning of finding such d_Λ for a general conditioning variable Λ is seen from the fact that we have on the set $\{\Lambda \geq d_\Lambda\}$ the relation:

$$E^Q[(\mathbb{S}-nK)_+ \mid \Lambda] = E^Q[\mathbb{S}-nK \mid \Lambda] = \left(\mathbb{S}^{\ell}-nK\right)_+ . \tag{36}$$

The following theorem can be seen as a generalization of the corresponding result in Nielsen and Sandmann (2003). Whereas Nielsen and Sandmann (2003) derived their result directly for Λ given by (14), we extend this approach to any normally distributed conditioning random variable Λ.

Theorem 6. *Let $\mathbb{S}$ be given by* (6)-(8) *and Λ is a normally distributed conditioning variable such that $(\sigma B(T-i), \Lambda)$ are bivariate normally distributed for all i. Suppose there exists a $d_\Lambda \in \mathbb{R}$ such that $\Lambda \geq d_\Lambda$ implies that $\mathbb{S} \geq nK$. Then an upper bound to the option price $AC(n,K,T)$ is given by*

$$\mathrm{UB}\Lambda_d = \mathrm{LB}\Lambda + \frac{e^{-rT}}{n}\varepsilon(d_\Lambda) \tag{37}$$

where the error bound $\varepsilon(d_\Lambda)$ is given by

$$\begin{aligned}\varepsilon(d_\Lambda) = \frac{S(0)}{2}\left\{\Phi(d_\Lambda^*)\right\}^{\frac{1}{2}} \times \\ \times \left\{\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} e^{r(2T-i-j)+\sigma^2\rho_{T-i}\rho_{T-j}\sqrt{T-i}\sqrt{T-j}}\Phi\left(d_\Lambda^* - \sigma(\rho_{T-i}\sqrt{T-i}+\rho_{T-j}\sqrt{T-j})\right)\times\right. \\ \left.\times\left(e^{\sigma^2\left(\min(T-i,T-j)-\rho_{T-i}\rho_{T-j}\sqrt{T-i}\sqrt{T-j}\right)}-1\right)\right\}^{\frac{1}{2}},\end{aligned} \tag{38}$$

with $d_\Lambda^* = \frac{d_\Lambda - E^Q[\Lambda]}{\sigma_\Lambda}$, $\Phi(\cdot)$ *the standard normal cdf and* $\rho_{T-i} = \mathrm{corr}(\sigma B(T-i), \Lambda) \geq 0$.

Proof. In general, for $d_\Lambda \in \mathbb{R}$ such that $\Lambda \geq d_\Lambda$ implies that $\mathbb{S} \geq nK$, it follows by (36) that:

$$\begin{aligned}
0 &\leq E^Q\left[E^Q[(\mathbb{S}-nK)_+ \mid \Lambda] - \left(\mathbb{S}^\ell - nK\right)_+\right] \\
&= \int_{-\infty}^{d_\Lambda} \left(E^Q[(\mathbb{S}-nK)_+ \mid \Lambda=\lambda] - \left(E^Q\left[\mathbb{S} \mid \Lambda=\lambda\right] - nK\right)_+\right) dF_\Lambda(\lambda) \\
&\leq \frac{1}{2}\int_{-\infty}^{d_\Lambda} \left(\mathrm{var}^Q\left(\mathbb{S} \mid \Lambda=\lambda\right)\right)^{\frac{1}{2}} dF_\Lambda(\lambda) \qquad (39) \\
&\leq \frac{1}{2}\left(E^Q\left[\mathrm{var}^Q\left(\mathbb{S} \mid \Lambda\right) 1_{\{\Lambda<d_\Lambda\}}\right]\right)^{\frac{1}{2}} \left(E^Q\left[1_{\{\Lambda<d_\Lambda\}}\right]\right)^{\frac{1}{2}} =: \varepsilon(d_\Lambda), \qquad (40)
\end{aligned}$$

where Hölder's inequality has been applied in the last inequality, where $1_{\{\Lambda<d_\Lambda\}}$ is the indicator function, and where $F_\Lambda(\cdot)$ denotes the normal cumulative distribution function of Λ.

The first expectation term in the product (40) can be expressed as

$$E^Q\left[\mathrm{var}^Q\left(\mathbb{S}|\Lambda\right) 1_{\{\Lambda<d_\Lambda\}}\right] = E^Q\left[E^Q[\mathbb{S}^2|\Lambda]1_{\{\Lambda<d_\Lambda\}}\right] - E^Q\left[(E^Q[\mathbb{S}|\Lambda])^2 1_{\{\Lambda<d_\Lambda\}}\right]. \qquad (41)$$

The second term of the right-hand side of (41) can according to (35) be rewritten as

$$\begin{aligned}
E^Q\left[(E^Q[\mathbb{S}|\Lambda])^2 1_{\{\Lambda<d_\Lambda\}}\right] &= \int_{-\infty}^{d_\Lambda} (E^Q[\mathbb{S}|\Lambda=\lambda])^2 dF_\Lambda(\lambda) \\
&= S(0)^2 \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} e^{r(2T-i-j)-\frac{\sigma^2}{2}\left(\rho_{T-i}^2(T-i)+\rho_{T-j}^2(T-j)\right)} \int_{-\infty}^{d_\Lambda} e^{\sigma\left(\rho_{T-i}\sqrt{T-i}+\rho_{T-j}\sqrt{T-j}\right)\Phi^{-1}(v)} dF_\Lambda(\lambda)
\end{aligned} \qquad (42)$$

where we recall that $\Phi^{-1}(v) = \frac{\lambda - E^Q[\Lambda]}{\sigma_\Lambda}$ and $\Phi(\cdot)$ is the cumulative distribution function of a standard normal variable. Applying the equality

$$\int_{-\infty}^{d_\Lambda} e^{b\Phi^{-1}(v)} dF_\Lambda(\lambda) = e^{\frac{b^2}{2}}\Phi\left(d_\Lambda^* - b\right), \quad d_\Lambda^* = \frac{d_\Lambda - E^Q[\Lambda]}{\sigma_\Lambda}, \qquad (43)$$

with $b = \sigma\left(\rho_{T-i}\sqrt{T-i} + \rho_{T-j}\sqrt{T-j}\right)$ we can express $E^Q\left[(E^Q[\mathbb{S}|\Lambda])^2 1_{\{\Lambda<d_\Lambda\}}\right]$ as

$$S(0)^2 \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} e^{r(2T-i-j)+\sigma^2\rho_{T-i}\rho_{T-j}\sqrt{T-i}\sqrt{T-j}} \Phi\left(d_\Lambda^* - \sigma(\rho_{T-i}\sqrt{T-i} + \rho_{T-j}\sqrt{T-j})\right). \qquad (44)$$

To transform the first term of the right-hand side of (41) we invoke (29)-(31) and apply (43) with $b = r_{ij}\sigma\sigma_{ij} = \sigma\left(\rho_{T-i}\sqrt{T-i} + \rho_{T-j}\sqrt{T-j}\right)$:

$$
\begin{aligned}
&E^Q\left[E^Q[\mathbb{S}^2 \mid \Lambda]1_{\{\Lambda<d_\Lambda\}}\right] \\
&= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\int_{-\infty}^{d_\Lambda} E^Q\left[S(T-i)S(T-j)|\Lambda=\lambda\right]dF_\Lambda(\lambda) \\
&= S(0)^2\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} e^{(r-\frac{\sigma^2}{2})(2T-i-j)+\frac{1}{2}\left(1-r_{ij}^2\right)\sigma^2\sigma_{ij}^2}\int_{-\infty}^{d_\Lambda} e^{r_{ij}\sigma\sigma_{ij}\Phi^{-1}(v)}dF_\Lambda(\lambda) \\
&= S(0)^2\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} e^{r(2T-i-j)+\sigma^2\min(T-i,T-j)}\Phi\left(d_\Lambda^* - \sigma(\rho_{T-i}\sqrt{T-i}+\rho_{T-j}\sqrt{T-j})\right). \qquad (45)
\end{aligned}
$$

The second expectation term in the product (40) equals $F_\Lambda(d_\Lambda) = \Phi(d_\Lambda^*)$.

Combining (44) and (45) into (41), and then substituting $\Phi(d_\Lambda^*)$ and (41) into (40) finally leads to expression (38). □

We stress that the error bound (40) and thus (38) hold for any conditioning normal random variable Λ that satisfies the assumptions of Theorem 1 and for which there exists an integration bound d_Λ such that $\Lambda \geq d_\Lambda$ implies $\mathbb{S} \geq nK$. For Λ given by (14), Nielsen and Sandmann found that the corresponding d_Λ is given by

$$
d_{GA} = \frac{n\ln\left(\frac{K}{S(0)}\right) - \sum_{i=0}^{n-1}\left(r-\frac{\sigma^2}{2}\right)(T-i)}{\sigma\sqrt{n^2T - \frac{1}{6}n(n-1)(4n+1)}}, \qquad (46)
$$

where the subscript GA is to remind the fact that Λ is the standardized logarithm of the geometric average. The error bound (38) coincides with the one found in Nielsen and Sandmann (2003) for the special choice (14) for Λ and the corresponding d_{GA} (46). Let us show that also for Λ given by (13) this technique works to strengthen the error bound (32) and hence to sharpen the upper bound (27). Using the property that $e^x \geq 1+x$ and relations (6)-(8) and (13), we obtain

$$
\mathbb{S} = \sum_{i=0}^{n-1}\alpha_i e^{Y_i} \geq \sum_{i=0}^{n-1}\alpha_i + S(0)\sigma\underbrace{\sum_{i=0}^{n-1} e^{(r-\frac{\sigma^2}{2})(T-i)}B(T-i)}_{=\Lambda}.
$$

Hence $\mathbb{S} \geq nK$ when Λ is larger than $\dfrac{nK - \sum_{i=0}^{n-1}\alpha_i}{S(0)\sigma}$. Thus in case of Λ being a linear transfor-

mation of the first order approximation (FA) of $\mathbb{S}$, we have

$$d_{FA} = \frac{nK - \sum_{i=0}^{n-1} S(0)e^{(r-\frac{\sigma^2}{2})(T-i)}}{S(0)\sigma}. \tag{47}$$

Let us also notice that the upper bound (27) corresponds to the limiting case of (39) where d_Λ equals infinity. Further note that in contrast to (32) the error bound now depends on K through d_Λ.

2.3 Partially exact/comonotonic upper bound

Next we combine the technique for obtaining an improved comonotonic upper bound by conditioning on some normally distributed random variable Λ and the idea of decomposing the calculations in an exact part and an approximating part which goes at least back to Curran (1994). This so-called partially exact/comonotonic upper bound consists of an exact part of the option price and some improved comonotonic upper bound for the remaining part. This upper bound improves the upper bound denoted by $C_A^{**,G}$ in the paper of Nielsen and Sandmann (2003), as will be explained at the end of this section.

Theorem 7. *Let $\mathbb{S}$ be given by* (6)-(8) *and Λ be a normally distributed conditioning variable such that $(\sigma B(T-i), \Lambda)$ are bivariate normally distributed for all i. Suppose there exists a $d_\Lambda \in \mathbb{R}$ such that $\Lambda \geq d_\Lambda$ implies that $\mathbb{S} \geq nK$. Then the partially exact/comonotonic upper bound to the option price $AC(n, K, T)$ is given by*

$$\begin{aligned}
&\mathrm{PECUB}\Lambda \\
&= \frac{S(0)}{n}\sum_{i=0}^{n-1} e^{-ri}\Phi(\rho_{T-i}\sigma\sqrt{T-i} - d_\Lambda^*) - e^{-rT}K\Phi(-d_\Lambda^*) \\
&\quad + \frac{S(0)}{n}\sum_{i=0}^{n-1} e^{-ri}e^{-\frac{\sigma^2}{2}\rho_{T-i}^2(T-i)} \\
&\qquad\qquad \times \int_0^{\Phi(d_\Lambda^*)} e^{\rho_{T-i}\,\sigma\sqrt{T-i}\,\Phi^{-1}(v)}\Phi\left(\sqrt{1-\rho_{T-i}^2}\,\sigma\sqrt{T-i} - \Phi^{-1}\left(F_{\mathbb{S}^u|V=v}(nK)\right)\right) dv \\
&\quad - e^{-rT}K\left(\Phi(d_\Lambda^*) - \int_0^{\Phi(d_\Lambda^*)} F_{\mathbb{S}^u|V=v}(nK)\, dv\right)
\end{aligned} \tag{48}$$

where $d^*_\Lambda = \dfrac{d_\Lambda - E^Q[\Lambda]}{\sigma_\Lambda}$ *and* $F_{\mathbb{S}^u|V=v}$ *is given by* (18) *and* $\rho_{T-i} = \mathrm{corr}(\sigma B(T-i), \Lambda)$.

Proof. For any normally distributed random variable Λ, with cdf $F_\Lambda(\cdot)$, for which there exists a d_Λ such that $\Lambda \geq d_\Lambda$ implies $\mathbb{S} \geq nK$ and which satisfies the assumptions of Theorem 1, we can write

$$\begin{aligned}
&\frac{e^{-rT}}{n} E^Q[(\mathbb{S} - nK)_+] = \frac{e^{-rT}}{n} E^Q[E^Q[(\mathbb{S} - nK)_+ \mid \Lambda]] \\
&= \frac{e^{-rT}}{n} \left\{ \int_{-\infty}^{d_\Lambda} E^Q[(\mathbb{S} - nK)_+ \mid \Lambda = \lambda] dF_\Lambda(\lambda) + \int_{d_\Lambda}^{+\infty} E^Q[\mathbb{S} - nK \mid \Lambda = \lambda] dF_\Lambda(\lambda) \right\}. \quad (49)
\end{aligned}$$

The second term in the equality (49) can be written in closed-form along similar lines as (42)-(44):

$$\begin{aligned}
&\frac{e^{-rT}}{n} \int_{d_\Lambda}^{+\infty} E^Q[\mathbb{S} \mid \Lambda = \lambda] dF_\Lambda(\lambda) - e^{-rT} K (1 - F_\Lambda(d_\Lambda)) \\
&= \frac{e^{-rT}}{n} \sum_{i=0}^{n-1} S(0) e^{(r - \frac{1}{2}\sigma^2 \rho^2_{T-i})(T-i)} \int_{d_\Lambda}^{+\infty} e^{\rho_{T-i}\sigma\sqrt{T-i}\,\Phi^{-1}(v)} dF_\Lambda(\lambda) - e^{-rT} K (1 - \Phi(d^*_\Lambda)) \\
&= \frac{S(0)}{n} \sum_{i=0}^{n-1} e^{-ri} \Phi(\rho_{T-i}\sigma\sqrt{T-i} - d^*_\Lambda) - e^{-rT} K \Phi(-d^*_\Lambda), \quad (50)
\end{aligned}$$

where $d^*_\Lambda = \dfrac{d_\Lambda - E^Q[\Lambda]}{\sigma_\Lambda}$ and $v = \dfrac{\lambda - E^Q[\Lambda]}{\sigma_\Lambda}$.

In the first term of (49) we replace $\mathbb{S}$ by $\mathbb{S}^u$ in order to obtain an upper bound and apply (16) but now with an integral from zero to $\Phi(d^*_\Lambda)$:

$$\begin{aligned}
&\frac{e^{-rT}}{n} \int_{-\infty}^{d} E^Q[(\mathbb{S} - nK)_+ \mid \Lambda = \lambda] dF_\Lambda(\lambda) \\
&\leq \frac{e^{-rT}}{n} \int_{-\infty}^{d} E^Q[(\mathbb{S}^u - nK)_+ \mid \Lambda = \lambda] dF_\Lambda(\lambda) = \frac{e^{-rT}}{n} \int_0^{\Phi(d^*_\Lambda)} E^Q[(\mathbb{S}^u - nK)_+ \mid V = v]\, dv \\
&= \frac{S(0)}{n} \sum_{i=0}^{n-1} e^{-ri} e^{-\frac{\sigma^2}{2}\rho^2_{T-i}(T-i)} \\
&\qquad \times \int_0^{\Phi(d^*_\Lambda)} e^{\rho_{T-i}\,\sigma\sqrt{T-i}\,\Phi^{-1}(v)} \Phi\left(\sqrt{1 - \rho^2_{T-i}}\,\sigma\sqrt{T-i} - \Phi^{-1}\left(F_{\mathbb{S}^u|V=v}(nK)\right) \right) dv \\
&\quad - e^{-rT} K \left(\Phi(d^*_\Lambda) - \int_0^{\Phi(d^*_\Lambda)} F_{\mathbb{S}^u|V=v}(nK)\, dv \right). \quad (51)
\end{aligned}$$

Adding (50) and (51) we obtain (48). □

Theorem 8. *For any conditioning variable Λ satisfying the assumptions of Theorem 7,*

$$\mathrm{PECUB}\Lambda \leq \mathrm{ICUB}\Lambda,$$

where $\mathrm{PECUB}\Lambda$ *and* $\mathrm{ICUB}\Lambda$ *are defined by* (48) *and* (16)*, respectively.*

Proof. Recall that according to the assumption of Theorem 7 there exists d_Λ such that $\Lambda \geq d_\Lambda \Rightarrow \mathbb{S} \geq nK$. Using this fact and by convex ordering of stop-loss premia of $\mathbb{S}$ and $\mathbb{S}^u$ we obtain

$$\begin{aligned}
ne^{rT}\mathrm{ICUB}\Lambda &= \int_{-\infty}^{+\infty} \mathrm{E}\left[(\mathbb{S}^u - nK)_+ \mid \Lambda = \lambda\right] dF_\Lambda(\lambda) \\
&= \int_{-\infty}^{d_\Lambda} \mathrm{E}\left[(\mathbb{S}^u - nK)_+ \mid \Lambda = \lambda\right] dF_\Lambda(\lambda) + \int_{d_\Lambda}^{+\infty} \mathrm{E}\left[(\mathbb{S}^u - nK)_+ \mid \Lambda = \lambda\right] dF_\Lambda(\lambda) \\
&\geq \int_{-\infty}^{d_\Lambda} \mathrm{E}\left[(\mathbb{S}^u - nK)_+ \mid \Lambda = \lambda\right] dF_\Lambda(\lambda) + \int_{d_\Lambda}^{+\infty} \mathrm{E}\left[(\mathbb{S} - nK)_+ \mid \Lambda = \lambda\right] dF_\Lambda(\lambda) \\
&= \int_{-\infty}^{d_\Lambda} \mathrm{E}\left[(\mathbb{S}^u - nK)_+ \mid \Lambda = \lambda\right] dF_\Lambda(\lambda) + \int_{d_\Lambda}^{+\infty} \mathrm{E}\left[\mathbb{S} - nK \mid \Lambda = \lambda\right] dF_\Lambda(\lambda) \\
&= ne^{rT}\mathrm{PECUB}\Lambda.
\end{aligned}$$

□

We stress that for two distinct conditioning variables Λ_1 and Λ_2 it does not necessarily hold that $\mathrm{PECUB}\Lambda_1 \leq \mathrm{ICUB}\Lambda_2$.

For the random variables Λ given by (13) and (14) we derived a d_Λ, see (47) and (46), and thus we can compute the new upper bound PECUBΛ, cfr. (48). Recall that these choices of Λ do not lead to the best improved comonotonic upper bound. The "best" choice is $\Lambda = B(T)$ for which we do not find the necessary d_Λ in this new upper bound. However, we expect that the contribution of the exact part (50) which is the second term in (49) will compensate for the somewhat lower quality of the $\mathbb{S}^u$.

Finally, we note that the upper bound $C_A^{**,G}$ in Nielsen and Sandmann (2003) was derived for the special conditioning variable Λ given by (14), with the usage of an optimization algorithm to find the weights a_i such that their upper bound for the first term in (49), namely

$$\frac{e^{-rT}}{n} \sum_{i=0}^{n-1} \int_{-\infty}^{d_\Lambda} E^Q[(S(T-i) - a_i nK)_+ \mid \Lambda = \lambda] dF_\Lambda(\lambda),$$

is minimized. In fact, they introduce a second approximation by bounding this expression from above using a portfolio of call options, following the presentation in Ross (1976). The expression obtained this way is then minimized with respect to the weights a_i. With our method, however, we directly have the explicit optimal solution of the original minimization problem, namely the optimal weights a_i for a given λ or v are:

$$\begin{aligned} a_i &= \frac{1}{nK} F^{-1}_{S(T-i)|\Lambda=\lambda}(F_{\mathbb{S}^u|V=v}(nK)) \\ &= \frac{S(0)}{nK} e^{(r-\frac{\sigma^2}{2})(T-i)+\rho_{T-i}\sigma\sqrt{T-i}\,\Phi^{-1}(v)+\sqrt{1-\rho^2_{T-i}}\,\sigma\sqrt{T-i}\,\Phi^{-1}(F_{\mathbb{S}^u|V=v}(nK))}. \end{aligned}$$

In this sense, the partially exact/comonotonic upper bound improves their upper bound $C_A^{**,G}$, see Table 2 for numerical results.

2.4 General remarks

In this section we summarize some general remarks:

1. Denoting the price of a European-style discrete arithmetic Asian *put* option with exercise date T, n averaging dates and fixed strike price K by $AP(n, K, T)$, we find from the put-call parity at the present:

$$AC(n, K, T) - AP(n, K, T) = \frac{S(0)}{n} \frac{1 - e^{-rn}}{1 - e^{-r}} - e^{-rT} K. \tag{52}$$

   Hence, we can derive bounds for the Asian put option from the bounds for the call. These bounds for the put option coincide with the bounds that are obtained by applying the theory of comonotonic bounds and the conditioning approach directly to Asian put options. This stems from the fact that the put-call parity also holds for these bounds.

2. Note that for numerical computations in (52), if n and T are expressed in days then r should be interpreted as a continuously compounded interest rate for one day which equals a continuously compounded interest rate for one year divided by the number of (trading) days per year.

3. The case of a continuous dividend yield δ can easily be dealt with by replacing the interest rate r by $r - \delta$.

4. When the number of averaging dates n equals 1, the Asian call option $AC(n, K, T)$ reduces to a European call option. It can be proven that in this case the upper and the lower bounds for the price of the Asian option both reduce to the Black and Scholes formula for the price of a European call option. For bounds based on a conditioning variable Λ this is true since for $n = 1$ we have that $\Lambda = \beta_0 B(T)$ while $\mathbb{S} = S(0) \exp\left((r - \frac{1}{2}\sigma^2)T + \sigma B(T)\right)$ implying that $\rho_T = 1$, and thus that $\mathbb{S}^u = \mathbb{S}^\ell = \mathbb{S}$.

5. The lower and upper bounds are derived for forward starting Asian options but they can easily be adapted to hold for Asian options in progress. In this case $T - n + 1 \leq 0$ and only the prices of $S(1), \ldots, S(T)$ remain random such that the price of the option is given by:

$$\begin{aligned} AC(n, K, T) &= \frac{e^{-rT}}{n} E^Q \left[\left(\sum_{i=0}^{n-1} S(T-i) - nK \right)_+ \right] \\ &= \frac{e^{-rT}}{n} E^Q \left[\left(\sum_{i=0}^{T-1} S(T-i) - \left(nK - \sum_{i=T}^{n-1} S(T-i) \right) \right)_+ \right]. \end{aligned}$$

   Thus substituting $nK - \sum_{i=T}^{n-1} S(T-i)$ for nK and summing for the average over i from zero to $T - 1$ instead of $n - 1$ the desired bounds follow.

6. The bounds can be extended to the case of deterministic volatility function $\sigma = \sigma(t)$ or $\sigma = \sigma(S(0), t)$ but are not applicable when we assume a stochastic volatility surface $\sigma = \sigma(S, t)$.

2.5 Numerical illustration

In this section we give a number of numerical examples in the Black & Scholes setting. We discuss our results and compare them to those found in the literature and to the Monte Carlo price. Further, we approximate $\mathbb{S}$ by a lognormal distribution which is the closest in the Kullback-Leibler sense. We also measure the closeness of the lower and upper bounds in the distributional sense.

2.5.1 Comparing bounds

In this section we discuss our results and compare them with those of Jacques (1996) where the distribution of the sum $\mathbb{S}$ of lognormals, see (6), entering in the European-style discrete arithmetic Asian option was approximated by means of the lognormal (LN) and the inverse Gaussian (IG) distribution. For the comparison we also included the upper bounds based on the lower bounds, see Theorem 5 and 6. We show here one set of numerical experiments where we consider a forward starting European-style discrete arithmetic Asian call option with fixed strike having the same data as in the paper of Jacques (1996): an initial stock price $S(0) = 100$, a nominal annual (daily discretely compounded) interest rate of 9% per year (corresponding to a continuously compounded interest rate $r = \ln\left(1 + \frac{0.09}{365}\right)$ per day[1] or 8,9989% per year), a maturity of 120 days and an averaging period n of 30 days. The values of the volatility σ are on annual basis. As a benchmark we included the price obtained via Monte Carlo simulation by adapting the control variate technique of Kemna and Vorst (1990) to European-style discrete arithmetic Asian options. The number of simulated Monte Carlo paths was $10\,000$.

We use the following notations where Λ can be GA, FA or B_T: LBΛ for lower bound, PECUBΛ for partially exact/comonotonic upper bound, UBΛ for upper bound based on lower bound (cfr. Theorem 5), and UBΛ_d for upper bound given by Theorem 6.

As we see from Table 1, the lower bounds LBFA and LBGA are equal up to five decimals. They both perform much better in comparison with Monte Carlo results than the lower bound LBB_T where we conditioned on $\Lambda = \sum_{k=1}^{T} W_k \stackrel{d}{=} B(T)$ (cfr. (24)). The bad performance is due to the fact that $B(T)$ differs much from $\mathbb{S}$ for n larger than one and hence $\mathrm{E}^Q\left[\sqrt{\mathrm{var}^Q(\mathbb{S} \mid B(T))}\right]$ is large, while for the Λ of (13) or (14) this term $\mathrm{E}^Q\left[\sqrt{\mathrm{var}^Q(\mathbb{S} \mid \Lambda)}\right]$ is very small because Λ en $\mathbb{S}$ are very much alike. It seems that the relative difference between a lower bound and its

[1] In the paper of Jacques (1996) this interest rate is reported and is used in our computations of the bounds. The actual computations in Jacques (1996) were made with a continuously compounded interest rate of $\frac{\ln(1+0.09)}{365}$ per day, where 9% is an effective annual interest rate. Due to this inconsistency, we recomputed LN and IG approximations with the interest rate as mentioned in that paper.

upper-bound-counterpart increases with K. For the upper bounds UBFA and UBB_T this is clear, since for different values of K a same constant is added while the value of the lower bound is decreasing. The upper bound UBGA_d which is based on the lower bound LBGA plus a pricing error cfr. (37)-(38) and (46), performs the best of all upper bounds considered. However, UBFA_d cfr. (37)-(38) and (47), performs good as well. For this set of parameters, the values for the partially exact/comonotonic upper bound PECUBGA, cfr. (48) and (46), are smaller than those for the improved comonotonic upper bound ICUBB_T but, as the results in Table 1 show for the case of Λ given by (14), they are not that good as we would have expected. Notice that we have included only PECUBGA in Table 1 since it was the best PECUBΛ upper bound for the two conditioning variables that we consider.

Comparing UBFA with UBFA_d, we note that making the error bound dependent on the strike price K has led to an improvement. Table 1 also reveals that in general the lognormal (LN) approximation as well as the inverse Gaussian (IG) approximation of Jacques (1996) fall within the interval given by the best lower bound and the best upper bound. The exception is the lognormal approximation in case when $K = 110$ for $\sigma = 0.2$ and $\sigma = 0.3$, and the inverse Gaussian approximation in case when $K = 80$ for $\sigma = 0.2$, $\sigma = 0.3$, and $\sigma = 0.4$ (in those cases the prices are smaller than the (comonotonic) lower bounds LBFA and LBGA). Notice that the approximations of Jacques (1996) (except of the cases mentioned above) are always higher than the respective Monte Carlo values, but nevertheless they all fall into the Monte Carlo price interval (MC $\pm$ SE). Further, note that the precision of the simulated prices decreases as the volatility σ increases. The Monte Carlo approach systematically seems to underestimate the true price, especially for at- and out-of-the-money options for which the Monte Carlo price falls slightly below the lower bounds.

Conclusion 1. *From Table 1 LBFA and LBGA perform equally well and are very close to the Monte Carlo values. The UBGA_d is the best upper bound for the parameters considered in this table.*

2.5.2 The effect of the averaging period and of interest rates on the bounds

In this section we compare bounds over several averaging periods and for different interest rates. For different sets of parameters, we have computed the lower and the upper bounds together with the price obtained by Monte Carlo simulation[2]. The latter is based on generating $10\,000$ paths. This has been done in particular for four different options: the first with expiration date at time $T = 120$ and 30 averaging days, the second with expiration at time $T = 60$ and 30 averaging days, the third one with again expiration time $T = 120$ but only 10 averaging days, and as the last one we considered the case where averaging was done over the whole period of 120 days. In all cases we considered the 4 following strike prices K: $80, 90$, 100 and 110, three values (0.2, 0.3 and 0.4) for the volatility σ, and the two different flat continuously compounded risk-free interest rates r: 5% and 9% yearly. The initial stock price was fixed at $S(0) = 100$.

The absolute and relative differences between the best upper and lower bound increase with the volatility and with the strike price, but decrease with the interest rate. The results further suggest that all intervals are sharper for options that are in-the-money. For fixed maturity, the length of the intervals reduces with the number of averaging dates. However for a fixed averaging period the effect of the maturity date seems to be less clear.

Conclusion 2. *The difference between the lower bounds LBGA and LBFA is overall practically zero. The upper bound UBGA_d is in general the best but for example when $r = 0.05$, $K = 100$ and $\sigma = 0.4$, UBFA_d turns out to be smaller than UBGA_d.*

[2]The tables with the results discussed in this paragraph are available from Liinev (2003).

$T = 120, n = 30, r = \ln(1 + 0.09/365)$ daily, $S(0) = 100$

σ	K	LN	IG	MC (SE $\times 10^4$)	LBB_T	**LBFA**	**LBGA**	**UBGA_d**	UBFA_d	UBFA	PECUBGA	ICUBB_T	UBB_T
0.2	80	22.0027	22.0022	22.00271 (2.5)	21.994822	22.002619	22.002619	22.002732	22.002849	22.014767	22.004625	22.006032	23.446236
	90	12.7603	12.7599	12.76012 (2.6)	12.691751	12.760052	12.760053	12.761283	12.761506	12.772219	12.778069	12.786728	14.143164
	100	5.5219	5.5236	5.521652 (2.5)	5.364993	5.521689	5.521689	5.526257	5.526389	5.533856	5.566340	5.580651	6.816407
	110	1.6526	1.6536	1.652697 (2.0)	1.518289	1.652807	1.652806	1.661491	1.661639	1.664974	1.695799	1.704168	2.969703
0.3	80	22.3102	22.3079	22.30976 (5.8)	22.250172	22.309736	22.309736	22.311225	22.311808	22.337168	22.325349	22.333495	24.428128
	90	13.9253	13.9268	13.92461 (5.9)	13.763614	13.924578	13.924579	13.929696	13.930099	13.952005	13.968496	13.985921	15.941570
	100	7.5351	7.5414	7.534506 (5.8)	7.295732	7.534676	7.534676	7.545641	7.545771	7.562103	7.603959	7.624473	9.473688
	110	3.5174	3.5225	3.517352 (5.1)	3.288965	3.517536	3.517535	3.534765	3.535066	3.544963	3.589000	3.604201	5.466921
0.4	80	23.0359	23.0339	23.03488 (10.7)	22.894509	23.034765	23.034765	23.039974	23.041030	23.083564	23.072463	23.088993	25.800008
	90	15.4251	15.4330	15.42367 (10.8)	15.172741	15.423789	15.423789	15.435454	15.435878	15.472586	15.493971	15.518613	18.078240
	100	9.5649	9.5805	9.563843 (10.5)	9.244120	9.564114	9.564114	9.584043	9.584080	9.612911	9.658116	9.684280	12.149619
	110	5.5176	5.5318	5.517215 (9.7)	5.199653	5.517573	5.517573	5.545909	5.546323	5.566370	5.616391	5.637784	8.105152

Table 1: LN and IG approximations of Jacques (1996) compared with our bounds.

$T = 3$ years, $n = 3$ years, $r = 0.04$ yearly, $S(0) = 100$

σ	K	MC (SE $\times 10^3$)	LBFA	LBGA	UBGA_d	UBFA_d	UBFA	UBGA	PECUBGA	ICUBB_T	$C_A^{**,G}$	$C_A^{*,G}$	$C_A^{N,G}$
0.25	50	50.0506 (5.6)	50.0473	50.0472	**50.0488**	50.0599	50.5557	50.6536	50.0517	50.0565	50.0518	50.0535	50.0641
	80	24.7540 (5.7)	24.7457	24.7471	**24.8222**	24.8342	25.2542	25.3535	25.0299	25.2125	25.0424	25.0931	25.2908
	90	17.9405 (5.8)	17.9312	17.9343	**18.0582**	18.0632	18.4396	18.5406	18.4047	18.6367	18.4309	18.4950	18.6188
	100	12.4799 (5.9)	12.4759	12.4743	**12.6490**	12.6565	12.9843	13.0807	13.1149	13.3350	13.1516	13.2158	13.2088
	110	8.3887 (6.0)	8.3860	8.3830	**8.6110**	8.6206	8.8944	8.9894	9.1259	9.2843	9.1717	9.2261	9.1827
	200	0.1214 (2.9)	0.1183	0.1159	0.6962	0.6104	0.6267	0.7223	0.2514	**0.2081**	0.2662	0.2666	0.5922

Table 2: Comparing bounds in Nielsen and Sandmann (2003) with our results.

σ :	yearly volatility
K :	strike price
LN :	lognormal approximation of a sum of lognormals
IG :	inverse gaussian approximation of a sum of lognormals
MC :	Monte Carlo price with its standard error (SE) based on 10 000 paths
LBB_T :	lower bound with $\Lambda = \sum_{k=1}^{T} W_k \stackrel{d}{=} B(T)$
LBFA :	lower bound with $\Lambda = \sum_{j=0}^{n-1} \exp[(r - \frac{\sigma^2}{2})(T-j)]B(T-j)$
LBGA :	lower bound with $\Lambda = (\ln \mathbb{G} - E^Q[\ln \mathbb{G}])/\sqrt{\mathrm{var}^Q(\ln \mathbb{G})}$
UBGA_d :	upper bound equal to lower bound LBGA plus $\varepsilon(d_{GA})\mathrm{e}^{-rT}/n$
UBFA_d :	upper bound equal to lower bound LBFA plus $\varepsilon(d_{FA})\mathrm{e}^{-rT}/n$
UBFA :	upper bound equal to lower bound LBFA plus constant $\varepsilon\mathrm{e}^{-rT}/n$
UBGA :	upper bound equal to lower bound LBGA plus constant $\varepsilon\mathrm{e}^{-rT}/n$
UBB_T :	upper bound equal to lower bound LBB_T plus constant $\varepsilon\mathrm{e}^{-rT}/n$
PECUBGA :	partially exact/comonotonic upper bound with $\Lambda = (\ln \mathbb{G} - E^Q[\ln \mathbb{G}])/\sqrt{\mathrm{var}^Q(\ln \mathbb{G})}$
ICUBB_T :	improved comonotonic upper bound with $\Lambda = \sum_{k=1}^{T} W_k \stackrel{d}{=} B(T)$
$C_A^{**,G}$:	$C_A^{u,G}$ upper bound of Nielsen and Sandmann with optimal weights
$C_A^{*,G}, C_A^{N,G}$:	$C_A^{u,G}$ with special choice for weights, with equal weights

2.5.3 Comparison of lower and upper bounds as in Nielsen and Sandmann (2003) with our bounds

In this section we use the data from Nielsen and Sandmann (2003) in order to compare their different upper bounds with our results. They give as input data: $\sigma = 0.25$, $r = 0.04$, $S(0) = 100$, $T = 3$ years. Note that they use price averaging over the whole period ($n = 3$ years) where averaging takes place each month (in the previous sections the averaging was done daily).

The first column of Table 2 shows the selection of strike prices from Nielsen and Sandmann (2003). In addition to the strike prices used in the above sections we also included $K = 50$ and $K = 200$ as examples of extreme in- and out-of-the-money options.

The bounds LBGA, UBGA and UBGA_d in Table 2 were reported in Nielsen and Sandmann (2003) and we recall that these three bounds are the special cases of the more general bounds LBΛ, UBΛ and UBΛ_d, respectively. Nielsen and Sandmann (2003) also derive another upper bound $C_A^{u,G}$ which depends on coefficients a_i satisfying $\sum_{i=1}^n a_i = 1$. The last three columns in Table 2 show the bounds $C_A^{u,G}$ for different choices of coefficients a_i. The columns labelled as $C_A^{*,G}$ and $C_A^{N,G}$ are computed for the choice of $a_i = a_i^*$ (special choice by Nielsen and Sandmann) and $a_i = \frac{1}{n}$, respectively. The column $C_A^{**,G}$ presents the results for the optimal sequence of the weights a_i in relation to the $C_A^{u,G}$ bound (i.e. the sequence which minimizes the upper bound $C_A^{u,G}$). From this table it is clear that the PECUBGA indeed improves $C_A^{**,G}$ as explained in Section 2.3.

We note again that the partially exact/comonotonic upper bound PECUBGA is smaller and thus better than the improved comonotonic upper bound ICUBB_T for strike prices in the range 50 to 150 (not all values are reported in Table 2), but for deeply out-of-the-money options there is a switch and ICUBB_T becomes better and even for $K = 200$ outperforms all other the upper bounds including the choices of Nielsen and Sandmann. Note that this is an example of the case when for two distinct conditioning variables Λ_1 and Λ_2 it does not follow that PECUB$\Lambda_1 \leq$ ICUBΛ_2.

Conclusion 3. *We can conclude that the best upper bound is again given by UBGA_d. Notice also that the lower bounds LBFA and LBGA are very close and equal up to two decimals.*

2.5.4 Distributional distance between the bounds and lognormal approximation of $\mathbb{S}$

As already mentioned, the sum of lognormal random variables is not lognormally distributed. However, in practice it is often claimed to be approximately lognormal. In this section we aim to quantify the distance between the distribution of $\mathbb{S} = \sum_{i=1}^{n} X_i$, (7), which is a sum of lognormal random variables, and the lognormal family of distributions by means of the so-called Kullback-Leibler information. We also use the Hellinger distance in order to measure the closeness of the derived lower and upper bounds. This section uses the ideas from Brigo and Liinev (2002) and we refer to Liinev (2003) for more details. See also Brigo et al. (2003) in the context of basket options.

Firstly, note that it is possible to calculate the Kullback-Leibler distance (KLI) of the distribution of the sum $\mathbb{S}$ from the lognormal family of distributions $\mathcal{L}$ in the following way

$$\begin{aligned} KLI(p(x), \mathcal{L}) &= E_p[\ln p(x)] + \frac{1}{2} + E_p\left[\ln\left(\frac{x}{S(0)}\right)\right] \\ &\quad + \frac{1}{2}\ln\left(2\pi S(0)^2\left[E_p\left[\ln^2\left(\frac{x}{S(0)}\right)\right] - \left(E_p\left[\ln\left(\frac{x}{S(0)}\right)\right]\right)^2\right]\right), \end{aligned} \tag{53}$$

where $p(x)$ denotes the density function of $\mathbb{S}$, and $E_p[\phi(x)] = \int \phi(x)p(x)dx$. This distance is readily computed, once one has an estimate of the true $\mathbb{S}$ density and of its first two log-moments. The distance (53) can be interpreted as the distance of the distribution of $\mathbb{S}$ from the closest lognormal distribution in Kullback-Leibler sense. The latter is the distribution which shares the same log-moments $E_p[(\ln(\cdot))^i], i = 1, 2$ with the distribution of $\mathbb{S}$.

This provides an alternative way to the lognormal approximation of Jacques (1996) in order to compute the price of the Asian call option $AC(n, K, T)$. Namely, we can estimate the parameters of the closest lognormal distribution based on the simulated $\mathbb{S}$, and then apply the standard Black & Scholes technique in order to find the price. This method is considerably easier to implement than that of Jacques (1996). However, to obtain a correct price approximation, more simulations are needed than for the usual Monte-Carlo price estimate.

In Table 3 we present the results obtained in evaluating the Kullback-Leibler distance for the sum of lognormals $\mathbb{S}$ through a standard Monte Carlo method with $10\,000$ antithetic paths, for the

parameters in Table 1. In the brackets we show the sample standard errors (SE) for both quantities. In order to have an idea for what it means to have a KLI distance of about 0.003 between two distributions, we may resort to the KLI distance of two lognormals, which can be easily computed analytically. It appears that we find a KLI distance comparable in size to our distances below if we consider for example two lognormal densities with the same mean but different standard deviations. Then a KLI distance of approximately 0.003 amounts to a percentage difference in standard deviations of about 0.29%. This gives a feeling for the size of the distributional discrepancy our distance implies.

σ	$\mathbb{S}$ (SE)	KLI (SE)
0.2	$3079.000\,(3.255429)$	$0.0032712\,(0.0001183)$
0.3	$3078.555\,(4.905087)$	$0.0033344\,(0.0001144)$
0.4	$3078.558\,(6.579753)$	$0.0032950\,(0.0001277)$

Table 3: Distance analysis.

In Table 4 we show the corresponding lognormal price approximation (for the respective Monte Carlo values we refer to Table 1). These values seem to indicate that this method underestimates the price. This indicates that even the optimal lognormal distribution (in KLI sense) does not attribute enough weight to the upper tail.

K	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.4$
80	22.00133	22.30572	23.02679
90	12.75699	13.91766	15.41261
100	5.515920	7.525337	9.550753
110	1.647747	3.508497	5.504232

Table 4: Price approximation based on the closest lognormal distribution in Kullback-Leibler sense.

In Table 5 we display the Hellinger distances HD between the densities p_ℓ of $\mathbb{S}^\ell$, (35), when

the conditioning variable Λ is given by (13) (hereafter denoted as $\mathbb{S}^{\ell}_{FA}$), and p_c of the comonotonic sum $\mathbb{S}^c$, defined as

$$HD(\mathbb{S}^{\ell}_{FA};\mathbb{S}^c) := 2 - 2\int \sqrt{p_\ell(x)p_c(x)}dx.$$

It appears that increasing the volatility σ the densities tend to move further away from each other.

σ	$HD(\mathbb{S}^{\ell}_{FA};\mathbb{S}^c)$
0.2	0.001756845
0.3	0.001831938
0.4	0.001949698

Table 5: Hellinger distance between comonotonic lower and upper bound of $\mathbb{S}$.

We also computed the distance between the densities of $\mathbb{S}^{\ell}_{FA}$ and of $\mathbb{S}^{\ell}_{GA}$ which is $\mathbb{S}^{\ell}$ with conditioning variable Λ (14). This distance was found to be of the magnitude of 10^{-13}, and also increasing with increasing σ.

2.6 Hedging the fixed strike Asian option

Hedging is an important concept for managing risks in the market. Most traders use quite sophisticated hedging schemes which involve calculating several "measures" in order to characterize risk exposure. These measures are referred to as "Greek letters", or "Greeks". Each Greek measures a different aspect of the risk in an option position. Delta represents the sensitivity with respect to $S(0)$, the initial value of the underlying asset. It is defined as a rate of change of the option price w.r.t. the price of the underlying asset. Gamma of a portfolio of derivatives is a rate of change of the portfolio's Delta w.r.t. the asset price. Vega characterizes the rate of change of the value of the portfolio w.r.t. the volatility of the underlying asset.

In this Section we show that from the analytical expressions in terms of Black and Scholes prices for the lower and the upper bounds we can easily obtain the hedging Greeks which are summarized by the following proposition. Note, however, that these expressions for the Greeks do

Bound	Delta (Δ)
LBΛ	$\sum_{i=0}^{n-1} \frac{e^{-ri}}{n} \Phi(\rho_{T-i}\sigma\sqrt{T-i} - \Phi^{-1}(F_{\mathbb{S}^\ell}(nK)))$
ICUBΛ	$\int_0^1 I_2(v)dv$
PECUBΛ	$\sum_{i=0}^{n-1} \frac{e^{-ri}}{n} \Phi(\rho_{T-i}\sigma\sqrt{T-i} - d_\Lambda^*) + \int_0^{\Phi(d_\Lambda^*)} I_2(v)dv + [\chi(d_\Lambda^*) + I_1(\Phi(d_\Lambda^*))\varphi(d_\Lambda^*)]\frac{\partial d_\Lambda^*}{\partial S(0)}$
UBΛ	$\Delta_{\mathrm{LB}\Lambda} + \frac{e^{-rT}}{2n}\int_0^1 \sqrt{q(v)}dv$
UBΛ_d	$\Delta_{\mathrm{LB}\Lambda} + g(d_\Lambda^*)[1+\eta(d_\Lambda^*)]$
	Gamma (Γ)
LBΛ	$e^{-rT}n\left[\frac{K}{S(0)}\right]^2 \varphi(\Phi^{-1}(F_{\mathbb{S}^\ell}(nK)))\left[\sum_{i=0}^{n-1}\tilde{K}_i\sigma\rho_{T-i}\sqrt{T-i}\right]^{-1}$
ICUBΛ	$e^{-rT}n\left[\frac{K}{S(0)}\right]^2 \int_0^1 \varphi(\Phi^{-1}(F_{\mathbb{S}^u\mid V=v}(nK)))\left[\sum_{i=0}^{n-1}\tilde{K}_i(v)p_i(v)e^{-r(T-i)}\sigma\sqrt{1-\rho_{T-i}^2}\sqrt{T-i}\right]^{-1} dv$
PECUBΛ	$\left[-\frac{e^{-rT}}{n}\sum_{i=0}^{n-1}p_i(\Phi(d_\Lambda^*)) + 2I_2(\Phi(d_\Lambda^*)) + \frac{\partial\chi(d_\Lambda^*)}{\partial S(0)}\right]\varphi(d_\Lambda^*)\frac{\partial d_\Lambda^*}{\partial S(0)} + \chi(d_\Lambda^*)\frac{\partial^2 d_\Lambda^*}{\partial S(0)^2} + \gamma(d_\Lambda^*)I_1(\Phi(d_\Lambda^*))$ $+e^{-rT}n\left[\frac{K}{S(0)}\right]^2 \int_0^{\Phi(d_\Lambda^*)} \varphi(\Phi^{-1}(F_{\mathbb{S}^u\mid V=v}(nK)))\left[\sum_{i=0}^{n-1}\tilde{K}_i(v)p_i(v)e^{-r(T-i)}\sigma\sqrt{1-\rho_{T-i}^2}\sqrt{T-i}\right]^{-1} dv$
UBΛ	$\Gamma_{\mathrm{LB}\Lambda}$
UBΛ_d	$\Gamma_{\mathrm{LB}\Lambda} + \frac{1}{2}g(d_\Lambda^*)\left[(1+\eta(d_\Lambda^*))\eta(d_\Lambda^*) + \frac{\partial\eta(d_\Lambda^*)}{\partial d_\Lambda^*}\right]\frac{\partial d_\Lambda^*}{\partial S(0)}$
	Vega ($\mathcal{V}$)
LBΛ	$\frac{e^{-rT}}{n}\left[\sum_{i=0}^{n-1}\tilde{K}_i\rho_{T-i}\sqrt{T-i}\right]\varphi(\Phi^{-1}(F_{\mathbb{S}^\ell}(nK)))$
ICUBΛ	$\int_0^1 \frac{\partial I_1(v)}{\partial\sigma}dv$
PECUBΛ	$[\chi(d_\Lambda^*) + I_1(\Phi(d_\Lambda^*))\varphi(d_\Lambda^*)]\frac{\partial d_\Lambda^*}{\partial\sigma} + \frac{e^{-rT}}{n}S(0)\varphi(d_\Lambda^*)\sum_{i=0}^{n-1}p_i(\Phi(d_\Lambda^*))\rho_{T-i}\sqrt{T-i} + \int_0^{\Phi(d_\Lambda^*)}\frac{\partial I_1(v)}{\partial\sigma}dv$
UBΛ	$\mathcal{V}_{\mathrm{LB}\Lambda} + \frac{e^{-rT}}{4n}S(0)\int_0^1 \frac{1}{\sqrt{q(v)}}\frac{\partial q(v)}{\partial\sigma}dv$
UBΛ_d	$\mathcal{V}_{\mathrm{LB}\Lambda} + \frac{S(0)}{2}g(d_\Lambda^*)\,\zeta(d_\Lambda^*)\frac{\partial d_\Lambda^*}{\partial\sigma}$

Notations

$$p_i(v) = e^{(r-\frac{\sigma^2}{2}\rho_{T-i}^2)(T-i)}e^{\rho_{T-i}\sigma\sqrt{T-i}\,\Phi^{-1}(v)}$$

$$I_1(v) = \frac{1}{n}\sum_{i=0}^{n-1}\frac{p_i(v)}{e^{r(T-i)}}\left[e^{-ri}S(0)\Phi(d_{1,i}(v)) - e^{-rT}\tilde{K}_i(v)\Phi(d_{2,i}(v))\right]$$

$$I_2(v) = \frac{\partial I_1(v)}{\partial S(0)} = \frac{e^{-rT}}{n}\sum_{i=0}^{n-1}p_i(v)\Phi(d_{1,i}(v))$$

$$c_{ij} = e^{\sigma^2(\min(T-i,T-j)-\rho_{T-i}\rho_{T-j}\sqrt{T-i}\sqrt{T-j})}$$

$$q_{ij} = e^{r(2T-i-j)+\sigma^2\rho_{T-i}\rho_{T-j}\sqrt{T-i}\sqrt{T-j}}(c_{ij}-1)$$

$$h(d_\Lambda^*) = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}q_{ij}\Phi(d_\Lambda^* - \sigma(\rho_{T-i}\sqrt{T-i} + \rho_{T-j}\sqrt{T-j}))$$

$$g(d_\Lambda^*) = \frac{e^{-rT}}{2n}\Phi(d_\Lambda^*)^{1/2}h(d_\Lambda^*)^{1/2}$$

$$\zeta(d_\Lambda^*) = \frac{\varphi(d_\Lambda^*)}{\Phi(d_\Lambda^*)} + \frac{1}{h(d_\Lambda^*)}\frac{\partial h(d_\Lambda^*)}{\partial d_\Lambda^*}$$

$$\eta(d_\Lambda^*) = \frac{S(0)}{2}\zeta(d_\Lambda^*)\frac{\partial d_\Lambda^*}{\partial S(0)}$$

$$\gamma(d_\Lambda^*) = \varphi(d_\Lambda^*)\left[-d_\Lambda^*(\frac{\partial d_\Lambda^*}{\partial S(0)})^2 + \frac{\partial^2 d_\Lambda^*}{\partial S(0)^2}\right]$$

$$\chi(d_\Lambda^*) = e^{-rT}\varphi(d_\Lambda^*)\left[K - \frac{S(0)}{n}\sum_{i=0}^{n-1}p_i(\Phi(d_\Lambda^*))\right]$$

$$q(v) = \frac{\mathrm{var}^Q(\mathbb{S}|\Lambda)}{S(0)^2} = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}p_i(v)p_j(v)c_{ij} - \left[\sum_{i=0}^{n-1}p_i(v)\right]^2$$

Table 6: Delta, Gamma and Vega for bounds.

not represent the bounds for the hedging parameters. Instead, they can be considered as an approximation to the hedging Greeks. Nielsen and Sandmann (2003) also derived the Greeks for their bounds, noticing that this approximation was quite good in numerical examples.

Proposition 1. *The Delta, Gamma and Vega positions of the bounds* (9), (16), (48), (27), *and* (37) *are given by the expressions in Table 6.*

The proof for obtaining the hedging Greeks is a straightforward application of partial differentiation of the combinations of Black and Scholes type prices that we found for the bounds (cf. Theorems 2 and 4).

In the next section we discuss different methods for pricing European-style discrete arithmetic Asian options with floating strike through the bounds developed in previous sections.

3 Floating strike Asian options in a Black & Scholes settings

By arbitrage arguments, the price at current time $t = 0$ of a floating strike Asian put option with percentage β is given by

$$APF(n, \beta, T) = \frac{e^{-rT}}{n} E^Q \left[\left(\sum_{i=0}^{n-1} S(T-i) - n\beta S(T) \right)_+ \right]$$

under the risk-neutral probability measure Q. In the Black & Scholes model, the following change of measure leads to results dealt with in Section 2. Let us define the probability $\tilde{Q}$ equivalent to Q by the Radon-Nikodym derivative

$$\frac{d\tilde{Q}}{dQ} = \frac{S(T)}{S(0)e^{rT}} = \exp(-\frac{\sigma^2}{2}T + \sigma B(T)). \tag{54}$$

Under this probability $\tilde{Q}$, $\tilde{B}(t) = B(t) - \sigma t$ is a Brownian motion and therefore, the dynamics of the share under $\tilde{Q}$ are given by

$$\frac{dS(t)}{S(t)} = (r + \sigma^2)dt + \sigma d\tilde{B}(t). \tag{55}$$

Let us exemplarily consider the case of a forward starting floating strike Asian put option with $T - n + 1 > 0$.

Using the probability $\tilde{Q}$, the corresponding option price is given by

$$APF(n, \beta, T) = \frac{S(0)}{n} E^{\tilde{Q}} \left[\left(\frac{\sum_{i=0}^{n-1} S(T-i)}{S(T)} - \beta n \right)_+ \right].$$

From this formula, one can conjecture that a floating strike Asian put option can be interpreted as a fixed strike Asian call with strike price $\beta S(0)$. Henderson and Wojakowski (2002) have obtained symmetry results between the floating and fixed strike Asian options in the forward starting case of continuous averaging. They considered the Black & Scholes dynamics for the underlying asset with a continuous dividend yield δ. In Section 3.1, we prove similar results in case of the European-style discrete arithmetic Asian options. The symmetry results become very useful for transferring knowledge about one type of an option to another. However, there does not exist such a symmetry relation for the options 'in progress'.

3.1 Symmetry results for arithmetic Asian options

In order to derive the similar results to Henderson and Wojakowski (2002) in case of discrete averaging, we introduce some generalized notation. For the fixed strike Asian call option we use the notation

$$AC(x_1, x_2, x_3, x_4, x_5, x_6, x_7),$$

where

$x_1 =$ strike price

$x_2 =$ initial value of the process $(S(t))_{t \geq 0}$

$x_3 =$ risk-free interest rate

$x_4 =$ dividend yield

$x_5 =$ option maturity

$x_6 =$ number of averaging terms

$$x_7 = \text{starting date of averaging.}$$

Analogously, for a put option we set $AP(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$. For example, $AP(K, S(0), r, \delta, T, n, T-n+1)$ denotes the Asian put option with fixed strike price K and maturity date T which is forward starting with n terms and with the first term being $S(T-n+1)$, where $(S(t))_{t\geq 0}$ denotes as usual a Black and Scholes process with initial value $S(0)$ and with dividend yield δ. The short-term constant interest rate equals r.

For floating strike options, we introduce a similar slightly modified notation. Namely, by

$$ACF(y_1, y_2, y_3, y_4, y_5, y_6, y_7)$$

we denote the floating strike Asian call option with

$$y_1 = \text{initial value of the process } (S(t))_{t\geq 0}$$

$$y_2 = \text{percentage}$$

$$y_3 = \text{risk-free interest rate}$$

$$y_4 = \text{dividend yield}$$

$$y_5 = \text{option maturity}$$

$$y_6 = \text{number of averaging terms in strike}$$

$$y_7 = \text{starting date of averaging.}$$

For example, $ACF(S(0), \frac{K}{S(0)}, \delta, r, T, n, 0)$ denotes the European-style floating strike Asian call option with percentage $\frac{K}{S(0)}$ and maturity date T which is forward starting with n terms and with the first term being $S(0)$, where $(S(t))_{t\geq 0}$ denotes as usual a Black and Scholes process with initial value $S(0)$ and with dividend yield r. The constant short-term interest rate is equal to δ. Analogously, for a floating strike put option we set $APF(y_1, y_2, y_3, y_4, y_5, y_6, y_7)$.

Using these notations, we obtain the following symmetry results, which are proved in Appendix B.

Theorem 9.

$$
\begin{aligned}
AP(K, S(0), r, \delta, T, n, T-n+1) &= ACF(S(0), \frac{K}{S(0)}, \delta, r, T, n, 0) \\
ACF(S(0), \beta, r, \delta, T, n, T-n+1) &= AP(\beta S(0), S(0), \delta, r, T, n, 0)
\end{aligned}
$$

and

$$
\begin{aligned}
AC(K, S(0), r, \delta, T, n, T-n+1) &= APF(S(0), \frac{K}{S(0)}, \delta, r, T, n, 0) \\
APF(S(0), \beta, r, \delta, T, n, T-n+1) &= AC(\beta S(0), S(0), \delta, r, T, n, 0).
\end{aligned}
$$

From the equalities above it is clear that by using the results of Section 2, one can obtain bounds for a floating strike Asian option through the bounds for a fixed strike Asian option. Note that the interest rate and the dividend yield have switched their roles when going from a floating to a fixed strike Asian option or vice versa.

3.2 Direct approach

In what follows we show that, instead of using symmetry, we can directly derive bounds for the floating strike Asian options. We also stress that these bounds can manage both 'in progress' and forward-starting floating strike Asian options as opposed to the approach using symmetry. Writing down the formulae for $S(T-i)$ and $S(T)$ in the Black & Scholes setting leads to

$$
\mathbb{S} = \frac{\sum_{i=0}^{n-1} S(T-i)}{S(T)} = \sum_{i=0}^{n-1} e^{-(r+\frac{\sigma^2}{2})i+\sigma(\tilde{B}(T-i)-\tilde{B}(T))} =: \sum_{i=0}^{n-1} \alpha_i \, e^{Y_i}
$$

with $\alpha_i = e^{-(r+\frac{\sigma^2}{2})i}$ and with $Y_i = \sigma(\tilde{B}(T-i) - \tilde{B}(T))$ a normally distributed random variable with mean $E^{\tilde{Q}}[Y_i] = 0$ and variance $\sigma^2_{Y_i} = i\sigma^2$. Note that $\alpha_0 e^{Y_0}$ is in fact a constant. Clearly $\mathbb{S}$ is a sum of lognormal variables and thus we can apply the results of Section 2.

Denoting the price of an European-style discrete arithmetic floating strike Asian *call* option with exercise date T, n averaging dates and percentage β by

$$
ACF(n, \beta, T) = \frac{e^{-rT}}{n} E^Q \left[\left(n\beta S(T) - \sum_{i=0}^{n-1} S(T-i) \right)_+ \right],
$$

we find from the put-call parity at the present:

$$APF(n,\beta,T) - ACF(n,\beta,T) = \frac{S(0)}{n}\frac{1-e^{-rn}}{1-e^{-r}} - \beta S(0). \tag{56}$$

Hence, we can derive bounds for the Asian floating strike call option from the bounds for the put.

In the remaining of the section, we only work out in detail the forward starting case as the 'in progress' case can be dealt with in a similar way.

3.2.1 Lower bound

In order to obtain a lower bound of good quality for the forward starting Asian option, we consider as conditioning variable a normal random variable Λ which is as much alike as $\mathbb{S}$. Inspired by the choice for the fixed case, we take

$$\Lambda = \sum_{i=0}^{n-1} \beta_i (\tilde{B}(T-i) - \tilde{B}(T)) \tag{57}$$

with some positive real numbers β_i. In particular for $\beta_i = e^{-(r+\frac{\sigma^2}{2})i}$ we find the first order approximation of $\mathbb{S}$. If β_i equals $\frac{1}{\sqrt{\frac{1}{3}n^3-\frac{1}{2}n^2+\frac{1}{6}n}}$ for all i, then $\Lambda = \frac{\ln \mathbb{G} - E^{\tilde{Q}}[\ln \mathbb{G}]}{\sqrt{\mathrm{var}^{\tilde{Q}}[\ln \mathbb{G}]}}$ is the standardized logarithm of the geometric average $\mathbb{G}$:

$$\mathbb{G} = \left(\prod_{i=0}^{n-1} \frac{S(T-i)}{S(T)}\right)^{1/n} = \left(\prod_{i=0}^{n-1} \exp\left[-(r+\frac{\sigma^2}{2})i + \sigma(\tilde{B}(T-i) - \tilde{B}(T))\right]\right)^{1/n}, \tag{58}$$

with

$$\begin{aligned} E^{\tilde{Q}}[\ln \mathbb{G}] &= -(r+\frac{\sigma^2}{2})\frac{n-1}{2} \\ \mathrm{var}^{\tilde{Q}}[\ln \mathbb{G}] &= \frac{\sigma^2}{n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \min(i,j) = \frac{\sigma^2}{n^2}\left(\frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n\right). \end{aligned}$$

This choice of Λ is similar to the choice (14) of Nielsen and Sandmann (2003) in the fixed strike setting.

For general β_i, we have that $Y_i \mid \Lambda = \lambda$ is normally distributed with mean $r_i \frac{\sigma\sqrt{i}}{\sigma_\Lambda}\lambda$ and variance $\sigma^2_{Y_i}(1-r_i^2)$ where $r_0 = 0$ and for $i \geq 1$

$$r_i = \frac{\mathrm{cov}\left(\tilde{B}(T-i) - \tilde{B}(T), \Lambda\right)}{\sqrt{i}\,\sigma_\Lambda} = \frac{\sum_{j=0}^{n-1}\beta_j \min(i,j)}{\sqrt{i}\sqrt{\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\beta_i\beta_j \min(i,j)}}. \tag{59}$$

For both choices of Λ that we consider, these correlations r_i are positive. We thus find analogously to Theorem 1 the following lower bound for the price of the forward starting Asian floating put option:

$$APF(n,\beta,T) \geq \frac{S(0)}{n}\sum_{i=0}^{n-1} e^{-ri}\Phi\left[\sigma r_i\sqrt{i} - \Phi^{-1}\left(F_{\mathbb{S}^\ell}(n\beta)\right)\right] - S(0)\beta\left(1 - F_{\mathbb{S}^\ell}(n\beta)\right),$$

where $F_{\mathbb{S}^\ell}(n\beta)$ is obtained from

$$\sum_{i=0}^{n-1}\exp\left[-\left(r + \frac{r_i^2\sigma^2}{2}\right)i + r_i\sigma\sqrt{i}\Phi^{-1}(F_{\mathbb{S}^\ell}(n\beta))\right] = n\beta.$$

3.2.2 Improved comonotonic upper bound

Analogously to the case of the improved comonotonic upper bound for the Asian fixed strike, we have found that also in the Asian floating strike case, the conditioning variable

$$\Lambda = -\sum_{k=1}^{T} W_k, \quad \text{with } W_k \text{ i.i.d. } N(0,1) \text{ such that } \tilde{B}(T-i) \stackrel{d}{=} \sum_{k=1}^{T-i} W_k,\ i = 0,\ldots,n-1,$$

leads to a sharper upper bound than other choices, for example the conditioning variable in the lower bound.

The theory of comonotonicity (see (23) and (21)) then leads to the following upper bound

$$\begin{aligned}
&\frac{S(0)}{n}E^{\tilde{Q}}\left[(\mathbb{S}^u - n\beta)_+\right] \\
&= \frac{S(0)}{n}\sum_{i=0}^{n-1} e^{-(r+\frac{\sigma^2}{2}r_i^2)i}\int_0^1 e^{r_i\sigma\sqrt{i}\,\Phi^{-1}(v)}\Phi\left(\sqrt{1-r_i^2}\,\sigma\sqrt{i} - \Phi^{-1}\left(F_{\mathbb{S}^u|V=v}(n\beta)\right)\right)dv \\
&\quad -S(0)\beta\left(1 - F_{\mathbb{S}^u}(n\beta)\right)
\end{aligned}$$

with the correlations given by $r_i = \sqrt{\frac{i}{T}}$, $i = 1,\ldots,n-1$ and $r_0 = 0$. Invoking (19)-(20), the conditional distribution $F_{\mathbb{S}^u|V=v}(x)$ and the cdf of $\mathbb{S}^u$ can be obtained.

3.2.3 Bounds based on the Rogers & Shi approach

By a similar reasoning as in Section 2.2, it is easy to derive an upper bound based on the lower bound by following the ideas of Rogers and Shi (1995) and Nielsen and Sandmann (2003). Indeed,

by using our conditioning variable Λ given by (57), we obtain

$$APF(n,\beta,T) \leq \frac{S(0)}{n}\left\{E^{\tilde{Q}}\left[(\mathbb{S}^{\ell}-n\beta)^{+}\right]+\varepsilon(d_{\Lambda})\right\}$$

where d_{Λ} is such that $\mathbb{S} \geq n\beta$ if $\Lambda \geq d_{\Lambda}$ and with

$$\varepsilon(d_{\Lambda}) = \frac{1}{2}\left\{\Phi(d_{\Lambda}^{*})\right\}^{\frac{1}{2}} \times$$
$$\times \left\{\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} e^{-r(i+j)+\sigma^2 r_i r_j \sqrt{i}\sqrt{j}}\Phi\left(d_{\Lambda}^{*}-\sigma(r_i\sqrt{i}+r_j\sqrt{j})\right)\left(e^{\sigma^2\left(\min(i,j)-r_i r_j\sqrt{i}\sqrt{j}\right)}-1\right)\right\}^{\frac{1}{2}}$$

where $d_{\Lambda}^{*} = \frac{d_{\Lambda}-E^{\tilde{Q}}[\Lambda]}{\sigma_{\Lambda}}$ and with correlations r_i defined in (59).

In particular for the linear transformation of the first order approximation (FA) of $\mathbb{S}$, namely $\Lambda = \sum_{i=0}^{n-1}\beta_i(\tilde{B}(T-i)-\tilde{B}(T))$ with $\beta_i = e^{-(r+\frac{\sigma^2}{2})i}$, one gets

$$\tilde{d}_{FA} = \frac{n\beta - \sum_{i=0}^{n-1} e^{-(r+\frac{\sigma^2}{2})i}}{\sigma}.$$

For $\beta_i = \frac{\sigma}{n}\frac{1}{\sqrt{\mathrm{var}^{\tilde{Q}}[\ln \mathbb{G}]}}$ with the geometric average (GA) $\mathbb{G}$ defined in (58), Λ equals the standardized logarithm of the geometric average and the corresponding d_{Λ} equals

$$\tilde{d}_{GA} = \frac{\ln(\beta)+(r+\frac{\sigma^2}{2})\frac{n-1}{2}}{\frac{\sigma}{n}\sqrt{\frac{1}{3}n^3-\frac{1}{2}n^2+\frac{1}{6}n}}.$$

Notice also, that analogously to Theorem 5 one can obtain an upper bound for $APF(n,\beta,T)$ in terms of a constant error ε.

3.2.4 Partially exact/comonotonic upper bound

Along similar lines as in Section 2.2.3, we can derive a partially exact/comonotonic upper bound by recalling that for some normally distributed variable Λ there exists a d_{Λ} such that $\Lambda \geq d_{\Lambda}$ implies $\mathbb{S} \geq n\beta$:

$$\begin{aligned}APF(n,\beta,T) \leq & \frac{S(0)}{n}\sum_{i=0}^{n-1} e^{-ri}\Phi(r_i\sigma\sqrt{i}-d_{\Lambda}^{*}) - S(0)\beta\Phi(-d_{\Lambda}^{*}) \\ & + \frac{S(0)}{n}\sum_{i=0}^{n-1} e^{-(r+\frac{\sigma^2}{2}r_i^2)i}\int_0^{\Phi(d_{\Lambda}^{*})} e^{r_i\,\sigma\sqrt{i}\,\Phi^{-1}(v)}\Phi\left(\sqrt{1-r_i^2}\,\sigma\sqrt{i}-\Phi^{-1}\left(F_{\mathbb{S}^u|V=v}(n\beta)\right)\right)dv \\ & - S(0)\beta\left(\Phi(d_{\Lambda}^{*})-\int_0^{\Phi(d_{\Lambda}^{*})} F_{\mathbb{S}^u|V=v}(n\beta)\,dv\right)\end{aligned}$$

where $d^*_\Lambda = \dfrac{d_\Lambda - E^{\tilde{Q}}[\Lambda]}{\sigma_\Lambda}$ and $v = \dfrac{\lambda - E^{\tilde{Q}}[\Lambda]}{\sigma_\Lambda}$.
The first two terms of the upper bound are composing the exact part of $\frac{S(0)}{n} E^{\tilde{Q}}[(\mathbb{S} - n\beta)_+]$, while the last two terms define the improved comonotonic upper bound for the remaining part of it.

3.3 Numerical illustration

In this section we shall give a numerical example of a floating strike Asian put option.

In Table 7 we display different lower and upper bounds for a floating strike Asian put option with an initial stock price $S(0) = 100$, a maturity of 120 days and an averaging period n of 30 days. The choices for volatility and risk-free interest rate are the same as in Section 2.5.2. The percentage β is chosen so that $\beta S(0)$ corresponds to the respective strike K in Section 2.5.2. We obtained Monte Carlo price estimates (based on $10\,000$ simulated paths) by adapting the Kemna and Vorst (1990) control variate technique. Indeed, by applying the change of measure (54), we can interpret a floating strike Asian put option as a fixed strike Asian call option with strike price $\beta S(0)$. Hence we can simulate the dynamics of the stock price according to (55), and use the geometric average $\mathbb{G}$ given by (58) as our control variate.
Note that by using the put-call parity result (56) one can easily obtain the price for the floating strike Asian call option. For example, consider the entry in Table 7 with $\beta = 1.0$, $\sigma = 0.2$, and $r = 0.05$. By applying (56), we obtain that LB$FA = 1.387410$, LB$GA = 1.387411$, UB$GA_d = 1.388847$, UB$FA_d = 1.388792$, PECUBGA =1.557532, and ICUBB_T =1.575395.

We observe similar behaviour of the lower and upper bounds as for the fixed strike Asian call option apart from some interesting particular cases:

1. for $\sigma = 0.2, 0.3, 0.4$ and $\beta = 0.8$ the lower and the best upper bounds coincide up to three or four decimals and thus give almost exact results. Although the Monte Carlo price estimate is slightly higher, the interval $[MC - SE, MC + SE]$ overlaps with the interval $[\text{LB}\Lambda, \text{UB}\Lambda_d]$ for $\Lambda = FA$ or $\Lambda = GA$. Notice that for $\beta = 0.8$ ($\sigma = 0.2, 0.3$) – which is a

$r = 0.09$

σ	β	MC (SE $\times 10^4$)	LBFA	LBGA	UBGA_d	UBFA_d	UBFA	PECUBGA	ICUBB_T
0.2	0.8	19.64351 (2.5)	19.643331	19.643331	19.643331	19.643331	19.652053	19.643118	19.643284
	0.9	9.644117 (2.5)	9.643903	9.643903	9.643923	9.643934	9.652625	9.645429	9.646147
	1.0	1.113866 (2.1)	1.113997	1.113998	1.119154	1.118720	1.122719	1.283311	1.301119
	1.1	0.001167 (0.6)	0.001154	0.001155	0.010306	0.010293	0.009876	0.004286	0.004762
0.3	0.8	19.64376 (5.6)	19.643332	19.643332	19.643333	19.643334	19.662815	19.642851	19.643255
	0.9	9.670844 (5.3)	9.670327	9.670324	9.671056	9.671175	9.689810	9.704453	9.708673
	1.0	1.752636 (5.0)	1.753406	1.753406	1.764434	1.763671	1.772889	2.008637	2.034843
	1.1	0.040901 (3.2)	0.040840	0.040844	0.060394	0.060568	0.060323	0.084571	0.089851
0.4	0.8	19.64452 (9.9)	19.643666	19.643666	19.643700	19.643762	19.678319	19.645280	19.645424
	0.9	9.784575 (9.1)	9.784545	9.784533	9.788040	9.788243	9.819198	9.891788	9.904717
	1.0	2.391664 (9.1)	2.393883	2.393884	2.412935	2.411692	2.428536	2.734542	2.769381
	1.1	0.191081 (7.4)	0.192114	0.192128	0.224217	0.224551	0.226767	0.320139	0.334277

$r = 0.05$

0.2	0.8	19.80180 (2.5)	19.801637	19.801637	19.801637	19.801637	19.810313	19.801423	19.801590
	0.9	9.802297 (2.5)	9.802114	9.802114	9.802131	9.802141	9.810790	9.803394	9.804074
	1.0	1.188935 (2.2)	1.189061	1.189061	1.193931	1.193664	1.197736	1.359169	1.377045
	1.1	0.001407 (0.7)	0.001377	0.001377	0.010502	0.010525	0.010052	0.004943	0.005479
0.3	0.8	19.80200 (5.6)	19.801638	19.801638	19.801638	19.801640	19.821132	19.801156	19.801557
	0.9	9.826784 (5.3)	9.826301	9.826299	9.826970	9.827101	9.845795	9.858436	9.862434
	1.0	1.830198 (5.1)	1.830953	1.830953	1.841571	1.841046	1.850447	2.086848	2.113107
	1.1	0.044667 (3.3)	0.044669	0.044671	0.064136	0.064355	0.064163	0.091056	0.096617
0.4	0.8	19.80267 (10.0)	19.801942	19.801942	19.801972	19.802032	19.836644	19.803444	19.803566
	0.9	9.935006 (9.2)	9.935044	9.935035	9.938357	9.938621	9.969747	10.038765	10.051296
	1.0	2.470755 (9.2)	2.473011	2.473011	2.491532	2.490598	2.507713	2.814307	2.849193
	1.1	0.202340 (7.6)	0.203494	0.203505	0.235379	0.235795	0.238196	0.335905	0.350466

Table 7: Comparing bounds for a floating strike Asian put option

$T = 120, n = 30,\ \sigma:$ yearly volatility, $\beta:$ percentage, $S(0) = 100$

MC :	Monte Carlo price together with its standard error (SE) based on 10 000 paths
LBFA :	lower bound with $\Lambda = \sum_{j=0}^{n-1} e^{(r-\frac{\sigma^2}{2})(T-j)} B(T-j)$
LBGA :	lower bound with $\Lambda = (\ln \mathbb{G} - E^{\tilde{Q}}[\ln \mathbb{G}])/\sqrt{\mathrm{var}^{\tilde{Q}}(\ln \mathbb{G})}$
UBGA_d :	upper bound equal to lower bound LBGA plus $\varepsilon(\tilde{d}_{GA})S(0)/n$
UBFA_d :	upper bound equal to lower bound LBFA plus $\varepsilon(\tilde{d}_{FA})S(0)/n$
UBFA :	upper bound equal to lower bound LBFA plus constant $\varepsilon S(0)/n$
PECUBGA :	partially exact/comonotonic upper bound with $\Lambda = (\ln \mathbb{G} - E^{\tilde{Q}}[\ln \mathbb{G}])/\sqrt{\mathrm{var}^{\tilde{Q}}(\ln \mathbb{G})}$
ICUBB_T :	improved comonotonic upper bound with $\Lambda = \sum_{k=1}^{T} W_k \stackrel{d}{=} B(T)$

case of theoretical interest as this rarely happens in practice – the values of PECUBGA and ICUBB_T suffer from numerical instabilities caused by the involved numerical integration.

2. for $\sigma = 0.2$ and 0.3, and $\beta = 1.1$ the value for upper bound UBFA_d is larger than the one for UBFA which must be caused by the additional Hölder inequality in the derivation of the error bound $\varepsilon(\tilde{d}_{FA})$.

3. the partially exact/comonotonic upper bound PECUBGA is the best of all upper bounds for $\sigma = 0.2$ and $\beta = 1.1$.

4 Conclusions and future research

We derived analytical lower and upper bounds for the price of European-style discrete arithmetic Asian options with fixed and floating strike. Hereto we used and combined different ideas and techniques such as firstly conditioning on some random variable as in Rogers and Shi (1995), secondly results based on comonotonic risks and bounds for stop-loss premiums of sums of dependent random variables as in Kaas, Dhaene and Goovaerts (2000), and finally adaptation of the error bound of Rogers and Shi as in Nielsen and Sandmann (2003). All bounds have analytical expressions. This allows a study of the hedging Greeks of these bounds. For the numerical experiments it was important to find and motivate a good choice for the conditioning variables appearing in the formulae. We note that the expressions found for the bounds are not only analytical but also easily computable. The numerical results in the tables show that the upper bounds UBGA_d or UBFA_d are in general the best ones except for extreme values of the strike price K or β; then ICUBB_T or PECUBGA outperforms all the other upper bounds. The lower bounds LBGA and LBFA are practically equal and very close to the Monte Carlo values.

This approach has also been used to derive upper and lower bounds for basket options and Asian basket options, see Deelstra et al. (2004). The derivation of bounds for Asian options by using binomial trees was investigated by Reynaerts et al. (2004).

We mention that in view of recent developments for modelling the asset prices by exponential Lévy process, Albrecher and Predota (2002, 2004) have applied the comonotonic upper bound of Kaas et al. (2000) when the asset price dynamics is driven by a Normal Inverse Gaussian (NIG) and Variance Gamma (VG) Lévy processes. Moreover, Albrecher et al. (2004) present a general case of this upper bound and illustrate super-hedging of Asian options using European call options in a buy-and-hold strategy. We note also that in context of Lévy processes the results on the

equivalence between fixed and floating strike Asian options are recently derived by Eberlein and Papapantoleon (2005).

Further research includes extending the conditioning approach to more general distributions than lognormal. For example, one candidate is the class of log-elliptic distributions which is a better choice from the point of view of providing a better fit to the real data (cfr. Valdez and Dhaene (2003)).

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Appendix A. Some theoretical results

In this section, we recall from Dhaene et al. (2002a) and the references therein the procedures for obtaining the lower and upper bounds for stop-loss premiums of sums $\mathbb{S}$ of dependent random variables by using the notion of comonotonicity and of convex ordering, see Section 2.1.

Improved comonotonic upper bound

As proven in Dhaene et al. (2002a), the convex-largest sum of the components of a random vector with given marginals is obtained by the comonotonic sum $\mathbb{S}^c$, see (2). In the following theorem

Dhaene et al. (2002a) have proved that the stop-loss premiums of a sum of comonotonic random variables can easily be obtained from the stop-loss premiums of the terms.

Theorem 10. *The stop-loss premiums of the sum $\mathbb{S}^c$ of the components of the comonotonic random vector $(X_1^c, X_2^c, \ldots, X_n^c)$ are given by*

$$E\left[(\mathbb{S}^c - d)_+\right] = \sum_{i=1}^{n} E\left[\left(X_i - F_{X_i}^{-1}\left(F_{\mathbb{S}^c}(d)\right)\right)_+\right], \qquad (F_{\mathbb{S}^c}^{-1}(0) < d < F_{\mathbb{S}^c}^{-1}(1)).$$

Let us now assume that we have some additional information available concerning the stochastic nature of $(X_1, \ldots, X_n)$. More precisely, we assume that there exists some random variable Λ with a given distribution function, such that we know the conditional cumulative distribution functions, given $\Lambda = \lambda$, of the random variables X_i, for all possible values of λ. In fact, Kaas et al. (2000) define the improved comonotonic upper bound $\mathbb{S}^u$ as in (3). Notice that

$$\mathbb{S}^u = \left(\sum_{i=1}^{n} X_i \mid \Lambda\right)^c.$$

In order to obtain the distribution function of $\mathbb{S}^u$, observe that given the event $\Lambda = \lambda$, the random variable $\mathbb{S}^u$ is a sum of comonotonic random variables. Hence,

$$F_{\mathbb{S}^u|\Lambda=\lambda}^{-1}(p) = \sum_{i=1}^{n} F_{X_i|\Lambda=\lambda}^{-1}(p), \quad p \in [0,1].$$

Given $\Lambda = \lambda$, the cdf of $\mathbb{S}^u$ is defined by

$$F_{\mathbb{S}^u|\Lambda=\lambda}(x) = \sup\left\{p \in [0,1] \mid \sum_{i=1}^{n} F_{X_i|\Lambda=\lambda}^{-1}(p) \leq x\right\}.$$

The cdf of $\mathbb{S}^u$ then follows from

$$F_{\mathbb{S}^u}(x) = \int_{-\infty}^{+\infty} F_{\mathbb{S}^u|\Lambda=\lambda}(x)\, dF_\Lambda(\lambda).$$

If the marginal cdfs $F_{X_i|\Lambda=\lambda}$ are strictly increasing and continuous, then $F_{\mathbb{S}^u|\Lambda=\lambda}(x)$ is a solution to

$$\sum_{i=1}^{n} F_{X_i \mid \Lambda=\lambda}^{-1}\left(F_{\mathbb{S}^u \mid \Lambda=\lambda}(x)\right) = x, \qquad x \in \left(F_{\mathbb{S}^u \mid \Lambda=\lambda}^{-1}(0), F_{\mathbb{S}^u \mid \Lambda=\lambda}^{-1}(1)\right). \tag{60}$$

In this case, we also find that for any $d \in \left(F^{-1}_{\mathbb{S}^u|\Lambda=\lambda}(0), F^{-1}_{\mathbb{S}^u|\Lambda=\lambda}(1)\right)$:

$$E\left[(\mathbb{S}^u - d)_+ \mid \Lambda = \lambda\right] = \sum_{i=1}^{n} E\left[\left(X_i - F^{-1}_{X_i|\Lambda=\lambda}\left(F_{\mathbb{S}^u|\Lambda=\lambda}(d)\right)\right)_+ \mid \Lambda = \lambda\right], \tag{61}$$

from which the stop-loss premium at retention d of $\mathbb{S}^u$ can be determined by integration with respect to λ.

Lower bound

Let $\underline{X} = (X_1, \ldots, X_n)$ be a random vector with given marginal cdfs $F_{X_1}, F_{X_2}, \ldots, F_{X_n}$. We assume as in the previous section that there exists some random variable Λ with a given distribution function, such that we know the conditional cdfs, given $\Lambda = \lambda$, of the random variables X_i, for all possible values of λ. This random variable Λ, however, should not be the same as in case of the upper bound. We recall from Kaas et al. (2000) how to obtain a lower bound, in the sense of convex order, for $\mathbb{S} = X_1 + X_2 + \cdots + X_n$ by conditioning on this random variable.

For the conditional expectation $\mathbb{S}^\ell$, see (4), let us further assume that the random variable Λ is such that all $E\left[X_i \mid \Lambda\right]$ are non-decreasing and continuous functions of Λ. The quantiles of the lower bound $\mathbb{S}^\ell$ then follow from

$$F^{-1}_{\mathbb{S}^\ell}(p) = \sum_{i=1}^{n} F^{-1}_{E[X_i|\Lambda]}(p) = \sum_{i=1}^{n} E\left[X_i \mid \Lambda = F^{-1}_{\Lambda}(p)\right], \quad p \in [0,1],$$

and the cdf of $\mathbb{S}^\ell$ is given by

$$F_{\mathbb{S}^\ell}(x) = \sup\left\{p \in [0,1] \mid \sum_{i=1}^{n} E\left[X_i \mid \Lambda = F^{-1}_{\Lambda}(p)\right] \leq x\right\}.$$

If we now additionally assume that the cdfs of the random variables $E\left[X_i \mid \Lambda\right]$ are strictly increasing and continuous, then the cdf of $\mathbb{S}^\ell$ is also strictly increasing and continuous, and we get for all $x \in \left(F^{-1}_{\mathbb{S}^\ell}(0), F^{-1}_{\mathbb{S}^\ell}(1)\right)$,

$$\sum_{i=1}^{n} F^{-1}_{E[X_i|\Lambda]}\left(F_{\mathbb{S}^\ell}(x)\right) = x \quad \Leftrightarrow \quad \sum_{i=1}^{n} E\left[X_i \mid \Lambda = F^{-1}_{\Lambda}\left(F_{\mathbb{S}^\ell}(x)\right)\right] = x, \tag{62}$$

which unambiguously determines the cdf of the convex order lower bound $\mathbb{S}^\ell$ for $\mathbb{S}$. Using Theorem 10, the stop-loss premiums of $\mathbb{S}^\ell$ can be computed as:

$$E\left[\left(\mathbb{S}^\ell - d\right)_+\right] = \sum_{i=1}^{n} E\left[\left(E\left[X_i \mid \Lambda\right] - E\left[X_i \mid \Lambda = F_\Lambda^{-1}\left(F_{\mathbb{S}^\ell}(d)\right)\right]\right)_+\right], \tag{63}$$

which holds for all retentions $d \in \left(F_{\mathbb{S}^\ell}^{-1}(0),\ F_{\mathbb{S}^\ell}^{-1}(1)\right)$.

So far, we considered the case that all $E\left[X_i \mid \Lambda\right]$ are non-decreasing functions of Λ. The case where all $E\left[X_i \mid \Lambda\right]$ are non-increasing and continuous functions of Λ also leads to a comonotonic vector $\left(E\left[X_1 \mid \Lambda\right], E\left[X_2 \mid \Lambda\right], \ldots, E\left[X_n \mid \Lambda\right]\right)$, and can be treated in a similar way.

Sums of lognormal variables

In this section, we study upper and lower bounds for $E\left[(\mathbb{S} - d)_+\right]$ where $\mathbb{S}$ is a linear combination of lognormal variables. Let us denote

$$\mathbb{S} = \sum_{i=1}^{n} X_i = \sum_{i=1}^{n} \alpha_i\, e^{Y_i}, \tag{64}$$

with Y_i a normally distributed random variable with mean $E\left[Y_i\right]$ and variance $\sigma_{Y_i}^2$, and $\alpha_i \in \mathbb{R}$.

In this case the stop-loss premium with some retention d_i, namely $\mathrm{E}[(X_i - d_i)_+]$, can be obtained from the following theorem.

Theorem 11. *Let X_i be a lognormal random variable of the form $X_i = \alpha_i e^{Y_i}$ with*

$$Y_i \sim N(\mathrm{E}[Y_i], \sigma_{Y_i})$$

and $\alpha_i \in \mathbb{R}$. Then the stop-loss premium with retention d_i equals for $\alpha_i d_i > 0$

$$E[(X_i - d_i)_+] = \mathrm{sign}\left(\alpha_i\right) e^{\mu_i + \frac{\sigma_i^2}{2}}\, \Phi(\mathrm{sign}\left(\alpha_i\right) d_{i,1}) - d_i\, \Phi(\mathrm{sign}\left(\alpha_i\right) d_{i,2}), \tag{65}$$

where Φ is the cdf of the $N(0,1)$ distribution, and $d_{i,1}$ and $d_{i,2}$ are determined by

$$d_{i,1} = \frac{\mu_i + \sigma_i^2 - \ln|d_i|}{\sigma_i}, \quad d_{i,2} = d_{i,1} - \sigma_i. \tag{66}$$

The cases $\alpha_i d_i < 0$ are trivial.

We now consider a normally distributed random variable Λ and we slightly generalize Theorem 1 of Dhaene et al. (2002b) to our more general settings.

Theorem 12. *Let $\mathbb{S}$ be given by* (64) *and consider a normally distributed random variable Λ such that (Y_i, Λ) is bivariate normally distributed for all i. Then the distributions of the improved comonotonic upper bound $\mathbb{S}^u$ and the lower bound $\mathbb{S}^\ell$ are given by*

$$\mathbb{S}^u \stackrel{d}{=} \sum_{i=1}^{n} F^{-1}_{X_i|\Lambda}(U) = \sum_{i=1}^{n} \alpha_i \, e^{E[Y_i]+r_i\sigma_{Y_i}\Phi^{-1}(V)+\mathrm{sign}(\alpha_i)\sqrt{1-r_i^2}\,\sigma_{Y_i}\Phi^{-1}(U)}, \tag{67}$$

$$\mathbb{S}^\ell \stackrel{d}{=} \sum_{i=1}^{n} E[X_i \mid \Lambda] = \sum_{i=1}^{n} \alpha_i \, e^{E[Y_i]+r_i\sigma_{Y_i}\Phi^{-1}(V)+\frac{1}{2}\left(1-r_i^2\right)\sigma^2_{Y_i}}, \tag{68}$$

where U and $V = \Phi\left(\frac{\Lambda - E[\Lambda]}{\sigma_\Lambda}\right)$ are mutually independent uniform(0,1) random variables, Φ is the cdf of the $N(0,1)$ distribution and r_i is defined by

$$r_i = \mathrm{corr}\,(Y_i, \Lambda) = \frac{\mathrm{cov}\,[Y_i, \Lambda]}{\sigma_{Y_i}\sigma_\Lambda}.$$

When for all i $\mathrm{sign}(\alpha_i) = \mathrm{sign}(r_i)$ for $r_i \neq 0$, or for all i $\mathrm{sign}(\alpha_i) = -\mathrm{sign}(r_i)$ for $r_i \neq 0$, then $\mathbb{S}^\ell$ is comonotonic.

Appendix B. Proof of symmetry results in Theorem 9

Proof. We only prove the first symmetry result since the others follow along similar lines.

$$\begin{aligned}
&AP(K, S(0), r, \delta, T, n, T-n+1) \\
&\quad = e^{-rT} E^Q\left[\left(K - \frac{1}{n}\sum_{i=0}^{n-1} S(T-i)\right)_+\right] \\
&\quad = e^{-\delta T} E^Q\left[\frac{e^{-(r-\delta)T}S(T)}{S(0)}\left(\frac{KS(0)}{S(T)} - \frac{1}{n}\sum_{i=0}^{n-1}\frac{S(T-i)S(0)}{S(T)}\right)_+\right] \\
&\quad = e^{-\delta T} E^{\tilde{Q}}\left[\left(\frac{KS(0)}{S(T)} - \frac{1}{n}\sum_{i=0}^{n-1} S(0)\exp\left[-(r-\delta+\frac{\sigma^2}{2})i + \sigma\left(\tilde{B}(T-i) - \tilde{B}(T)\right)\right]\right)_+\right],
\end{aligned}$$

where we defined as before the probability $\tilde{Q}$ equivalent to Q by the Radon-Nikodym derivative but now by stressing the dividend yield δ

$$\frac{d\tilde{Q}}{dQ} = \frac{S(T)}{S(0)e^{(r-\delta)T}} = \exp(-\frac{\sigma^2}{2}T + \sigma B(T)).$$

Under this probability $\tilde{Q}$, $\tilde{B}(t) = B(t) - \sigma t$ is a Brownian motion and therefore, the dynamics of the share under $\tilde{Q}$ are given by

$$\frac{dS(t)}{S(t)} = ((r-\delta) + \sigma^2)dt + \sigma d\tilde{B}(t).$$

Due to the independent increments, $\tilde{B}(T-i) - \tilde{B}(T)$ has the same distribution as $\tilde{B}(i)$ and $-\tilde{B}(i)$, and we can concentrate on the process $(S^*(t))_t$ defined by

$$S^*(i) = S(0)\exp\left[-(r-\delta+\frac{\sigma^2}{2})i + \sigma\tilde{B}(i)\right].$$

Indeed, then

$$\begin{aligned} AP(K, S(0), r, \delta, T, n, T-n+1) &= e^{-\delta T}E^{\tilde{Q}}\left[\left(\frac{KS^*(T)}{S(0)} - \frac{1}{n}\sum_{i=0}^{n-1} S^*(i)\right)_+\right] \\ &= e^{-\delta T}E^{Q}\left[\left(\frac{K\tilde{S}(T)}{S(0)} - \frac{1}{n}\sum_{i=0}^{n-1} \tilde{S}(i)\right)_+\right] \end{aligned}$$

with the process $(\tilde{S}(t))_t$ defined by

$$\tilde{S}(i) = S(0)\exp\left[-(r-\delta+\frac{\sigma^2}{2})i + \sigma B(i)\right]$$

with $(B(t))_t$ a Brownian motion under Q.

As a conclusion,

$$AP(K, S(0), r, \delta, T, n, T-n+1) = ACF(S(0), \frac{K}{S(0)}, \delta, r, T, n, 0),$$

which was to be shown. □

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