APPROXIMATION OF STOP-LOSS PREMIUMS INVOLVING SUMS OF LOGNORMALS BY CONDITIONING ON TWO VARIABLES

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Abstract

In literature lower and upper bounds were obtained for arithmetic Asian and basket options based on comonotonicity results and by conditioning upon one variable. In this paper, we derive analytical expressions for the comonotonic bounds of stop-loss premiums of sums of dependent random variables by conditioning upon two variables. We also use the idea of several conditioning variables to develop an approximation for cases for which it is cumbersome to obtain a comonotonic lower bound. The numerical analysis shows that conditioning on two variables leads to very sharp results.

Keywords: conditional expectation; stop-loss premiums; sum of dependent random variables; basket options

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1. Introduction

Pricing of arithmetic Asian and basket options where the underlying is modelled in a Black & Scholes (1973) setting, boils down to computing stop-loss premiums of sums of dependent lognormal random variables. In Vanmaele et al. (2002) and in Deelstra et al. (2004) we derived lower and upper bounds based on comonotonicity results and on conditioning on one variable. A natural extension is to condition on more than one variable, because one intuitively expects in this way to improve those bounds. Note however that conditioning on more than two variables introduces additional computational difficulties such as multiple integration. Therefore we restrict ourselves to the case of two conditioning variables, although the formulae presented in this paper could be generalized to n conditioning variables.

We derive analytical expressions for the comonotonic bounds of stop-loss premiums of sums of dependent random variables and generalize in this way the results of Dhaene et al. (2002a) by conditioning on two variables.

Following the ideas of Deelstra et al. (2004), the stop-loss premium of a sum of random variables is decomposed in two parts, one of which can be simplified and turns out to be an exact part in case of lognormal variables. As expected, the number of integrals in the comonotonic bounds increases with the number of conditioning variables. However in case of two conditioning variables, the lower bound uses only one integration if the conditional density function is known. We specify this lower bound in the case of a sum of lognormal variables. In this case, the exact part of the stop-loss premium of a sum of lognormals does not need numerical integration. Different technical complications show up, in particular with respect to constraints on the choice of the conditioning variables. However, numerical results show that conditioning on more variables leads to very sharp lower bounds.

We concentrate upon these comonotonicity conditions in case of basket options in the Black & Scholes (1973) setting. A basket option is an option whose payoff depends on the value of a portfolio (or basket) of assets (stocks). Thus, an arithmetic basket call option with exercise date T, n risky assets and exercise price K generates a payoff $\left(\sum_{i=1}^{n} a_i S_i(T) - K\right)_+$ at T, that is, if the sum $\mathbb{S} = \sum_{i=1}^{n} a_i S_i(T)$ of asset prices S_i weighted by positive constants a_i at date T is more than K, the payoff equals the difference; if not, the payoff is zero. The price of the basket option at current time $t = 0$ is given by

$$BC(n, K, T) = e^{-rT}\,\mathrm{E}^{Q}\left[\left(\sum_{i=1}^{n} a_i S_i(T) - K\right)_+\right] \tag{1}$$

under the risk-neutral martingale measure Q and with r the risk-free interest rate.

Next, we consider an approximation of the stop-loss premium of a sum of random variables by deriving an improved comonotonic upper bound (by using a conditioning variable Λ_2) of the sum of conditional expectations of the random variables with respect to a conditioning variable Λ_1. This approach is useful for example in the case of basket options where the basket contains more than two not-all-positively-correlated assets since in that case, it is difficult to obtain a comonotonic lower bound.

In the settings of Asian options where the sum of dependent variables in the option payoff is not far from being comonotonic itself, Vyncke et al. (2004) propose a convex combination of a lower bound and an upper bound, namely the comonotonic upper bound and the improved comonotonic upper bound. In this paper, we adapt their method to the settings of basket options, where the sum of dependent variables is far from being comonotonic, and improve it by using the partially exact/comonotonic upper bound from Deelstra et al. (2004).

The paper is organized as follows. Section 2 recalls from Dhaene et al. (2002a) procedures for obtaining the lower and upper bounds for prices by using the notion of comonotonicity in case of one conditioning variable. In Section 3, we derive bounds based on comonotonicity by conditioning on two variables. In Section 4, we analyze an approximation by a 'comonotonic upper bound of a non-comonotonic lower bound'. In Section 5, we study different convex approximations. Section 6 is devoted to numerical results. Section 7 concludes the paper.

2. Some theoretical results

In this section, we recall from Dhaene et al. (2002a) and the references therein the procedures for obtaining the lower and upper bounds for stop-loss premiums of sums $\mathbb{S}$ of dependent random variables by using the notion of comonotonicity. A random vector $(X_1^c, \ldots, X_n^c)$ is *comonotonic* if each two possible outcomes $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ of $(X_1^c, \ldots, X_n^c)$ are ordered componentwise.

In both financial and actuarial context one encounters quite often random variables of the type $\mathbb{S} = \sum_{i=1}^n X_i$ where the terms X_i are not mutually independent, but the multivariate distribution function of the random vector $\underline{X} = (X_1, X_2, \ldots, X_n)$ is not completely specified because one only knows the marginal distribution functions of the random variables X_i. In such cases, to be able to make decisions it may be helpful to find the dependence structure for the random vector $(X_1, \ldots, X_n)$ producing the least favourable aggregate claims $\mathbb{S}$ with given marginals. Therefore, given the marginal distributions of the terms in a random variable $\mathbb{S} = \sum_{i=1}^n X_i$, we shall look for the joint distribution with a smaller respectively larger sum, in the convex order sense. In short, the sum $\mathbb{S}$ is bounded below and above in convex order ($\preceq_{\mathrm{cx}}$) by sums of variables which will be introduced in the following paragraphs:

$$\mathbb{S}^\ell \preceq_{\mathrm{cx}} \mathbb{S} \preceq_{\mathrm{cx}} \mathbb{S}^u \preceq_{\mathrm{cx}} \mathbb{S}^c,$$

where X is said to precede Y in *convex order* sense, notation $X \preceq_{cx} Y$, if and only if

$$\begin{aligned} E[X] &= E[Y] \\ E[(X-b)_+] &\leq E[(Y-b)_+], \end{aligned}$$

for $-\infty < b < +\infty$. This definition of convex order implies that

$$\mathrm{E}[(\mathbb{S}^\ell - b)_+] \leq \mathrm{E}[(\mathbb{S} - b)_+] \leq E[(\mathbb{S}^u - b)_+] \leq E[(\mathbb{S}^c - b)_+]$$

for all b in $\mathbb{R}$, while $\mathrm{E}[\mathbb{S}^\ell] = \mathrm{E}[\mathbb{S}] = E[\mathbb{S}^u] = \mathrm{E}[\mathbb{S}^c]$ and $\mathrm{var}[\mathbb{S}^\ell] \leq \mathrm{var}[\mathbb{S}] \leq \mathrm{var}[\mathbb{S}^u] \leq \mathrm{var}[\mathbb{S}^c]$.

2.1. Comonotonic upper bound

As proven in Dhaene et al. (2002a), the convex-largest sum of the components of a random vector with given marginals is obtained by the comonotonic sum $\mathbb{S}^c = X_1^c + X_2^c + \cdots + X_n^c$ with

$$\mathbb{S}^c \stackrel{d}{=} \sum_{i=1}^{n} F_{X_i}^{-1}(U), \tag{2}$$

where U denotes in the following a uniform$(0,1)$ random variable and where the usual inverse of a distribution function, which is the non-decreasing and left-continuous function $F_X^{-1}(p)$, is defined by

$$F_X^{-1}(p) = \inf\{x \in \mathbb{R} \mid F_X(x) \geq p\}, \qquad p \in [0,1],$$

with $\inf \emptyset = +\infty$ by convention.

Kaas et al. (2000) have proved that the inverse distribution function of a sum of comonotonic random variables is simply the sum of the inverse distribution functions of the marginal distributions. Moreover, in case of strictly increasing and continuous marginals, the cumulative distribution function (cdf) $F_{\mathbb{S}^c}(x)$ is uniquely determined by

$$F_{\mathbb{S}^c}^{-1}\left(F_{\mathbb{S}^c}(x)\right) = \sum_{i=1}^{n} F_{X_i}^{-1}\left(F_{\mathbb{S}^c}(x)\right) = x, \qquad F_{\mathbb{S}^c}^{-1}(0) < x < F_{\mathbb{S}^c}^{-1}(1). \tag{3}$$

Hereafter we restrict ourselves to this case of strictly increasing and continuous marginals.

In the following theorem Dhaene et al. (2002a) have proved that the stop-loss premiums of a sum of comonotonic random variables can easily be obtained from the stop-loss premiums of the terms.

Theorem 1. *The stop-loss premiums of the sum* $\mathbb{S}^c$ *of the components of the comonotonic random vector* $(X_1^c, X_2^c, \ldots, X_n^c)$ *are given by*

$$\mathrm{E}\left[(\mathbb{S}^c - b)_+\right] = \sum_{i=1}^{n} \mathrm{E}\left[\left(X_i - F_{X_i}^{-1}\left(F_{\mathbb{S}^c}(b)\right)\right)_+\right], \qquad F_{\mathbb{S}^c}^{-1}(0) < b < F_{\mathbb{S}^c}^{-1}(1). \tag{4}$$

If the only information available concerning the multivariate distribution function of the random vector $(X_1, \ldots, X_n)$ are the marginal distribution functions of the X_i, then the distribution function of $\mathbb{S}^c = F_{X_1}^{-1}(U) + F_{X_2}^{-1}(U) + \cdots + F_{X_n}^{-1}(U)$ is a prudent choice for approximating the unknown distribution function of $\mathbb{S} = X_1 + \cdots + X_n$. It is a supremum in terms of convex order. It is the best upper bound that can be derived under the given conditions.

2.2. Improved comonotonic upper bound

Let us now assume that we have some additional information available concerning the stochastic nature of $(X_1, \ldots, X_n)$. More precisely, we assume that there exists some random variable Λ with a given distribution function, such that we know the conditional cumulative distribution functions, given $\Lambda = \lambda$, of the random variables X_i, for all possible values of λ. In fact, Kaas et al. (2000) define the improved comonotonic upper bound $\mathbb{S}^u$ as

$$\mathbb{S}^u = F^{-1}_{X_1|\Lambda}(U) + F^{-1}_{X_2|\Lambda}(U) + \cdots + F^{-1}_{X_n|\Lambda}(U),$$

where U is a uniform$(0,1)$ variable independent of Λ, $F^{-1}_{X_i|\Lambda}(U)$ is the notation for the random variable $f_i(U, \Lambda)$, with the function f_i defined by $f_i(u, \lambda) = F^{-1}_{X_i|\Lambda=\lambda}(u)$. In order to obtain the distribution function of $\mathbb{S}^u$, observe that given the event $\Lambda = \lambda$, the random variable $\mathbb{S}^u$ is a sum of comonotonic random variables. If the marginal cdfs $F_{X_i|\Lambda=\lambda}$ are strictly increasing and continuous, then $F_{\mathbb{S}^u|\Lambda=\lambda}(x)$ is a solution to

$$\sum_{i=1}^{n} F^{-1}_{X_i \mid \Lambda=\lambda}\left(F_{\mathbb{S}^u \mid \Lambda=\lambda}(x)\right) = x, \qquad x \in \left(F^{-1}_{\mathbb{S}^u \mid \Lambda=\lambda}(0), F^{-1}_{\mathbb{S}^u \mid \Lambda=\lambda}(1)\right), \tag{5}$$

and the cdf of $\mathbb{S}^u$ then follows from

$$F_{\mathbb{S}^u}(x) = \int_{-\infty}^{+\infty} F_{\mathbb{S}^u|\Lambda=\lambda}(x)\, dF_\Lambda(\lambda).$$

In this case, we also find that for any $b \in \left(F^{-1}_{\mathbb{S}^u|\Lambda=\lambda}(0), F^{-1}_{\mathbb{S}^u|\Lambda=\lambda}(1)\right)$:

$$\mathrm{E}\left[(\mathbb{S}^u - b)_+ \mid \Lambda = \lambda\right] = \sum_{i=1}^{n} \mathrm{E}\left[\left(X_i - F^{-1}_{X_i|\Lambda=\lambda}\left(F_{\mathbb{S}^u|\Lambda=\lambda}(b)\right)\right)_+ \mid \Lambda = \lambda\right], \tag{6}$$

from which the stop-loss premium at retention b of $\mathbb{S}^u$ can be determined by integration with respect to λ over the real line.

2.3. Lower bound

Let $\underline{X} = (X_1, \ldots, X_n)$ be a random vector with given marginal cdfs $F_{X_1}, F_{X_2}, \ldots, F_{X_n}$. Assume again that there exists some random variable Λ with a given distribution function, such that we know the conditional distribution, given $\Lambda = \lambda$, of the random variables X_i, for all possible values of λ. We recall from Kaas et al. (2000) that a lower bound, in the sense of convex order, for $\mathbb{S} = X_1 + X_2 + \cdots + X_n$ is

$$\mathbb{S}^\ell = \mathrm{E}\left[\mathbb{S} \mid \Lambda\right]. \tag{7}$$

This idea can also be found in Rogers and Shi (1995) for the continuous case. We stress that $\mathbb{S}^\ell$ is not necessarily a comonotonic sum.

Let us further assume that the random variable Λ is such that all $\mathrm{E}[X_i \mid \Lambda]$ are non-decreasing and continuous functions of Λ. In that case $\mathbb{S}^\ell$ is a comonotonic variable, see e.g. Dhaene et al. (2002a). If in addition we assume that the cdfs of the random variables $\mathrm{E}[X_i \mid \Lambda]$ are strictly increasing and continuous, then the cdf of $\mathbb{S}^\ell$ is also strictly increasing and continuous, and we get for all $x \in \left(F_{\mathbb{S}^\ell}^{-1}(0), F_{\mathbb{S}^\ell}^{-1}(1)\right)$,

$$\sum_{i=1}^{n} F_{\mathrm{E}[X_i|\Lambda]}^{-1}\left(F_{\mathbb{S}^\ell}(x)\right) = x \quad \Leftrightarrow \quad \sum_{i=1}^{n} \mathrm{E}\left[X_i \mid \Lambda = F_\Lambda^{-1}\left(F_{\mathbb{S}^\ell}(x)\right)\right] = x, \tag{8}$$

which unambiguously determines the cdf of the convex order lower bound $\mathbb{S}^\ell$ for $\mathbb{S}$. Using Theorem 1, the stop-loss premiums of $\mathbb{S}^\ell$ can be computed as:

$$\mathrm{E}\left[\left(\mathbb{S}^\ell - b\right)_+\right] = \sum_{i=1}^{n} \mathrm{E}\left[\left(\mathrm{E}[X_i \mid \Lambda] - \mathrm{E}\left[X_i \mid \Lambda = F_\Lambda^{-1}\left(F_{\mathbb{S}^\ell}(b)\right)\right]\right)_+\right], \tag{9}$$

which holds for all retentions $b \in \left(F_{\mathbb{S}^\ell}^{-1}(0),\ F_{\mathbb{S}^\ell}^{-1}(1)\right)$.

So far, we considered the case that all $\mathrm{E}[X_i \mid \Lambda]$ are non-decreasing functions of Λ. The case where all $\mathrm{E}[X_i \mid \Lambda]$ are non-increasing and continuous functions of Λ also leads to a comonotonic vector $(\mathrm{E}[X_1 \mid \Lambda], \mathrm{E}[X_2 \mid \Lambda], \ldots, \mathrm{E}[X_n \mid \Lambda])$, and can be treated in a similar way but will not be dealt with in this paper.

3. Bounds by conditioning on two variables

Now we study the case that the random variables X_i in the sum $\mathbb{S} = \sum_{i=1}^{n} X_i$ are conditioned on two random variables Λ_1 and Λ_2 of which the joint distribution is known.

3.1. Lower bound

A general lower bound for the stop-loss premium in case of two conditioning variables can be found from the following theorem.

Theorem 2. *Let Λ_1 and Λ_2 be two random variables with some known joint distribution. Then the lower bound for the stop-loss premium $\mathrm{E}[(\mathbb{S} - b)_+]$ is given by*

$$\mathrm{E}[(\mathbb{S} - b)_+] \geq \int_{-\infty}^{+\infty} \mathrm{E}[(\mathbb{S}_{\Lambda_1=\lambda_1}^\ell - b)_+] f_{\Lambda_1}(\lambda_1) d\lambda_1, \tag{10}$$

where f_{Λ_1} *denotes the probability density function (pdf) of* Λ_1, $b \in (F^{-1}_{\mathbb{S}^{\ell}_{\Lambda_1=\lambda_1}}(0), F^{-1}_{\mathbb{S}^{\ell}_{\Lambda_1=\lambda_1}}(1))$, *and where*

$$\mathbb{S}^{\ell}_{\Lambda_1=\lambda_1} = \mathrm{E}[\mathbb{S}|\Lambda_1 = \lambda_1, \Lambda_2] = \sum_{i=1}^{n} \mathrm{E}[X_i|\Lambda_1 = \lambda_1, \Lambda_2], \tag{11}$$

implying that the expectation under the integral is taken w.r.t. $\Lambda_2|\Lambda_1 = \lambda_1$.

Proof. By the tower property for conditional expectations the stop-loss premium $\mathrm{E}[(\mathbb{S} - b)_+]$ with $\mathbb{S} = \sum_{i=1}^{n} X_i$ equals

$$\mathrm{E}[\mathrm{E}[(\mathbb{S} - b)_+|\Lambda_1, \Lambda_2]] \tag{12}$$

for some conditioning variables Λ_1 and Λ_2. Next applying Jensen's inequality and taking into account the relation between the joint pdf $f_{\Lambda_1\Lambda_2}$, the marginal pdf f_{Λ_1} and the conditional pdf $f_{\Lambda_2|\Lambda_1}$ we obtain

$$\begin{aligned}
\mathrm{E}[(\mathbb{S} - b)_+] &= \mathrm{E}\left[\mathrm{E}[(\mathbb{S} - b)_+|\Lambda_1, \Lambda_2]\right] \\
&\geq \mathrm{E}\left[(\mathrm{E}[\mathbb{S}|\Lambda_1, \Lambda_2] - b)_+\right] \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (\mathrm{E}[\mathbb{S}|\Lambda_1 = \lambda_1, \Lambda_2 = \lambda_2] - b)_+ f_{\Lambda_1\Lambda_2}(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2 \\
&= \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} (\sum_{i=1}^{n} \mathrm{E}[X_i|\Lambda_1 = \lambda_1, \Lambda_2 = \lambda_2] - b)_+ f_{\Lambda_2|\Lambda_1=\lambda_1}(\lambda_2|\lambda_1) d\lambda_2\right) f_{\Lambda_1}(\lambda_1) d\lambda_1
\end{aligned}$$

where the inner integral represents the stop-loss premium of $\mathbb{S}^{\ell}_{\Lambda_1=\lambda_1}$ (11), i.e.

$$\mathrm{E}[(\mathbb{S}^{\ell}_{\Lambda_1=\lambda_1} - b)_+] = \int_{-\infty}^{+\infty} (\sum_{i=1}^{n} \mathrm{E}[X_i|\Lambda_1 = \lambda_1, \Lambda_2 = \lambda_2] - b)_+ f_{\Lambda_2|\Lambda_1=\lambda_1}(\lambda_2|\lambda_1) d\lambda_2. \tag{13}$$

Note that (13) always holds independent of whether $\mathbb{S}^{\ell}_{\Lambda_1=\lambda_1}$ is a comonotonic sum or not. □

3.2. Decomposition and bounds for the stop-loss premium

In this section we decompose the stop-loss premium starting from (12), an idea which goes back at least to Curran (1994).

Assuming that there exists a b_{Λ_1} such that $\Lambda_1 \geq b_{\Lambda_1}$ implies that $\mathbb{S} \geq b$ for any value of Λ_2 or such that $\Lambda_1 \leq b_{\Lambda_1}$ implies that $\mathbb{S} \geq b$ for any value of Λ_2, the stop-loss premium can be split in two parts. We concentrate upon the decomposition in the first case, the second case can be treated in a similar way with the appropriate integration bounds:

$$\begin{aligned}\mathrm{E}[(\mathbb{S}-b)_+] &= \int_{-\infty}^{b_{\Lambda_1}}\int_{-\infty}^{+\infty}\mathrm{E}[(\mathbb{S}-b)_+|\Lambda_1=\lambda_1,\Lambda_2=\lambda_2]f_{\Lambda_1\Lambda_2}(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2\\ &+\int_{b_{\Lambda_1}}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{E}[(\mathbb{S}-b)|\Lambda_1=\lambda_1,\Lambda_2=\lambda_2]f_{\Lambda_1\Lambda_2}(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2\\ &=: I_1+I_2 \end{aligned}\tag{14}$$

where $f_{\Lambda_1\Lambda_2}(\lambda_1,\lambda_2)$ is the joint pdf of Λ_1 and Λ_2.

The second integral can further be simplified to

$$\begin{aligned} I_2 &= \int_{b_{\Lambda_1}}^{+\infty}\int_{-\infty}^{+\infty}\sum_{i=1}^{n}\mathrm{E}[X_i|\Lambda_1=\lambda_1,\Lambda_2=\lambda_2]f_{\Lambda_1\Lambda_2}(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2\\ &- b\underbrace{\int_{b_{\Lambda_1}}^{+\infty}\underbrace{\left(\int_{-\infty}^{+\infty}f_{\Lambda_1\Lambda_2}(\lambda_1,\lambda_2)d\lambda_2\right)}_{f_{\Lambda_1}(\lambda_1)}d\lambda_1}_{1-F_{\Lambda_1}(b_{\Lambda_1})}, \end{aligned}\tag{15}$$

where F_{Λ_1} stands for the cdf of Λ_1, while for the first integral I_1 we develop bounds.

Note that we assumed that there exists a b_{Λ_1} independent of Λ_2. In this way the integration bounds are constant which simplifies the problem under consideration.

Another technique is to approximate $\mathbb{S}|_{\Lambda_1,\Lambda_2}$ by a known distribution with the same mean and variance. However this leads to a double integral requiring involved numerical computations.

Starting from the decomposition (14) we can construct the lower bound for the first integral I_1 by using the following theorem.

Theorem 3. *Let Λ_1 and Λ_2 be two random variables with some known joint distribution. Suppose there exists a b_{Λ_1} such that $\Lambda_1 \geq b_{\Lambda_1}$ implies that $\mathbb{S}\geq b$ for any value of Λ_2. Then the lower bound for I_1 in the decomposition (14) of the stop-loss premium $\mathrm{E}[(\mathbb{S}-b)_+]$ is given by*

$$\mathrm{LB}(I_1)=\int_{-\infty}^{b_{\Lambda_1}}\mathrm{E}[(\mathbb{S}^{\ell}_{\Lambda_1=\lambda_1}-b)_+]f_{\Lambda_1}(\lambda_1)d\lambda_1,\tag{16}$$

where $\mathbb{S}^{\ell}_{\Lambda_1=\lambda_1}$ is defined in (11) and its stop-loss premium is given by (13), where f_{Λ_1} denotes the pdf of Λ_1 and $b\in(F^{-1}_{\mathbb{S}^{\ell}_{\Lambda_1=\lambda_1}}(0),F^{-1}_{\mathbb{S}^{\ell}_{\Lambda_1=\lambda_1}}(1))$.

When $\mathbb{S}^{\ell}_{\Lambda_1=\lambda_1}$ is moreover a comonotonic sum then its stop-loss premium (13) is given by

$$\mathrm{E}[(\mathbb{S}^{\ell}_{\Lambda_1=\lambda_1}-b)_+]=\sum_{i=1}^{n}\mathrm{E}\left[\left(\mathrm{E}[X_i|\Lambda_1=\lambda_1,\Lambda_2]-\mathrm{E}[X_i|\Lambda_1=\lambda_1,\Lambda_2=F^{-1}_{\Lambda_2}(F_{\mathbb{S}^{\ell}_{\Lambda_1=\lambda_1}}(b))]\right)_+\right],\tag{17}$$

where the outer expectation is taken w.r.t. $\Lambda_2|\Lambda_1 = \lambda_1$.

Proof. Relation (16) is the analogue of the lower bound (10) in Theorem 2 with the appropriate upper integration bound b_{Λ_1}.

When for given $\Lambda_1 = \lambda_1$ the sum $\sum_{i=1}^n \mathrm{E}[X_i|\Lambda_1 = \lambda_1, \Lambda_2]$, denoted by $\mathbb{S}^\ell_{\Lambda_1=\lambda_1}$, is comonotonic, the quantiles of this sum follow from

$$F^{-1}_{\mathbb{S}^\ell_{\Lambda_1=\lambda_1}}(p) = \sum_{i=1}^n F^{-1}_{\mathrm{E}[X_i|\Lambda_1=\lambda_1,\Lambda_2]}(p) \qquad p \in (0,1). \tag{18}$$

If moreover the random variable Λ_2 is such that $g_i(\Lambda_2) \equiv \mathrm{E}[X_i|\Lambda_1 = \lambda_1, \Lambda_2]$ are strictly increasing and continuous functions of Λ_2, these quantiles read for all $p \in (0,1)$

$$F^{-1}_{\mathbb{S}^\ell_{\Lambda_1=\lambda_1}}(p) = \sum_{i=1}^n F^{-1}_{g_i(\Lambda_2)}(p) = \sum_{i=1}^n g_i(\Lambda_2 = F^{-1}_{\Lambda_2}(p)) = \sum_{i=1}^n \mathrm{E}[X_i|\Lambda_1 = \lambda_1, \Lambda_2 = F^{-1}_{\Lambda_2}(p)]. \tag{19}$$

If additionally the cdf's of the random variables $\mathrm{E}[X_i|\Lambda_1 = \lambda_1, \Lambda_2]$ are strictly increasing and continuous, then the cdf of $\mathbb{S}^\ell_{\Lambda_1=\lambda_1}$ is also strictly increasing and continuous and we get from (18) and (19) for all $x \in (F^{-1}_{\mathbb{S}^\ell_{\Lambda_1=\lambda_1}}(0), F^{-1}_{\mathbb{S}^\ell_{\Lambda_1=\lambda_1}}(1))$

$$\sum_{i=1}^n F^{-1}_{\mathrm{E}[X_i|\Lambda_1=\lambda_1,\Lambda_2]}(F_{\mathbb{S}^\ell_{\Lambda_1=\lambda_1}}(x)) = x,$$

or equivalently

$$\sum_{i=1}^n \mathrm{E}[X_i|\Lambda_1 = \lambda_1, \Lambda_2 = F^{-1}_{\Lambda_2}(F_{\mathbb{S}^\ell_{\Lambda_1=\lambda_1}}(x))] = x, \tag{20}$$

which unambigously determines the cdf of $\mathbb{S}^\ell_{\Lambda_1=\lambda_1}$.

Under these assumptions, applying (20) to the retention b in the stop-loss premium (13) of $\mathbb{S}^\ell_{\Lambda_1=\lambda_1}$ and invoking Theorem 1 leads to the expression (17). □

In order to further develop the expressions for I_2 and for the lower bound of I_1 we need to know the distribution of X_i and of $X_i|_{\Lambda_1=\lambda_1,\Lambda_2=\lambda_2}$.

Note that when it is not possible to find an integration bound b_{Λ_1} in order to split the stop-loss premium into two parts, one can find a lower bound from Theorem 2. This lower bound equals the sum of I_2, (15), and the lower bound (16) of I_1.

Remark 1. When for given $\Lambda_1 = \lambda_1$, the conditional distribution of $X_i|_{\Lambda_1=\lambda_1,\Lambda_2}$ is known, then by convex ordering it holds that

$$\sum_{i=1}^n X_i|_{\Lambda_1=\lambda_1,\Lambda_2} \preceq_{\mathrm{cx}} \sum_{i=1}^n F^{-1}_{X_i|\Lambda_1=\lambda_1,\Lambda_2}(U),$$

and thus the integral I_1 can be bounded above using the comonotonic upper bound and applying (3) and Theorem 1. However, since this leads to a double integral and hence to the problem of a numerical integration of a multiple integral, we will not further proceed with this upper bound.

3.3. Case of sum of lognormal random variables

In this paragraph we further develop the expressions for the lower bound when the random variables X_i in the sum $\mathbb{S}$ are lognormal.

We assume that $X_i = \alpha_i e^{Y_i}$ with $Y_i \sim \mathcal{N}(\mathrm{E}[Y_i], \sigma_{Y_i})$ and $\alpha_i \in \mathbb{R}$. In this case the stop-loss premium with some retention b_i, namely $\mathrm{E}[(X_i - b_i)_+]$, is well-known from the following lemma.

Lemma 1. *Let X be a lognormal random variable of the form αe^Y with $Y \sim \mathcal{N}(\mathrm{E}[Y], \sigma_Y)$ and $\alpha \in \mathbb{R}$. Then the stop-loss premium with retention b equals for $\alpha b > 0$*

$$\mathrm{E}[(X-b)_+] = \mathrm{sign}\,(\alpha)\, e^{\mu + \frac{\sigma^2}{2}}\ \Phi(\mathrm{sign}\,(\alpha)\, d_1) - b\Phi(\mathrm{sign}\,(\alpha)\, d_2), \tag{21}$$

where

$$\mu = \ln|\alpha| + \mathrm{E}[Y] \qquad \sigma = \sigma_Y \tag{22}$$

$$d_1 = \frac{\mu + \sigma^2 - \ln|b|}{\sigma} \qquad d_2 = d_1 - \sigma \tag{23}$$

and where Φ stands for the cdf of a standard normal random variable. The cases $\alpha b < 0$ are trivial.

We now develop the conditional density function of a normally distributed random variable given n normally distributed conditioning variables. This result will be crucial in what follows where it is applied for two conditioning variables.

Lemma 2. *Let $X,\ \Lambda_1, \ldots, \Lambda_n$ denote $n+1$ random variables which are multivariate normally distributed, with*

$$X \sim \mathcal{N}(\mu_X, \sigma_X) \quad \text{and} \quad \Lambda_i \sim \mathcal{N}(\mu_{\Lambda_i}, \sigma_{\Lambda_i}) \qquad i = 1, \ldots, n.$$

Introducing the vectors $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ and $\boldsymbol{\mu}_\Lambda = (\mu_{\Lambda_1}, \ldots, \mu_{\Lambda_n})$, the conditional density function of X given $\Lambda_1 = \lambda_1,\ \ldots,\ \Lambda_n = \lambda_n$ is given by

$$f_{X|\Lambda_1=\lambda_1,\ldots,\Lambda_n=\lambda_n}(x|\lambda_1,\ldots,\lambda_n) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\sigma_X^2 - \boldsymbol{R}_X^T \boldsymbol{\Sigma}_{1n}^{-1} \boldsymbol{R}_X}} \exp\left[-\frac{1}{2} \left(\frac{x - \mu_X - \boldsymbol{R}_X^T \boldsymbol{\Sigma}_{1n}^{-1} (\boldsymbol{\lambda} - \boldsymbol{\mu}_\Lambda)}{\sqrt{\sigma_X^2 - \boldsymbol{R}_X^T \boldsymbol{\Sigma}_{1n}^{-1} \boldsymbol{R}_X}} \right)^2 \right] \tag{24}$$

where $\mathbf{\Sigma}_{1n}$ *is the variance-covariance matrix of* $\Lambda_1, \ldots, \Lambda_n$*:*

$$\mathbf{\Sigma}_{1n} = \begin{pmatrix} \sigma^2_{\Lambda_1} & \mathrm{cov}(\Lambda_1, \Lambda_2) & \cdots & \mathrm{cov}(\Lambda_1, \Lambda_n) \\ \mathrm{cov}(\Lambda_1, \Lambda_2) & \sigma^2_{\Lambda_2} & \cdots & \mathrm{cov}(\Lambda_2, \Lambda_n) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{cov}(\Lambda_1, \Lambda_n) & \mathrm{cov}(\Lambda_2, \Lambda_n) & \cdots & \sigma^2_{\Lambda_n} \end{pmatrix} \tag{25}$$

and

$$\boldsymbol{R}_X^T = \begin{pmatrix} \mathrm{cov}(X, \Lambda_1) & \cdots & \mathrm{cov}(X, \Lambda_n) \end{pmatrix}. \tag{26}$$

Proof. The conditional density function of X given $\Lambda_1 = \lambda_1, \ldots, \Lambda_n = \lambda_n$ is defined as the ratio of the joint probability density function of $X, \Lambda_1, \ldots, \Lambda_n$ and the joint pdf of $\Lambda_1, \ldots, \Lambda_n$:

$$f_{X|\Lambda_1=\lambda_1,\ldots,\Lambda_n=\lambda_n}(x|\lambda_1, \ldots, \lambda_n) = \frac{f_{X,\Lambda_1,\ldots,\Lambda_n}(x, \lambda_1, \ldots, \lambda_n)}{f_{\Lambda_1\ldots\Lambda_n}(\lambda_1, \ldots, \lambda_n)}. \tag{27}$$

The joint pdf of X, $\Lambda_1, \ldots, \Lambda_n$ is given by

$$f_{X,\Lambda_1,\ldots,\Lambda_n}(x, \lambda_1, \ldots, \lambda_n) = \frac{1}{(2\pi)^{\frac{n+1}{2}}} \frac{1}{\sqrt{|\mathbf{\Sigma}|}}\, e^{\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T \mathbf{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})}, \tag{28}$$

where $\boldsymbol{x} = (x, \lambda_1, \ldots, \lambda_n)$, $\boldsymbol{\mu} = (\mu_X, \mu_{\Lambda_1}, \ldots, \mu_{\Lambda_n})$, $|\boldsymbol{A}| =$ determinant of a square matrix $\boldsymbol{A}$ and where $\mathbf{\Sigma}$ is a block matrix:

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma^2_X & \boldsymbol{R}_X^T \\ \boldsymbol{R}_X & \mathbf{\Sigma}_{1n} \end{pmatrix} \quad \text{with} \quad |\mathbf{\Sigma}| = |\mathbf{\Sigma}_{1n}| \left(\sigma^2_X - \boldsymbol{R}_X^T \mathbf{\Sigma}_{1n}^{-1} \boldsymbol{R}_X \right). \tag{29}$$

The joint pdf of $\Lambda_1, \ldots, \Lambda_n$ reads

$$f_{\Lambda_1\ldots\Lambda_n}(\lambda_1, \ldots, \lambda_n) = \frac{1}{(2\pi)^{\frac{n}{2}}} \frac{1}{\sqrt{|\mathbf{\Sigma}_{1n}|}}\, e^{-\frac{1}{2}(\boldsymbol{\lambda}-\boldsymbol{\mu}_\Lambda)^T \mathbf{\Sigma}_{1n}^{-1}(\boldsymbol{\lambda}-\boldsymbol{\mu}_\Lambda)}. \tag{30}$$

Using the technique of block inversion one obtains the following block matrix as the inverse matrix $\mathbf{\Sigma}^{-1}$:

$$\mathbf{\Sigma}^{-1} = \frac{|\mathbf{\Sigma}_{1n}|}{|\mathbf{\Sigma}|} \begin{pmatrix} 1 & -\boldsymbol{R}_X^T \mathbf{\Sigma}_{1n}^{-1} \\ -\mathbf{\Sigma}_{1n}^{-1} \boldsymbol{R}_X & \sigma^2_X \mathbf{\Sigma}_{1n}^{-1} - \mathbf{\Sigma}_{1n}^{-1} \boldsymbol{R}_X^T \mathbf{\Sigma}_{1n}^{-1} \boldsymbol{R}_X + \mathbf{\Sigma}_{1n}^{-1} \boldsymbol{R}_X \boldsymbol{R}_X^T \mathbf{\Sigma}_{1n}^{-1} \end{pmatrix}. \tag{31}$$

Applying block matrix multiplication we arrive at

$$\begin{aligned} (\boldsymbol{x}-\boldsymbol{\mu})^T \mathbf{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}) &= \begin{pmatrix} x - \mu_X & (\boldsymbol{\lambda} - \boldsymbol{\mu}_\Lambda)^T \end{pmatrix} \mathbf{\Sigma}^{-1} \begin{pmatrix} x - \mu_X \\ \boldsymbol{\lambda} - \boldsymbol{\mu}_\Lambda \end{pmatrix} \\ &= \frac{|\mathbf{\Sigma}_{1n}|}{|\mathbf{\Sigma}|}(x - \mu_X - \boldsymbol{R}_X^T \mathbf{\Sigma}_{1n}^{-1} \boldsymbol{\lambda})^2 + (\boldsymbol{\lambda} - \boldsymbol{\mu}_\Lambda)^T \mathbf{\Sigma}_{1n}^{-1}(\boldsymbol{\lambda} - \boldsymbol{\mu}_\Lambda). \end{aligned} \tag{32}$$

Finally, dividing the right-hand-side of (28) by the right-hand-side of (30) while taking into account (29) and (32), relation (24) follows from (27). □

The following theorem gives a closed-form expression for the lower bound (10). Hereto we concentrate on the stop-loss premium of $\mathbb{S}^{\ell}_{\Lambda_1=\lambda_1}$.

We denote

$$\boldsymbol{R}_i^T = \begin{pmatrix} R_{i1} & R_{i2} \end{pmatrix} = \begin{pmatrix} \text{cov}(Y_i, \Lambda_1) & \text{cov}(Y_i, \Lambda_2) \end{pmatrix}.$$

Theorem 4. *Consider the normal random variables $\Lambda_1 \sim \mathcal{N}(\mu_{\Lambda_1}, \sigma_{\Lambda_1})$ and $\Lambda_2 \sim \mathcal{N}(\mu_{\Lambda_2}, \sigma_{\Lambda_2})$ which satisfy the assumptions of Theorem 2, and such that $(Y_i, \Lambda_1, \Lambda_2)$ are multivariate normally distributed for all i. Assume that $\mathbb{S}^{\ell}_{\Lambda_1=\lambda_1}$ given by (11) is a comonotonic sum, i.e. for all i it holds that* $\text{sign}(\alpha_i) = \text{sign}((\boldsymbol{R}_i^T \boldsymbol{\Sigma}_{12}^{-1})_2)$ *when $(\boldsymbol{R}_i^T \boldsymbol{\Sigma}_{12}^{-1})_2 \neq 0$. Then the lower bound in (10) is given by*

$$\text{LB} = \sum_{i=1}^{n} \int_0^1 f_i(u) du$$

with

$$\begin{aligned} f_i(u) =& \alpha_i \mathrm{e}^{\mathrm{E}[Y_i] + \frac{1}{2}(\sigma_{Y_i}^2 - \boldsymbol{R}_i^T \boldsymbol{\Sigma}_{12}^{-1} \boldsymbol{R}_i) + [(\boldsymbol{R}_i^T \boldsymbol{\Sigma}_{12}^{-1})_1 \sigma_{\Lambda_1} + (\boldsymbol{R}_i^T \boldsymbol{\Sigma}_{12}^{-1})_2 \sigma_{\Lambda_2} \tilde{r}] \Phi^{-1}(u) \sigma_{\Lambda_1}} \times \\ & \times \mathrm{e}^{\frac{1}{2}(\boldsymbol{R}_i^T \boldsymbol{\Sigma}_{12}^{-1})_2^2 \sigma_{\Lambda_2}^2 (1-\tilde{r}^2)} \Phi \left[\text{sign}(\alpha_i) \left((\boldsymbol{R}_i^T \boldsymbol{\Sigma}_{12}^{-1})_2 \sigma_{\Lambda_2} \sqrt{1 - \tilde{r}^2} + d \right) \right] - b_i \Phi \left(\text{sign}(\alpha_i) d \right), \end{aligned} \tag{33}$$

where $U = \Phi\left(\frac{\Lambda_1 - \mu_{\Lambda_1}}{\sigma_{\Lambda_1}}\right)$ and $\tilde{r} = corr(\Lambda_1, \Lambda_2)$,

$$\begin{aligned} b_i &= \alpha_i e^{\mathrm{E}[Y_i] + \frac{1}{2}(\sigma_{Y_i}^2 - \boldsymbol{R}_i^T \boldsymbol{\Sigma}_{12}^{-1} \boldsymbol{R}_i) + (\boldsymbol{R}_i^T \boldsymbol{\Sigma}_{12}^{-1})_1 \sigma_{\Lambda_1} \Phi^{-1}(u) + (\boldsymbol{R}_i^T \boldsymbol{\Sigma}_{12}^{-1})_2 \sigma_{\Lambda_2} \Phi^{-1}(F_{\mathbb{S}^{\ell}_{U=u}}(b))}, \\ d &= \frac{\tilde{r} \Phi^{-1}(u) - \Phi^{-1}(F_{\mathbb{S}^{\ell}_{U=u}}(b))}{\sqrt{1 - \tilde{r}^2}}, \end{aligned}$$

and $F_{\mathbb{S}^{\ell}_{U=u}}(b)$ being the solution to

$$\sum_{i=1}^{n} \alpha_i e^{\mathrm{E}[Y_i] + \frac{1}{2}(\sigma_{Y_i}^2 - \boldsymbol{R}_i^T \boldsymbol{\Sigma}_{12}^{-1} \boldsymbol{R}_i) + (\boldsymbol{R}_i^T \boldsymbol{\Sigma}_{12}^{-1})_1 \sigma_{\Lambda_1} \Phi^{-1}(u) + (\boldsymbol{R}_i^T \boldsymbol{\Sigma}_{12}^{-1})_2 \sigma_{\Lambda_2} \Phi^{-1}(F_{\mathbb{S}^{\ell}_{U=u}}(b))} = b. \tag{34}$$

Proof. From Lemma 2 with $n = 2$ the distribution of Y_i given $\Lambda_1 = \lambda_1$ and $\Lambda_2 = \lambda_2$ follows:

$$Y_i|_{\Lambda_1=\lambda_1, \Lambda_2=\lambda_2} \sim \mathcal{N}(\mathrm{E}[Y_i] + \boldsymbol{R}_i^T \boldsymbol{\Sigma}_{12}^{-1}(\boldsymbol{\lambda} - \boldsymbol{\mu}_{\Lambda}), \sqrt{\sigma_{Y_i}^2 - \boldsymbol{R}_i^T \boldsymbol{\Sigma}_{12}^{-1} \boldsymbol{R}_i}) \tag{35}$$

and hence, recalling (30) for $n = 2$, we obtain

$$\begin{aligned} & \mathrm{E}[e^{Y_i} | \Lambda_1 = \lambda_1, \Lambda_2 = \lambda_2] f_{\Lambda_1 \Lambda_2}(\lambda_1, \lambda_2) \\ & = e^{\mathrm{E}[Y_i] + \boldsymbol{R}_i^T \boldsymbol{\Sigma}_{12}^{-1}(\boldsymbol{\lambda} - \boldsymbol{\mu}_{\Lambda}) + \frac{1}{2}(\sigma_{Y_i}^2 - \boldsymbol{R}_i^T \boldsymbol{\Sigma}_{12}^{-1} \boldsymbol{R}_i)} \cdot \frac{1}{2\pi \sqrt{|\boldsymbol{\Sigma}_{12}|}} e^{-\frac{1}{2}(\boldsymbol{\lambda} - \boldsymbol{\mu}_{\Lambda})^T \boldsymbol{\Sigma}_{12}^{-1} (\boldsymbol{\lambda} - \boldsymbol{\mu}_{\Lambda})}. \end{aligned} \tag{36}$$

From (35) and the first factor in (36) it follows that

$$
\begin{aligned}
\mathbb{S}^{\ell}_{\Lambda_1=\lambda_1} &= \sum_{i=1}^{n} \mathrm{E}[X_i|\Lambda_1=\lambda_1,\Lambda_2] \\
&= \sum_{i=1}^{n} \alpha_i e^{\mathrm{E}[Y_i]+\frac{1}{2}(\sigma^2_{Y_i}-\boldsymbol{R}_i^T\boldsymbol{\Sigma}_{12}^{-1}\boldsymbol{R}_i)+(\boldsymbol{R}_i^T\boldsymbol{\Sigma}_{12}^{-1})_1(\lambda_1-\mu_{\Lambda_1})+(\boldsymbol{R}_i^T\boldsymbol{\Sigma}_{12}^{-1})_2(\Lambda_2-\mu_{\Lambda_2})} \\
&= \sum_{i=1}^{n} \alpha_i e^{\mathrm{E}[Y_i]+\frac{1}{2}(\sigma^2_{Y_i}-\boldsymbol{R}_i^T\boldsymbol{\Sigma}_{12}^{-1}\boldsymbol{R}_i)+(\boldsymbol{R}_i^T\boldsymbol{\Sigma}_{12}^{-1})_1(\lambda_1-\mu_{\Lambda_1})+(\boldsymbol{R}_i^T\boldsymbol{\Sigma}_{12}^{-1})_2\sigma_{\Lambda_2}\Phi^{-1}(V)},
\end{aligned}
\tag{37}
$$

where $V=\Phi(\frac{\Lambda_2-\mu_{\Lambda_2}}{\sigma_{\Lambda_2}})$ is a (0,1) uniform random variable and $(\boldsymbol{x})_i$ denotes the ith component of the vector $\boldsymbol{x}$.

The sum $\mathbb{S}^{\ell}_{\Lambda_1=\lambda_1}$ is comonotonic as a non-decreasing function of Λ_2, or equivalently of $\sigma_{\Lambda_2}\Phi^{-1}(V)$, when for all i it holds that $\mathrm{sign}(\alpha_i)=\mathrm{sign}((\boldsymbol{R}_i^T\boldsymbol{\Sigma}_{12}^{-1})_2)$ when $(\boldsymbol{R}_i^T\boldsymbol{\Sigma}_{12}^{-1})_2\neq 0$. Then an analytical expression for the stop-loss premium (17) is obtained by adapting Lemma 1 taking into account that the expectation is taken with respect to $\Lambda_2|\Lambda_1=\lambda_1$ and denoting $\tilde{r}=\mathrm{corr}(\Lambda_1,\Lambda_2)$:

$$
\begin{aligned}
&\mathrm{E}[(\mathbb{S}^{\ell}_{\Lambda_1=\lambda_1}-b)_+] \\
&= \sum_{i=1}^{n}\mathrm{E}\left[\Big(\mathrm{E}[X_i|\Lambda_1=\lambda_1,\Lambda_2]-\mathrm{E}[X_i|\Lambda_1=\lambda_1,\Lambda_2=F^{-1}_{\Lambda_2}(F_{\mathbb{S}^{\ell}_{\Lambda_1=\lambda_1}}(b))]\Big)_+\right] \\
&= \sum_{i=1}^{n}\alpha_i e^{\mathrm{E}[Y_i]+\frac{1}{2}(\sigma^2_{Y_i}-\boldsymbol{R}_i^T\boldsymbol{\Sigma}_{12}^{-1}\boldsymbol{R}_i)+[(\boldsymbol{R}_i^T\boldsymbol{\Sigma}_{12}^{-1})_1\sigma_{\Lambda_1}+(\boldsymbol{R}_i^T\boldsymbol{\Sigma}_{12}^{-1})_2\sigma_{\Lambda_2}\tilde{r}]\frac{\lambda_1-\mu_{\Lambda_1}}{\sigma_{\Lambda_1}}+\frac{1}{2}(\boldsymbol{R}_i^T\boldsymbol{\Sigma}_{12}^{-1})_2^2\sigma^2_{\Lambda_2}(1-\tilde{r}^2)}\times \\
&\quad\times\Phi\left[\mathrm{sign}(\alpha_i)\left((\boldsymbol{R}_i^T\boldsymbol{\Sigma}_{12}^{-1})_2\sigma_{\Lambda_2}\sqrt{1-\tilde{r}^2}+d\right)\right]-\sum_{i=1}^{n}b_i\Phi\left(\mathrm{sign}(\alpha_i)d\right),
\end{aligned}
\tag{38}
$$

where

$$
\begin{aligned}
b_i &= \alpha_i e^{\mathrm{E}[Y_i]+\frac{1}{2}(\sigma^2_{Y_i}-\boldsymbol{R}_i^T\boldsymbol{\Sigma}_{12}^{-1}\boldsymbol{R}_i)+(\boldsymbol{R}_i^T\boldsymbol{\Sigma}_{12}^{-1})_1(\lambda_1-\mu_{\Lambda_1})+(\boldsymbol{R}_i^T\boldsymbol{\Sigma}_{12}^{-1})_2\sigma_{\Lambda_2}\Phi^{-1}(F_{\mathbb{S}^{\ell}_{\Lambda_1=\lambda_1}}(b))} \\
d &= \frac{\tilde{r}\frac{\lambda_1-\mu_{\Lambda_1}}{\sigma_{\Lambda_1}}-\Phi^{-1}(F_{\mathbb{S}^{\ell}_{\Lambda_1=\lambda_1}}(b))}{\sqrt{1-\tilde{r}^2}},
\end{aligned}
$$

and with $F_{\mathbb{S}^{\ell}_{\Lambda_1=\lambda_1}}(b)$ according to (20) and (37) the solution to

$$
\sum_{i=1}^{n}\alpha_i e^{\mathrm{E}[Y_i]+\frac{1}{2}(\sigma^2_{Y_i}-\boldsymbol{R}_i^T\boldsymbol{\Sigma}_{12}^{-1}\boldsymbol{R}_i)+(\boldsymbol{R}_i^T\boldsymbol{\Sigma}_{12}^{-1})_1(\lambda_1-\mu_{\Lambda_1})+(\boldsymbol{R}_i^T\boldsymbol{\Sigma}_{12}^{-1})_2\sigma_{\Lambda_2}\Phi^{-1}(F_{\mathbb{S}^{\ell}_{\Lambda_1=\lambda_1}}(b))}=b. \tag{39}
$$

When, in addition, we put $U=\Phi\left(\frac{\Lambda_1-\mu_{\Lambda_1}}{\sigma_{\Lambda_1}}\right)$ or, equivalently, $\Lambda_1-\mu_{\Lambda_1}=\sigma_{\Lambda_1}\Phi^{-1}(U)$ in relations (38) and (39), we obtain the lower bound (33)-(34). □

Remark 2. When the sum $\mathbb{S}^{\ell}_{\Lambda_1=\lambda_1}$ is not comonotonic due to a bad choice of the conditioning variables Λ_1 and Λ_2, it is clear from relation (13) in Theorem 2 that the lower bound in (10) contains a double integral. This case is not preferred since the error made by numerical integration is hard to control and could spoil the property that we are dealing with a lower bound.

The following theorem states an expression for the exact part and the lower bound for the remaining part in the decomposition (14) of the stop-loss premium $\mathrm{E}[(\mathbb{S}-b)_+]$.

Theorem 5. *Consider the normal random variables $\Lambda_1 \sim \mathcal{N}(\mu_{\Lambda_1}, \sigma_{\Lambda_1})$ and $\Lambda_2 \sim \mathcal{N}(\mu_{\Lambda_2}, \sigma_{\Lambda_2})$ such that $(Y_i, \Lambda_1, \Lambda_2)$ are multivariate normally distributed for all i. Further assume that the assumptions of Theorem 3 are satisfied. Then the lower bound of $\mathrm{E}[(\mathbb{S}-b)_+]$ (14) is given by*

$$\mathrm{LB} = I_2 + \mathrm{LB}(I_1),$$

where

$$I_2 = \sum_{i=1}^{n} \alpha_i e^{\mathrm{E}[Y_i]+\frac{1}{2}\sigma^2_{Y_i}} \Phi\left(\frac{R_{i1}}{\sigma_{\Lambda_1}} - b^*_{\Lambda_1}\right) - b\Phi(-b^*_{\Lambda_1}) \tag{40}$$

and

$$\mathrm{LB}(I_1) = \sum_{i=1}^{n} \int_0^{\Phi^{-1}(b^*_{\Lambda_1})} f_i(u)du, \tag{41}$$

with $b^*_{\Lambda_1} = \frac{b_{\Lambda_1}-\mu_{\Lambda_1}}{\sigma_{\Lambda_1}}$ and the function $f_i(u)$ given by (33).

Proof. We prove only (40) as (41) directly follows from Theorem 4 by considering the appropriate integration bound.

The expression (36) can be rewritten as

$$\begin{aligned}
&\mathrm{E}[e^{Y_i}|\Lambda_1=\lambda_1, \Lambda_2=\lambda_2] f_{\Lambda_1\Lambda_2}(\lambda_1,\lambda_2) \\
&= \frac{e^{\mathrm{E}[Y_i]+\frac{1}{2}\sigma^2_{Y_i}}}{2\pi\sqrt{|\boldsymbol{\Sigma}_{12}|}} e^{-\frac{1}{2}[(\boldsymbol{\lambda}-\boldsymbol{\mu}_\Lambda)^T\boldsymbol{\Sigma}^{-1}_{12}(\boldsymbol{\lambda}-\boldsymbol{\mu}_\Lambda)-2\boldsymbol{R}_i^T\boldsymbol{\Sigma}^{-1}_{12}(\boldsymbol{\lambda}-\boldsymbol{\mu}_\Lambda)+\boldsymbol{R}_i^T\boldsymbol{\Sigma}^{-1}_{12}\boldsymbol{R}_i]}.
\end{aligned} \tag{42}$$

The exponent can further be rearranged as follows

$$\begin{aligned}
&(\boldsymbol{\lambda}-\boldsymbol{\mu}_\Lambda)^T\boldsymbol{\Sigma}^{-1}_{12}(\boldsymbol{\lambda}-\boldsymbol{\mu}_\Lambda) - 2\boldsymbol{R}_i^T\boldsymbol{\Sigma}^{-1}_{12}(\boldsymbol{\lambda}-\boldsymbol{\mu}_\Lambda) + \boldsymbol{R}_i^T\boldsymbol{\Sigma}^{-1}_{12}\boldsymbol{R}_i \\
&= (\boldsymbol{\lambda}-\boldsymbol{\mu}_\Lambda-\boldsymbol{R}_i)^T\boldsymbol{\Sigma}^{-1}_{12}(\boldsymbol{\lambda}-\boldsymbol{\mu}_\Lambda) - \boldsymbol{R}_i^T\boldsymbol{\Sigma}^{-1}_{12}(\boldsymbol{\lambda}-\boldsymbol{\mu}_\Lambda-\boldsymbol{R}_i) \\
&= (\boldsymbol{\lambda}-\boldsymbol{\mu}_\Lambda-\boldsymbol{R}_i)^T\boldsymbol{\Sigma}^{-1}_{12}(\boldsymbol{\lambda}-\boldsymbol{\mu}_\Lambda) - (\boldsymbol{\lambda}-\boldsymbol{\mu}_\Lambda-\boldsymbol{R}_i)^T\boldsymbol{\Sigma}^{-1}_{12}\boldsymbol{R}_i \\
&= (\boldsymbol{\lambda}-\boldsymbol{\mu}_\Lambda-\boldsymbol{R}_i)^T\boldsymbol{\Sigma}^{-1}_{12}(\boldsymbol{\lambda}-\boldsymbol{\mu}_\Lambda-\boldsymbol{R}_i).
\end{aligned} \tag{43}$$

We also note that

$$F_{\Lambda_1}(b_{\Lambda_1}) = P(\Lambda_1 \leq b_{\Lambda_1}) = P\left(\frac{\Lambda_1 - \mathrm{E}[\Lambda_1]}{\sigma_{\Lambda_1}} \leq \frac{b_{\Lambda_1} - \mathrm{E}[\Lambda_1]}{\sigma_{\Lambda_1}}\right) = P(Z \leq b^*_{\Lambda_1}) = \Phi(b^*_{\Lambda_1}) \tag{44}$$

with $Z \sim \mathcal{N}(0,1)$, $b^*_{\Lambda_1} = (b_{\Lambda_1} - \mathrm{E}[\Lambda_1])/\sigma_{\Lambda_1}$.

Combining relations (42)-(44) into (15) yields

$$\begin{aligned} I_2 &= \int_{b_{\Lambda_1}}^{+\infty} \int_{-\infty}^{+\infty} \sum_{i=1}^{n} \alpha_i \frac{e^{\mathrm{E}[Y_i]+\frac{1}{2}\sigma^2_{Y_i}}}{2\pi\sqrt{|\boldsymbol{\Sigma}_{12}|}} e^{-\frac{1}{2}[(\boldsymbol{\lambda}-\boldsymbol{\mu}_\Lambda-\boldsymbol{R}_i)^T \boldsymbol{\Sigma}_{12}^{-1}(\boldsymbol{\lambda}-\boldsymbol{\mu}_\Lambda-\boldsymbol{R}_i)]} d\lambda_1 d\lambda_2 - b(1-\Phi(b^*_{\Lambda_1})) \\ &= \sum_{i=1}^{n} \alpha_i e^{\mathrm{E}[Y_i]+\frac{1}{2}\sigma^2_{Y_i}} \int_{b_{\Lambda_1}}^{+\infty} \left(\int_{-\infty}^{+\infty} \frac{e^{-\frac{1}{2}[(\boldsymbol{\lambda}-\boldsymbol{\mu}_\Lambda-\boldsymbol{R}_i)^T \boldsymbol{\Sigma}_{12}^{-1}(\boldsymbol{\lambda}-\boldsymbol{\mu}_\Lambda-\boldsymbol{R}_i)]}}{2\pi\sqrt{|\boldsymbol{\Sigma}_{12}|}} d\lambda_2 \right) d\lambda_1 \\ &\quad - b(1-\Phi(b^*_{\Lambda_1})), \end{aligned} \tag{45}$$

where the integrand is the joint pdf of two normally distributed random variables with mean $\boldsymbol{\mu}_\Lambda + \boldsymbol{R}_i$ and variance-covariance matrix $\boldsymbol{\Sigma}_{12}$. Since the second random variable is integrated out over the real line we end up with the marginal distribution of the first random variable which is normally distributed with mean $\mu_{\Lambda_1} + R_{i1}$ where $R_{i1} = \mathrm{cov}(Y_i, \Lambda_1) = \sigma_{Y_i}\sigma_{\Lambda_1}\mathrm{corr}(Y_i, \Lambda_1)$ and variance σ_{Λ_1}. After standardization of this normal random variable we end up with an exact expression for I_2:

$$\begin{aligned} I_2 &= \sum_{i=1}^{n} \alpha_i e^{\mathrm{E}[Y_i]+\frac{1}{2}\sigma^2_{Y_i}} \int_{\frac{b_{\Lambda_1}-\mu_{\Lambda_1}-R_{i1}}{\sigma_{\Lambda_1}}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz - b(1-\Phi(b^*_{\Lambda_1})) \\ &= \sum_{i=1}^{n} \alpha_i e^{\mathrm{E}[Y_i]+\frac{1}{2}\sigma^2_{Y_i}} \left(1 - \Phi(\frac{b_{\Lambda_1}-\mu_{\Lambda_1}}{\sigma_{\Lambda_1}} - \frac{R_{i1}}{\sigma_{\Lambda_1}})\right) - b(1-\Phi(b^*_{\Lambda_1})) \\ &= \sum_{i=1}^{n} \alpha_i e^{\mathrm{E}[Y_i]+\frac{1}{2}\sigma^2_{Y_i}} \Phi\left(\frac{R_{i1}}{\sigma_{\Lambda_1}} - b^*_{\Lambda_1}\right) - b\Phi(-b^*_{\Lambda_1}). \end{aligned}$$

□

Note that Λ_2 plays no role anymore in this exact part.

We now take a closer look at the condition for comonotonicity: $\mathrm{sign}(\alpha_i) = \mathrm{sign}((\boldsymbol{R}_i^T \boldsymbol{\Sigma}_{12}^{-1})_2)$ for all i. According to (25) and (26) we have

$$\begin{aligned} \boldsymbol{R}_i^T &= \begin{pmatrix} \mathrm{cov}(Y_i, \Lambda_1) & \mathrm{cov}(Y_i, \Lambda_2) \end{pmatrix} \\ &= \begin{pmatrix} \sigma_{Y_i}\sigma_{\Lambda_1} r_{i1} & \sigma_{Y_i}\sigma_{\Lambda_2} r_{i2} \end{pmatrix} \end{aligned}$$

with $r_{ij} = \mathrm{corr}(Y_i, \Lambda_j)$, $i = 1, \ldots, n$, $j = 1, 2$, and

$$\boldsymbol{\Sigma}_{12}^{-1} = \frac{1}{|\boldsymbol{\Sigma}_{12}|}\begin{pmatrix} \sigma_{\Lambda_2}^2 & -\mathrm{cov}(\Lambda_1, \Lambda_2) \\ -\mathrm{cov}(\Lambda_1, \Lambda_2) & \sigma_{\Lambda_1}^2 \end{pmatrix}$$

$$= \frac{1}{|\boldsymbol{\Sigma}_{12}|}\begin{pmatrix} \sigma_{\Lambda_2}^2 & -\sigma_{\Lambda_1}\sigma_{\Lambda_2}\tilde{r} \\ -\sigma_{\Lambda_1}\sigma_{\Lambda_2}\tilde{r} & \sigma_{\Lambda_1}^2 \end{pmatrix}$$

with $|\boldsymbol{\Sigma}_{12}| = \sigma_{\Lambda_1}^2\sigma_{\Lambda_2}^2(1-\tilde{r}^2)$ and $\tilde{r} = \mathrm{corr}(\Lambda_1, \Lambda_2)$, and thus

$$\begin{aligned}(\boldsymbol{R}_i^T\boldsymbol{\Sigma}_{12}^{-1})_2 &= \frac{1}{|\boldsymbol{\Sigma}_{12}|}[\sigma_{Y_i}\sigma_{\Lambda_1}r_{i1}(-\sigma_{\Lambda_1}\sigma_{\Lambda_2}\tilde{r}) + \sigma_{Y_i}\sigma_{\Lambda_2}r_{i2}\sigma_{\Lambda_1}^2] \\ &= \frac{\sigma_{Y_i}\sigma_{\Lambda_1}^2\sigma_{\Lambda_2}}{|\boldsymbol{\Sigma}_{12}|}[r_{i2} - r_{i1}\tilde{r}] \\ &= \frac{\sigma_{Y_i}}{\sigma_{\Lambda_2}(1-\tilde{r}^2)}[r_{i2} - r_{i1}\tilde{r}].\end{aligned}$$

When for example $\mathrm{sign}(\alpha_i) = 1$ for all i, we require that

$$r_{i2} - r_{i1}\tilde{r} \geq 0, \qquad i = 1, \ldots, n, \tag{46}$$

which is not trivially fulfilled as is illustrated for basket options.

Example (Basket options). We consider basket options with payoff $(\sum_{i=1}^n a_i S_i(T) - K)_+$ at time T and therefore with price

$$BC(n, K, T) = e^{-rT}\,\mathrm{E}^Q\left[\left(\sum_{i=1}^n a_i S_i(T) - K\right)_+\right]$$

under the risk-neutral martingale measure Q and with r the risk-free interest rate. We assume that under this measure Q

$$dS_i(t) = rS_i dt + \sigma_i S_i dW_i(t),$$

where $\{W_i(t), t \geq 0\}$ is a standard Brownian motion associated with the price process of asset i. Further, we assume the different asset prices to be instantaneously correlated according to

$$\mathrm{corr}(dW_i, dW_j) = \rho_{ij}dt. \tag{47}$$

For basket options the weights a_i are always positive which implies that the coefficients α_i are also positive.

Let us consider the simple case of $2(=n)$ conditioning variables Λ_1 and Λ_2 with

$$\Lambda_1 = \sum_{i=1}^{2} \beta_i \sigma_i W_i, \qquad \Lambda_2 = \sum_{i=1}^{2} \gamma_i \sigma_i W_i.$$

Then, we find that the comonotonicity conditions (46) for $n = 2$ lead to the following two-dimensional system of inequalities

$$\begin{cases} -\beta_2(\beta_1\gamma_2 - \beta_2\gamma_1) \geq 0 \\ \beta_1(\beta_1\gamma_2 - \beta_2\gamma_1) \geq 0, \end{cases}$$

which are independent of $\tilde{r}$. Obviously, β_1 and β_2 should have different signs. Which sign is accorded to β_1 (and hence the opposite to β_2) does not matter since conditioning on Λ or upon $-\Lambda$ will lead to the same results. This symmetry in the sign pattern of the coefficient of β_i does of course no longer hold when considering basket options with more than two underlying assets.

4. Approximation by a 'comonotonic upper bound of a non-comonotonic lower bound'

4.1. Introduction

Let us consider again a sum $\mathbb{S}$ of n dependent random variables $X_1, \ldots, X_n$, i.e.

$$\mathbb{S} = \sum_{i=1}^{n} X_i.$$

As seen in paragraph 2.3, this sum can be bounded below in the convex order sense by $\mathbb{S}^\ell$, i.e.

$$\mathbb{S}^\ell = \sum_{i=1}^{n} \mathrm{E}[X_i|\Lambda_1] \preceq_{cx} \mathbb{S} = \sum_{i=1}^{n} X_i$$

where we take a conditional expectation w.r.t. Λ_1. Let us denote $\mathrm{E}[X_i|\Lambda_1]$ by $\tilde{X}_i$, then $\mathbb{S}^\ell$ is again a sum of n dependent random variables $\tilde{X}_i$. When Λ_1 is properly chosen, this sum can be comonotonic. However, this procedure for finding a comonotonic lower bound can be rather involved. We think hereby at the case of a basket option where the basket consists of more than two assets which are not all positively correlated.

When $\mathbb{S}^\ell$ is not comonotonic, we can construct an improved comonotonic upper bound $(\mathbb{S}^\ell)^u$ as in paragraph 2.2.

Assume that given $\Lambda_2 = \lambda_2$ the cdf of $\tilde{X}_i$ is known, then:

$$\mathbb{S}^\ell = \sum_{i=1}^{n} \tilde{X}_i \preceq_{cx} \sum_{i=1}^{n} F^{-1}_{\tilde{X}_i|\Lambda_2}(U) =: (\mathbb{S}^\ell)^u.$$

Note that when $\mathbb{S}^\ell$ is a comonotonic sum $(\mathbb{S}^\ell)^u = \mathbb{S}^\ell$. If not, then $(\mathbb{S}^\ell)^u$ is an approximation to $\mathbb{S}$. For the stop-loss premia it holds that

$$\mathrm{E}[(\mathbb{S}^\ell - b)_+] \leq \mathrm{E}[(\mathbb{S} - b)_+] \leq \mathrm{E}[(\mathbb{S}^u - b)_+] \qquad \text{and} \qquad \mathrm{E}[(\mathbb{S}^\ell - b)_+] \leq \mathrm{E}[((\mathbb{S}^\ell)^u - b)_+] \leq \mathrm{E}[(\mathbb{S}^u - b)_+],$$

where the first set of inequalities follows by definition of $\mathbb{S}^\ell$, $\mathbb{S}$, $\mathbb{S}^u$ and the definition of convex order, while in the second set of inequalities the last inequality is shown below in Theorem 6.

Theorem 6. *Let* $(\mathbb{S}^\ell)^u = \sum_{i=1}^n F^{-1}_{\tilde{X}_i|\Lambda_2}(U)$ *and* $\mathbb{S}^u = \sum_{i=1}^n F^{-1}_{X_i|\Lambda_2}(U)$, *where the uniform* $(0,1)$ *random variable* U *is independent of* Λ_1 *and* Λ_2. *Then*

$$\mathrm{E}[((\mathbb{S}^\ell)^u - b)_+] \leq \mathrm{E}[(\mathbb{S}^u - b)_+].$$

Proof. First note that $(\mathbb{S}^\ell)^u = (\sum_{i=1}^n E[X_i|\Lambda_1]|\Lambda_2)^c = (\sum_{i=1}^n \tilde{X}_i|\Lambda_2)^c$. By setting $b = \sum_{i=1}^n b_i$ and using obvious properties of the maximum function $(\cdot)_+$, we obtain:

$$\begin{aligned}
\mathrm{E}[((\mathbb{S}^\ell)^u - b)_+] &= \mathrm{E}[\mathrm{E}[((\sum_{i=1}^n \tilde{X}_i|\Lambda_2)^c - b)_+]] \\
&\leq \mathrm{E}[\sum_{i=1}^n \mathrm{E}[((\tilde{X}_i|\Lambda_2)^c - b_i)_+]] \\
&= \sum_{i=1}^n \mathrm{E}[\mathrm{E}[(\tilde{X}_i - b_i)_+|\Lambda_2]] \\
&\leq \sum_{i=1}^n \mathrm{E}[\mathrm{E}[(X_i - b_i)_+|\Lambda_2]],
\end{aligned}$$

where the second equality is true since the components of a comonotonic counterpart have the same marginals as the original ones, and where the second inequality is based upon the fact that $\tilde{X}_i \preceq_{cx} X_i$. Since we can choose $b_i = F^{-1}_{X_i|\Lambda_2=\lambda_2}(F_{\mathbb{S}^u|\Lambda_2=\lambda_2}(b))$, Theorem 1 leads to:

$$\mathrm{E}[((\mathbb{S}^\ell)^u - b)_+] \leq \sum_{i=1}^n \mathrm{E}[\mathrm{E}[(X_i - b_i)_+|\Lambda_2]] = \mathrm{E}[(\mathbb{S}^u - b)_+].$$

□

In fact, since the expectations of the sums $\mathbb{S}^\ell$, $\mathbb{S}$ and $\mathbb{S}^u$ are the same we even have the following result in convex order sense:

$$\mathbb{S}^\ell \preceq_{cx} \mathbb{S} \preceq_{cx} \mathbb{S}^u \qquad \text{and} \qquad \mathbb{S}^\ell \preceq_{cx} (\mathbb{S}^\ell)^u \preceq_{cx} \mathbb{S}^u.$$

This motivates why one can consider $(\mathbb{S}^\ell)^u$ as an approximation to $\mathbb{S}$. As a consequence, in case $\mathbb{S}^\ell$ is not comonotonic, the stop-loss premium $\mathrm{E}[((\mathbb{S}^\ell)^u - b)_+]$ can be considered as an approximation to $\mathrm{E}[(\mathbb{S} - b)_+]$ which is, according to (6) and (5), given by

$$\begin{aligned}\mathrm{E}[((\mathbb{S}^\ell)^u - b)_+] &= \int_{-\infty}^{+\infty} \mathrm{E}[((\mathbb{S}^\ell)^u - b)_+ | \Lambda_2 = \lambda_2] f_{\Lambda_2}(\lambda_2) d\lambda_2 \\ &= \sum_{i=1}^n \int_{-\infty}^{+\infty} \mathrm{E}\left[\left(\tilde{X}_i - F^{-1}_{\tilde{X}_i|\Lambda_2=\lambda_2}(F_{(\mathbb{S}^\ell)^u|\Lambda_2=\lambda_2}(b))\right)_+\right] f_{\Lambda_2}(\lambda_2) d\lambda_2, \qquad (48)\end{aligned}$$

where for each λ_2, $F_{(\mathbb{S}^\ell)^u|\Lambda_2=\lambda_2}(b)$ is a solution to

$$\sum_{i=1}^n F^{-1}_{\tilde{X}_i|\Lambda_2=\lambda_2}(F_{(\mathbb{S}^\ell)^u|\Lambda_2=\lambda_2}(b)) = b.$$

The relation (48) can be further developed when a distribution for the X_i and for Λ_1, Λ_2 is assumed. We study the case of lognormal random variables X_i and normally distributed conditional variables.

4.2. Case of lognormal random variables X_i

Assume that $X_i = \alpha_i e^{Y_i}$ with $Y_i \sim \mathcal{N}(\mathrm{E}[Y_i], \sigma_{Y_i})$ and $\alpha_i \in \mathbb{R}$, i.e. the random variables X_i are lognormal random variables.

Given a normal random variable $\Lambda_1 \sim \mathcal{N}(\mathrm{E}[\Lambda_1], \sigma_{\Lambda_1})$ such that (Y_i, Λ_1) is bivariate normal for all i, then $Y_i|_{\Lambda_1=\lambda_1}$ is again normally distributed with mean $\mathrm{E}[Y_i] + r_{i1}\frac{\sigma_{Y_i}}{\sigma_{\Lambda_1}}(\lambda_1 - \mathrm{E}[\Lambda_1])$ and variance $(1 - r_{i1}^2)\sigma_{Y_i}^2$, where we recall that r_{i1} is the correlation between Y_i and Λ_1:

$$r_{i1} = \mathrm{corr}(Y_i, \Lambda_1) = \frac{\mathrm{cov}(Y_i, \Lambda_1)}{\sigma_{Y_i}\sigma_{\Lambda_1}}. \qquad (49)$$

Hence, we find

$$\mathrm{E}[X_i|\Lambda_1] = \alpha_i e^{\mathrm{E}[Y_i] + r_{i1}\frac{\sigma_{Y_i}}{\sigma_{\Lambda_1}}(\Lambda_1 - \mathrm{E}[\Lambda_1]) + \frac{1}{2}(1-r_{i1}^2)\sigma_{Y_i}^2}, \qquad (50)$$

or, when noting that $\frac{\Lambda_1 - \mathrm{E}[\Lambda_1]}{\sigma_{\Lambda_1}} \sim \mathcal{N}(0,1)$,

$$\mathrm{E}[X_i|\Lambda_1] = \alpha_i e^{\mathrm{E}[Y_i] + r_{i1}\sigma_{Y_i}\Phi^{-1}(V_1) + \frac{1}{2}(1-r_{i1}^2)\sigma_{Y_i}^2},$$

where the random variable $V_1 = \Phi\left(\frac{\Lambda_1 - \mathrm{E}[\Lambda_1]}{\sigma_{\Lambda_1}}\right)$ is uniformly distributed on $(0,1)$.

The lower bound $\mathbb{S}^\ell = \mathrm{E}[\mathbb{S} \mid \Lambda_1]$ is not comonotonic when there exists some i with $\mathrm{sign}(\alpha_i) = \mathrm{sign}(r_{i1})$ and some j with $\mathrm{sign}(\alpha_j) \neq \mathrm{sign}(r_{j1})$. In that case we proceed by deriving the upper bound $(\mathbb{S}^\ell)^u$ using a second conditioning variable Λ_2 which is correlated with Λ_1 according to $\tilde{r} = \mathrm{corr}(\Lambda_1, \Lambda_2)$.

Theorem 7. *Suppose $\mathbb{S}^\ell$ is not a comonotonic sum. Then the upper bound* $\mathrm{E}[((\mathbb{S}^\ell)^u - b)_+]$ *is an approximation to the stop-loss premium* $\mathrm{E}[(\mathbb{S} - b)_+]$ *and is given by*

$$\begin{aligned}
&\mathrm{E}[((\mathbb{S}^\ell)^u - b)_+] \\
&= \sum_{i=1}^{n} \alpha_i e^{\mathrm{E}[Y_i]+\frac{1}{2}(1-r_{i1}^2\tilde{r}^2)\sigma_{Y_i}^2} \times \\
&\quad \times \int_0^1 e^{\tilde{r} r_{i1}\sigma_{Y_i}\Phi^{-1}(v_2)} \Phi\left(\sqrt{1-\tilde{r}^2}|r_{i1}|\sigma_{Y_i}\mathrm{sign}(\alpha_i) - \Phi^{-1}(F_{(\mathbb{S}^\ell)^u|V_2=v_2}(b))\right) dv_2 \\
&\quad - b(1 - F_{(\mathbb{S}^\ell)^u}(b)),
\end{aligned} \tag{51}$$

where the cdf of $(\mathbb{S}^\ell)^u$ *is the integral*

$$F_{(\mathbb{S}^\ell)^u}(b) = \int_0^1 F_{(\mathbb{S}^\ell)^u|V_2=v_2}(b)dv_2, \tag{52}$$

with the integrand being the solution to the equation

$$\sum_{i=1}^{n} \alpha_i e^{\mathrm{E}[Y_i]+\frac{1}{2}(1-r_{i1}^2)\sigma_{Y_i}^2+\tilde{r} r_{i1}\sigma_{Y_i}\Phi^{-1}(v_2)+\mathrm{sign}(\alpha_i)\sqrt{1-\tilde{r}^2}|r_{i1}|\sigma_{Y_i}\Phi^{-1}(F_{(\mathbb{S}^\ell)^u|V_2=v_2}(b))} = b. \tag{53}$$

Proof. As explained in Section 4.1, $\mathrm{E}[((\mathbb{S}^\ell)^u - b)_+]$ is an approximation to the stop-loss premium $\mathrm{E}[(\mathbb{S} - b)_+]$.

Since each term $\tilde{X}_i = \mathrm{E}[X_i|\Lambda_1]$ in the sum $\mathbb{S}^\ell$ is again a lognormal random variable it can be expressed in the form

$$\tilde{X}_i = \tilde{\alpha}_i e^{\tilde{Y}_i}$$

with

$$\tilde{\alpha}_i = \alpha_i e^{\mathrm{E}[Y_i]+\frac{1}{2}(1-r_{i1}^2)\sigma_{Y_i}^2} \tag{54}$$

$$\tilde{Y}_i = r_{i1}\sigma_{Y_i}\frac{\Lambda_1 - \mathrm{E}[\Lambda_1]}{\sigma_{\Lambda_1}} \sim \mathcal{N}(0, |r_{i1}|\sigma_{Y_i}). \tag{55}$$

Conditionally, given $\Lambda_2 = \lambda_2$ with $\Lambda_2 \sim \mathcal{N}(\mathrm{E}[\Lambda_2], \sigma_{\Lambda_2})$, $\tilde{Y}_i$ is also normally distributed with mean $\tilde{\mu}(i)$ and variance $\tilde{\sigma}(i)^2$ given by

$$\begin{aligned}
\tilde{\mu}(i) &= \mathrm{E}[\tilde{Y}_i] + \tilde{r}_{i2}\frac{\sigma_{\tilde{Y}_i}}{\sigma_{\Lambda_2}}(\lambda_2 - \mathrm{E}[\Lambda_2]) = \tilde{r}_{i2}|r_{i1}|\sigma_{Y_i}\frac{\lambda_2 - \mathrm{E}[\Lambda_2]}{\sigma_{\Lambda_2}} \\
\tilde{\sigma}(i)^2 &= (1-\tilde{r}_{i2}^2)\sigma_{\tilde{Y}_i}^2 = (1-\tilde{r}_{i2}^2)r_{i1}^2\sigma_{Y_i}^2
\end{aligned} \tag{56}$$

where

$$\tilde{r}_{i2} = \mathrm{corr}(\tilde{Y}_i, \Lambda_2) = \frac{r_{i1}\sigma_{Y_i}}{\sigma_{\Lambda_1}} \frac{\mathrm{cov}(\Lambda_1, \Lambda_2)}{\sigma_{\tilde{Y}_i}\sigma_{\Lambda_2}} = \mathrm{sign}(r_{i1})\mathrm{corr}(\Lambda_1, \Lambda_2). \tag{57}$$

Recalling the notation

$$\tilde{r} = \mathrm{corr}(\Lambda_1, \Lambda_2),$$

(57) implies

$$\tilde{r}_{i2} \cdot |r_{i1}| = \tilde{r} \cdot r_{i1} \qquad \text{and} \qquad \tilde{r}_{i2}^2 = \tilde{r}^2.$$

Thus $\tilde{X}_i|_{\Lambda_2=\lambda_2}$ is distributed as

$$\tilde{\alpha}_i e^{\tilde{\mu}(i)+\mathrm{sign}(\alpha_i)\tilde{\sigma}(i)\Phi^{-1}(U)}$$

with U a uniform random variable on $(0,1)$. Therefore, the inverse distribution functions of the marginal distributions read for $p \in (0,1)$

$$\begin{aligned} F^{-1}_{\tilde{X}_i|\Lambda_2=\lambda_2}(p) &= \tilde{\alpha}_i e^{\tilde{\mu}(i)+\mathrm{sign}(\alpha_i)\tilde{\sigma}(i)\Phi^{-1}(p)} \\ &= \alpha_i e^{\mathrm{E}[Y_i]+\frac{1}{2}(1-r_{i1}^2)\sigma_{Y_i}^2+\tilde{r}r_{i1}\sigma_{Y_i}\frac{\lambda_2-\mathrm{E}[\Lambda_2]}{\sigma_{\Lambda_2}}+\mathrm{sign}(\alpha_i)\sqrt{1-\tilde{r}^2}|r_{i1}|\sigma_{Y_i}\Phi^{-1}(p)} \end{aligned}$$

or, equivalently, when putting $V_2 = \Phi\left(\frac{\Lambda_2-\mathrm{E}[\Lambda_2]}{\sigma_{\Lambda_2}}\right)$,

$$F^{-1}_{\tilde{X}_i|\Lambda_2=\lambda_2}(p) = \alpha_i e^{\mathrm{E}[Y_i]+\frac{1}{2}(1-r_{i1}^2)\sigma_{Y_i}^2+\tilde{r}r_{i1}\sigma_{Y_i}\Phi^{-1}(v_2)+\mathrm{sign}(\alpha_i)\sqrt{1-\tilde{r}^2}|r_{i1}|\sigma_{Y_i}\Phi^{-1}(p)} =: F^{-1}_{\tilde{X}_i|V_2=v_2}(p).$$

For the approximating stop-loss premium we find from (48) and Lemma 1:

$$\begin{aligned} &\mathrm{E}[((\mathbb{S}^\ell)^u - b)_+] \\ &= \sum_{i=1}^n \int_0^1 \mathrm{E}\left[\left(\tilde{X}_i - F^{-1}_{\tilde{X}_i|V_2=v_2}(F_{(\mathbb{S}^\ell)^u|V_2=v_2}(b))\right)_+\right] dv_2 \\ &= \sum_{i=1}^n \tilde{\alpha}_i e^{\frac{1}{2}r_{i1}^2\sigma_{Y_i}^2(1-\tilde{r}^2)} \int_0^1 e^{\tilde{r}r_{i1}\sigma_{Y_i}\Phi^{-1}(v_2)} \Phi(\sqrt{1-\tilde{r}^2}|r_{i1}|\sigma_{Y_i}\mathrm{sign}(\alpha_i) - \Phi^{-1}(F_{(\mathbb{S}^\ell)^u|V_2=v_2}(b)))dv_2 \\ &\quad - b(1 - F_{(\mathbb{S}^\ell)^u}(b)), \end{aligned} \tag{58}$$

where the cdf of $(\mathbb{S}^\ell)^u$ can be expressed as

$$F_{(\mathbb{S}^\ell)^u}(b) = \int_0^1 F_{(\mathbb{S}^\ell)^u|V_2=v_2}(b)dv_2, \tag{59}$$

with, according to (5), the integrand being the solution to the equation

$$
\begin{aligned}
& F^{-1}_{(\mathbb{S}^\ell)^u|V_2=v_2}\left(F_{(\mathbb{S}^\ell)^u|V_2=v_2}(b)\right) = b \\
\Leftrightarrow \quad & \sum_{i=1}^{n} F^{-1}_{\tilde{X}_i|V_2=v_2}\left(F_{(\mathbb{S}^\ell)^u|V_2=v_2}(b)\right) = b \\
\Leftrightarrow \quad & \sum_{i=1}^{n} \alpha_i e^{\mathrm{E}[Y_i]+\frac{1}{2}(1-r_{i1}^2)\sigma_{Y_i}^2+\tilde{r} r_{i1}\sigma_{Y_i}\Phi^{-1}(v_2)+\mathrm{sign}(\alpha_i)\sqrt{1-\tilde{r}^2}|r_{i1}|\sigma_{Y_i}\Phi^{-1}\left(F_{(\mathbb{S}^\ell)^u|V_2=v_2}(b)\right)} = b.
\end{aligned}
$$

In (58) the factor $\tilde{\alpha}_i e^{\frac{1}{2}r_{i1}^2\sigma_{Y_i}^2(1-\tilde{r}^2)}$ equals

$$
\alpha_i e^{\mathrm{E}[Y_i]+\frac{1}{2}(1-r_{i1}^2)\sigma_{Y_i}^2+\frac{1}{2}r_{i1}^2\sigma_{Y_i}^2(1-\tilde{r}^2)} = \alpha_i e^{\mathrm{E}[Y_i]+\frac{1}{2}(1-r_{i1}^2\tilde{r}^2)\sigma_{Y_i}^2}.
$$

□

When dealing with basket options, we assume that the different asset prices are instantaneously correlated according to (47) (see Example on basket options). Conditioning variables are taken of the form

$$
\sum_{j=1}^{n} \beta_j \sigma_j W_j(T). \tag{60}
$$

When the correlations ρ_{ij} are not all positive, it is not a priori guaranteed that for these choices all r_{i1} in (50) will be positive (of course it is possible to construct a conditioning variable Λ such that all r_{i1} are positive and thus the sum $\mathbb{S}^\ell$ is comonotonic, see Deelstra et al. (2004) Theorem 2), and hence this new approximation can be applied.

5. Convex approximations

Vyncke et al. (2004) used a moment matching technique of a convex combination of a lower bound and an upper bound in case of Asian options. In particular, they concentrate upon the comonotonic upper bound and the improved comonotonic upper bound. Since Asian options usually can be approximated by precise lower bounds and the convex combination puts much weight upon the lower bound, the convex combination leads to good approximations but do not improve the lower bounds significantly. Moreover, there is almost no difference among a convex combination with the comonotonic upper bound and the convex combination with the improved comonotonic upper bound.

Since a basket option is a stop-loss premium of a sum of dependent random variables which is far from being a comonotonic sum, numerical results show that convex combinations with different upper bounds lead to different convex approximations. In this section, we therefore improve the method of Vyncke et al. (2004) by using better upper bounds.

The approach of Vyncke et al. (2004) can be shortly explained as follows. Recalling that the sum $\mathbb{S}$ is bounded below and above in convex order ($\preceq_{\text{cx}}$):

$$\mathbb{S}^\ell \preceq_{\text{cx}} \mathbb{S} \preceq_{\text{cx}} \mathbb{S}^u \preceq_{\text{cx}} \mathbb{S}^c,$$

one can consider convex combinations like

$$\mathbb{S}^m = z\mathbb{S}^\ell + (1-z)\mathbb{S}^c \quad \text{or} \quad \mathbb{S}^m = z\mathbb{S}^\ell + (1-z)\mathbb{S}^u \tag{61}$$

for $z \in [0,1]$. The first case is referred to as a convex combination of the lower bound and the comonotonic upper bound, whereas the second case uses a convex combination of the lower bound and the improved comonotonic upper bound. Clearly, $E[\mathbb{S}^m] = E[\mathbb{S}]$ and Vyncke et al. (2004) determine z such that $\text{var}[\mathbb{S}^m] = \text{var}[\mathbb{S}]$, namely

$$z^c = \frac{\text{var}[\mathbb{S}^c] - \text{var}[\mathbb{S}]}{\text{var}[\mathbb{S}^c] - \text{var}[\mathbb{S}^\ell]} \quad \text{or} \quad z^u = \frac{\text{var}[\mathbb{S}^u] - \text{var}[\mathbb{S}]}{\text{var}[\mathbb{S}^u] - \text{var}[\mathbb{S}^\ell]}. \tag{62}$$

In this paper, we use the partially exact/comonotonic upper bound (PECUB) from Deelstra et al. (2004) which turned out to be a better upper bound than the improved comonotonic upper bound. Similarly to the construction of the upper bound based on the decomposition in Section 3.2, the upper bound PECUBΛ_1 of $E[(\mathbb{S}-K)_+]$ consists in an exact part and an upper bound to the remaining part by using the improved comonotonic upper bound $\mathbb{S}^u$:

$$\text{PECUB}\Lambda_1 = \text{exact part } \Lambda_1 + \int_{-\infty}^{b_{\Lambda_1}} E[(\mathbb{S}^u - K)_+ \mid \Lambda_1 = \lambda_1] dF_{\Lambda_1}(\lambda_1),$$

where b_{Λ_1} is chosen such that $\Lambda_1 \geq b_{\Lambda_1}$ implies that $\mathbb{S} \geq K$ and where

$$\text{exact part } \Lambda_1 = \int_{b_{\Lambda_1}}^{+\infty} E[\mathbb{S} \mid \Lambda_1 = \lambda_1] dF_{\Lambda_1}(\lambda_1) - K(1 - F_{\Lambda_1}(b_{\lambda_1})). \tag{63}$$

In case of lognormal random variables, this exact part can be reformulated in a closed-form by using the cumulative distribution function of the normal distribution.

Also each lower bound based on say Λ_2 can be decomposed in an exact part and the rest:

$$\text{LB}\Lambda_2 = \text{exact part } \Lambda_2 + \int_{-\infty}^{b_{\Lambda_2}} (E[\mathbb{S} \mid \Lambda_2 = \lambda_2] - K)_+ dF_{\Lambda_2}(\lambda_2),$$

where this exact part is given by substituting Λ_2 in (63).

Therefore, a convex combination of the lower bound with the partially exact/comonotonic upper bound leads to a convex combination of the different exact parts and integrals:

$$(1-z) \text{ exact part } \Lambda_1 + z \text{ exact part } \Lambda_2 + \tag{64}$$
$$+(1-z)\int_{-\infty}^{b_{\Lambda_1}} E[(\mathbb{S}^u - K)_+ \mid \Lambda_1 = \lambda_1] dF_{\Lambda_1}(\lambda_1) + z\int_{-\infty}^{b_{\Lambda_2}} (E[\mathbb{S} \mid \Lambda_2 = \lambda_2] - K)_+ dF_{\Lambda_2}(\lambda_2).$$

In the special case that $\Lambda_1 = \Lambda_2 =: \Lambda$, both the exact parts are the same and one clearly sees that the approximation by the convex combination is only due to a convex combination of the approximating integrals. Since the exact parts take more than 90% of the value into account, the use of PECUB upper bounds should intuitively lead to better approximations — a conjecture that has been proved by numerical results. However in the case of PECUB upper bounds, it is difficult to determine the convex weight which leads to equal variances. But since the non-exact part is constructed by using $\mathbb{S}^u$ for given Λ_1, a first idea is to use the z^u of Vyncke et al. (2004) explained above.

For completeness we state the expressions of the different terms in numerator and denominator of (62) in case of a sum of lognormal variables as in Section 3.3 such that they can be easily applied below to the basket option case in the Black & Scholes framework:

$$\text{var}[\mathbb{S}] = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\alpha_j e^{\frac{1}{2}(\sigma_{Y_i}^2+\sigma_{Y_j}^2)}\left[e^{\sigma_{Y_i}\sigma_{Y_j}\rho_{ij}} - 1\right]$$

$$\text{var}[\mathbb{S}^\ell] = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\alpha_j e^{\frac{1}{2}(r_i^2\sigma_{Y_i}^2+r_j^2\sigma_{Y_j}^2)}\left[e^{r_i r_j\sigma_{Y_i}\sigma_{Y_j}} - 1\right]$$

$$\text{var}[\mathbb{S}^c] = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\alpha_j e^{\frac{1}{2}(\sigma_{Y_i}^2+\sigma_{Y_j}^2)}\left[e^{\sigma_{Y_i}\sigma_{Y_j}} - 1\right]$$

$$\text{var}[\mathbb{S}^u] = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\alpha_j e^{\frac{1}{2}(\sigma_{Y_i}^2+\sigma_{Y_j}^2)}\left[e^{(r_i r_j\sigma_{Y_i}\sigma_{Y_j}+\sqrt{1-r_i^2}\sqrt{1-r_j^2}\sigma_{Y_i}\sigma_{Y_j})} - 1\right].$$

Here and in what follows, we drop the second subscript j in r_{ij} when only one conditioning variable Λ is considered, i.e. $r_i = \text{corr}(Y_i, \Lambda)$.

In the special case that $\Lambda_1 = \Lambda_2 =: \Lambda$, one can apply a moment matching technique to a convex combination of the terms under the integrals for each fixed λ. In other words we will define the convex approximation

$$\text{exact part } \Lambda + \int_{-\infty}^{b_\Lambda} z(\lambda)(E[\mathbb{S} \mid \Lambda = \lambda] - K)_+ dF_\Lambda(\lambda) + \int_{-\infty}^{b_\Lambda} (1 - z(\lambda))E[(\mathbb{S}^u - K)_+ \mid \Lambda = \lambda] dF_\Lambda(\lambda), \quad (65)$$

where we define $z(\lambda)$ such that $\text{var}[\mathbb{S} \mid \Lambda = \lambda] = z(\lambda)\text{var}[E[\mathbb{S} \mid \Lambda = \lambda]] + (1 - z(\lambda))\text{var}[\mathbb{S}^u \mid \Lambda = \lambda]$, namely

$$z(\lambda) = \frac{\text{var}[\mathbb{S}^u \mid \Lambda = \lambda] - \text{var}[\mathbb{S} \mid \Lambda = \lambda]}{\text{var}[\mathbb{S}^u \mid \Lambda = \lambda]}, \quad (66)$$

since $\text{var}[E[\mathbb{S} \mid \Lambda = \lambda]] = 0$, or equivalently

$$1 - z(\lambda) = \frac{\text{var}[\mathbb{S} \mid \Lambda = \lambda]}{\text{var}[\mathbb{S}^u \mid \Lambda = \lambda]}.$$

In case of lognormal variables as in Section 3.3, the numerator equals

$$\text{var}[\mathbb{S} \mid \Lambda = \lambda] = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \alpha_j e^{\frac{1}{2}(1-r_i^2)\sigma_{Y_i}^2 + \frac{1}{2}(1-r_j^2)\sigma_{Y_j}^2 + (r_i\sigma_{Y_i} + r_j\sigma_{Y_j})\Phi^{-1}(v)} \left[e^{\sigma_{Y_i}\sigma_{Y_j}(\rho_{ij} - r_i r_j)} - 1\right]$$

and the denominator equals

$$\text{var}[\mathbb{S}^u \mid \Lambda = \lambda] = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \alpha_j e^{\frac{1}{2}((1-r_i^2)\sigma_{Y_i}^2 + (1-r_j^2)\sigma_{Y_j}^2) + (r_i\sigma_{Y_i} + r_j\sigma_{Y_j})\Phi^{-1}(v)} \left[e^{\sqrt{1-r_i^2}\sqrt{1-r_i^2}\sigma_{Y_i}\sigma_{Y_j}} - 1\right].$$

6. Numerical illustration: basket options

In this section, we start by observing some numerical results in case of basket options in the Black & Scholes model. This choice is motivated by the fact that such a basket option forms a typical example of a (discounted) stop-loss premium involving a sum of lognormals which is usually far from being comonotonic. Therefore the approach of conditioning on one variable as in Deelstra et al. (2004) does sometimes lead to lower bounds and upper bounds which do not always have the desired precision that one might need as an approximation. We show in this section that conditioning on two variables improves the results in a promising way. Note that our theory can also be applied to the pricing of Asian options but as the bounds found in Dhaene et al. (2002b) and in Vanmaele et al. (2002) by conditioning on one variable were already very precise, the improvements of conditioning on more variables can only be noticed by observing the sixth decimal, see Liinev (2003) for details.

Another reason for concentrating upon basket options is that the Rogers and Shi (1995) lower bound is not necessarily a comonotonic sum as needed in paragraph 2.3 and therefore our approach of approximating the price by a comonotonic upper bound of a non-comonotonic lower bound turns out to be useful.

We continue with the example on basket options of Section 3.3. As concerns the conditioning variables $\sum_{j=1}^{n} \beta_j \sigma_j W_j(T)$, we consider three choices, namely two first order approximations of the sum $\sum_{i=1}^{n} a_i S_i(T)$ and the standardized logarithm of the geometric average, with corresponding integration bound b_Λ:

$$
\begin{aligned}
FA1 &= \sum_{j=1}^{n} e^{(r-\frac{\sigma_j^2}{2})T} a_j S_j(0)\sigma_j W_j(T) & b_{FA1} &= K - \sum_{j=1}^{n} a_j S_j(0) e^{(r-\frac{\sigma_j^2}{2})T}, \\
FA2 &= \sum_{j=1}^{n} a_j S_j(0)\sigma_j W_j(T) & b_{FA2} &= K - \sum_{j=1}^{n} a_j S_j(0)(1+(r-\frac{\sigma_j^2}{2})T), \\
GA &= \frac{\sum_{j=1}^{n} a_j \sigma_j W_j(T)}{\sqrt{\sum_{j=1}^{n}\sum_{k=1}^{n} a_j a_k \sigma_j \sigma_k \rho_{jk} T}} & b_{GA} &= \frac{\ln(K) - \sum_{j=1}^{n} a_j \left(\ln(S_j(0)) + (r-\frac{\sigma_j^2}{2})T\right)}{\sqrt{\sum_{j=1}^{n}\sum_{k=1}^{n} a_j a_k \sigma_j \sigma_k \rho_{jk} T}}.
\end{aligned}
\tag{67}
$$

We introduce the following notations where Λ can be *FA*1, *FA*2, *GA*: LBΛ for lower bound, PECUBΛ for partially exact/comonotonic upper bound, and UBΛ for an upper bound which equals a lower bound plus an error term based on Λ. See Deelstra et al. (2004) for the introduction of these definitions and a motivation of the choices of the conditioning variables. In the following, not all numerical results for all possible (combinations of) conditioning variables are reported but they are available from the authors.

The first set of input data is taken from Brigo et al. (2003), which was also used for illustration in Deelstra et al. (2004). Here we consider two assets with weights 0.5956 and 0.4044, and spot prices of 26.3 and 42.03, respectively. Maturity is equal to 5 years. The discount factor at payoff is 0.783895779. This example refers to a realistic basket, for which we allow the volatilities and correlations of individual assets to vary in order to facilitate the comparative price analysis.

TABLE 1: Comparing bounds, different weights and spot prices, different volatilities

In Table 1, we compare the lower bound LB2 obtained by conditioning on two variables with some values LB*opt* and UB from Deelstra et al. (2004) found by conditioning on one variable. The optimal comonotonic lower bound LB*opt* is the solution to an optimization procedure over all conditioning variables of the form (60). UB is the best upper bound obtained in these settings, as discussed in Deelstra et al. (2004). The lower bound LB2 is based upon a first conditioning variable $\Lambda_1 = e^{(r-\frac{\sigma_1^2}{2})T} a_1 S_1(0)\sigma_1 W_1(T) - e^{(r-\frac{\sigma_2^2}{2})T} a_2 S_2(0)\sigma_2 W_2(T)$ and a second conditioning variable Λ_2, being one of the three choices in (60) which all lead to the same result when rounded to 8 digits. We could not use $\Lambda_1 = FA1$ itself (or one of the other conditioning variables of (60)) since then the comonotonicity condition (46) is not satisfied. One clearly sees that the lower bound LB2 obtained by conditioning on two variables significantly improves the optimal lower bound LBopt and is very close to the Monte Carlo values. Only for fairly low volatility and very high correlation ($\rho_{12} = 0.99$), the LBopt is already so precise that one can no further improve.

As one might have the false impression that the amazing improvement of conditioning on two variables is only due to the fact that the basket option contains only just two underlying assets, we concentrate upon a second data set which has been taken from Milevsky and Posner (1998) (see also Beißer 2001) which is an average over seven stock indices. Indeed, the underlying asset of the basket option is the weighted average of the normalized G-7 stock indices

$$A(T) = \sum_{i=1}^{7} \tilde{a}_i \frac{S_i(T)}{S_i(0)},$$

where the influencing parameters are given in Table 2 and Table 3.

TABLE 2: G-7 index-linked guaranteed investment certificate weightings

TABLE 3: Correlation structure

The risk-free interest rate r equals 6.3% and we compute bounds and approximations to the value of the option for four different maturity dates (half a year, one, five and ten years). We use the conditioning variables in (60) with $a_j = \frac{\tilde{a}_j}{S_j(0)}$. As a consequence the correlations r_i in (49) in case of $FA2$ and GA coincide but the integration bounds still differ. This implies that both the lower bound and the improved comonotonic upper bound are indifferent to the choice of $FA2$ or GA, which is however not the case for the partially exact/comonotonic upper bound.

TABLE 4: Comparing bounds for 100 basket options of the G-7 stock indices

Table 4 mentions the best lower bound and upper bound when conditioning on one of our conditioning variables (67), namely the lower bound based upon the first order approximation $FA1$ and the upper bound based upon the lower bound in case of the standardized logarithm of the geometric average GA (see Deelstra et al. (2004)).

We further include the best lower bound LB2 in case of conditioning upon two conditioning variables. The first conditioning variable is one of our traditional conditioning variables (67) where the different terms in the signs are multiplied by the signs in the 7-tuple: for example $FA1(1,\ldots,1,-1)$ is the conditioning variable

$$FA1(1,\ldots,1,-1) = \sum_{j=1}^{6} \beta_j \sigma_j W_j(T) - \beta_7 \sigma_7 W_7(T)$$

with $\beta_j = e^{(r-\frac{\sigma_j^2}{2})T} a_j S_j(0)$. The second conditioning variable is just one of the unadapted announced conditioning variables (67). The adaptation of the signs of the β_j for the first conditioning variable is necessary in order to obtain a comonotonic sum. The easiest way to find a possible combination of plus- and minus- signs is to check the comonotonicity condition (46) by a simple numerical routine. However, an intuitive reasoning can explain why especially the two last terms need an adaptation in this case. Indeed, we have noticed that the terms with the most important weight profit from a different sign. In this example this is the case for the last two terms that have weights of respectively 20% and 25%. Clearly, the lower bound LB2 improves also in this case the lower bound based on one variable and can compete with the Monte Carlo estimates and this by using only a few seconds of computation time.

In Table 5, we compare different convex approximations for the G-7 basket. These convex approximations are based on the method of Vyncke et al. (2004), applied to basket options and modified as explained in Section 5. We use different notations for the convex approximations like LBPECUBGA which means that both the lower bound and the partially exact/comonotonic upper bound are based on the standardized logarithm of the geometric average GA. We state in a first column the results using $z(\lambda)$ as in (65) and (66), and in a second column the values found when discounting formula (64) which calls for z^u from (62). In comparison with the Monte Carlo estimates, the first case seems to underestimate a little bit, whereas the second case rather overestimates.

TABLE 5: Comparing convex approximations for 100 basket options of the G-7 stock indices

In Table 5, we further present under the names LBGACUB and LBICUBGA the results which are both based on the lower bound and then respectively upon the comonotonic upper bound and the improved comonotonic upper bound, all by using the standardized logarithm of the geometric average. One easily notices that both our convex approximations by using the partially exact/comonotonic upper bound improve in a significant way the convex combinations LB*GA*CUB and LBICUB*GA* adapted from Vyncke et al. (2004).

As expected, the idea of using a moment matching method to instruments like basket options where the underlying sum is far from comonotonic, leads to approximations which might be useful in comparison with a lower bound based on one conditioning variable. Especially for small horizons, the approximations are fairly good. This is due to the fact that for maturities up to 5 years, the exact part represents a very important part of the option price. The lower bounds based on two conditioning variables, however, are much more precise (even for large maturities) and have the advantage that they are known to be lower bounds. This is in contrast with the convex combination approximation which can deliver either smaller or larger values than the true price.

This basket option leads as in the previous data set to a comonotonic lower bound and therefore the technique of Section 4 cannot be applied. To illustrate the 'Comonotonic upper bound on a non-comonotonic lower bound', we consider a third example taken from Deelstra et al. (2004), namely with weights $a_1 = 0.3$ and $a_2 = 0.7$, spot prices of 130 and 70 and volatilities 0.2 and 0.3, respectively. The correlation among the assets equals $\rho_{12} = -0.7$. Maturity is equal to 1 year and the risk-free interest rate is 0.05. The strike prices in Table 6 are chosen in such a way that the option moneyness ranges from 10% in-the-money to 10% out-of-the-money.

In this example, the lower bound cannot be calculated by the usual comonotonicity formula since the sum of conditional expectations is not a comonotonic sum. Therefore, one is forced to apply a numerical recipe to calculate

$$E[(\mathbb{S}^{\ell} - b)_+] = \text{exact part } \Lambda + \int_{-\infty}^{b_\Lambda} \left(\sum_{i=1}^{n} E[X_i \mid \Lambda = \lambda] - b \right)_+ dF_\Lambda(\lambda),$$

which is an adaptation of a formula of Dhaene et al. (2002a). Indeed, we have split off the exact part (63) such that we only need a numerical integration of the second part.

TABLE 6: Case of a non-comonotonic lower bound

In Table 6, the non-comonotonic LB1 are calculated according to these formulae and the best result for $K = 83.26$ is obtained by using the conditioning variable $FA1$, respectively $FA2$ for $K = 92.51$ and GA for $K = 101.76$. The optimization program explained in Deelstra et al. (2004) leads to the optimal comonotonic lower bound LBopt when restricting to one conditioning variable of the form (60). LB2 shows the lower bounds based upon two conditioning variables, where the first conditioning variable takes the form $FA1(1, -1)$ while the second one can be either $FA1$, $FA2$ or GA since the results will only differ in the 14th decimal. 'ICUB after LB$FA1$' is the comonotonic upper bound based upon the non-comonotonic lower bound. LB1PECUBGA denotes the convex combination of the lower bound LB1 and PECUBGA according to (64) by using z^u from (62). Clearly, our conditioning methods on two variables lead to the best results.

7. Conclusion

In this paper, we studied methods of conditioning on two variables when valuing stop-loss premiums of a sum of dependent random variables. In particular, we derived analytical expressions for the comonotonic bounds of these stop-loss premiums. We applied our methods to the case of lognormally distributed random variables. Especially the lower bound leads to a very useful result.

Confronted with the inconvenience that for some sums it is cumbersome to obtain a comonotonic lower bound for the stop-loss premium, we used a comonotonic upper bound of the non-comonotonic lower bound as an approximation.

We further adapted the method of Vyncke et al. (2004) to the case that the sum of dependent random variables is far from being comonotonic, and this by studying convex approximations based on a lower bound and the partially exact/comonotonic upper bound from Deelstra et al. (2004).

In this paper we concentrated mainly upon basket option evaluation. The lower bounds based on two variables lead to very sharp results — even for large underlying baskets. Even when the underlying assets in the basket have a negative correlation, the quality of the lower bound based on two variables remains splendid. In this difficult case when a comonotonic lower bound is not straightforward to obtain, we propose the comonotonic upper bound of the non-comonotonic lower bound as a very satisfying approximation for the basket option price. The only disadvantage is that we no longer know whether it is a lower or an upper bound and therefore it is difficult to judge which conditioning variables lead to the best approximation.

Further, our numerical results show that our convex approximations improve the convex combination methodology of Vyncke et al. (2004) in case of the basket options. The convex approximations being no specific bounds, however, we recommend to use the comonotonic lower bound based on two variables. We further conclude that in the case of basket option pricing, which is an example of a stop-loss premium of a sum of variables that is far from being comonotonic, both our methods based upon more conditioning variables and our improved convex approximations lead to significant improvements in comparison with the existing techniques.

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TABLE 1: Comparing bounds, different weights and spot prices, different volatilities

σ_1	σ_2	corr	MC	SE	LB*opt*	LB2	UB
0.1	0.3	0.2	26.2824	0.002923	26.2380	26.2840	26.7612
		0.6	27.4454	0.003296	27.4356	27.4470	27.7353
		0.99	28.5055	0.003681	28.5084	28.5084	28.5213
0.3	0.6	0.2	39.8587	0.011675	38.9640	39.8740	42.6078
		0.6	42.5568	0.013208	42.3318	42.5685	43.9641
		0.99	45.1953	0.013762	45.1926	45.1931	45.2273

TABLE 2: G-7 index-linked guaranteed investment certificate weightings

country	index	weight (in %)	volatility (in %)	dividend yield (in %)
Canada	TSE 100	10	11.55	1.69
Germany	DAX	15	14.53	1.36
France	CAC 40	15	20.68	2.39
U.K.	FSTE 100	10	14.62	3.62
Italy	MIB 30	5	17.99	1.92
Japan	Nikkei 225	20	15.59	0.81
U.S.	S&P 500	25	15.68	1.66

TABLE 3: Correlation structure

	Canada	Germany	France	U.K.	Italy	Japan	U.S.
Canada	1.00	0.35	0.10	0.27	0.04	0.17	0.71
Germany	0.35	1.00	0.39	0.27	0.50	−0.08	0.15
France	0.10	0.39	1.00	0.53	0.70	−0.23	0.09
U.K.	0.27	0.27	0.53	1.00	0.45	−0.22	0.32
Italy	0.04	0.50	0.70	0.46	1.00	−0.29	0.13
Japan	0.17	−0.08	−0.23	−0.22	−0.29	1.00	−0.03
U.S.	0.71	0.15	0.09	0.32	0.13	−0.03	1.00

Table 4: Comparing bounds for 100 basket options of the G-7 stock indices

T	K	MC	SE	LB$FA1$	UBGA	LB2	Λ_1/Λ_2
0.5	0.95	7.37	0.001	7.3705	7.3938	7.3727	$FA2(1,\ldots,1,-1)/FA1$
	1	3.64	0.001	3.6351	3.7052	3.6354	$FA2(1,\ldots,1,-1)/FA1$
	1.05	1.34	0.001	1.3359	1.4616	1.3362	$FA2(1,\ldots,1,-1)/FA2$
1	0.95	9.56	0.002	9.5569	9.6144	9.5592	$FA2(1,\ldots,1,-1)/FA1$
	1	5.90	0.002	5.8938	6.0181	5.8957	$FA2(1,\ldots,-1,1)/FA1$
	1.05	3.19	0.002	3.1869	3.3909	3.1892	$FA2(1,\ldots,-1,1)/FA1$
5	0.95	22.75	0.011	22.7274	22.9426	22.7413	$FA1(1,\ldots,-1,1)/FA1$
	1	19.53	0.011	19.4680	19.7928	19.4894	$FA1(1,\ldots,-1,1)/FA1$
	1.05	16.47	0.012	16.4210	16.8772	16.4500	$FA1(1,\ldots,-1,1)/FA1$
10	0.95	33.64	0.022	33.6223	33.8701	33.6461	$FA2(1,\ldots,-1,1)/FA1$
	1	31.09	0.022	31.0761	31.4232	31.1103	$FA2(1,\ldots,-1,1)/FA1$
	1.05	28.62	0.022	28.5872	29.0530	28.6335	$FA2(1,\ldots,-1,1)/FA1$

Table 5: Comparing convex approximations for 100 basket options of the G-7 stock indices

T	K	MC	SE	LBPECUBGA		LBGACUB	LBICUBGA
				$z(\lambda)$	z^u		
0.5	0.95	7.37	0.001	7.3709	7.3720	7.3769	7.3769
	1	3.64	0.001	3.6364	3.6395	3.6446	3.6447
	1.05	1.34	0.001	1.3377	1.3424	1.3456	1.3457
1	0.95	9.56	0.002	9.5585	9.5621	9.5755	9.5755
	1	5.90	0.002	5.8969	5.9046	5.9187	5.9189
	1.05	3.19	0.002	3.1916	3.2034	3.2148	3.2151
5	0.95	22.75	0.011	22.7379	22.7583	22.8512	22.8508
	1	19.52	0.011	19.4838	19.5171	19.6205	19.6210
	1.05	16.47	0.012	16.4434	16.4926	16.6019	16.6031
10	0.95	33.64	0.022	33.6379	33.6543	33.7969	33.7953
	1	31.09	0.022	31.0971	31.1239	31.2868	31.2858
	1.05	28.62	0.022	28.6149	28.6554	28.8361	28.8361

Table 6: Case of a non-comonotonic lower bound

K	MC(SE$\times 10^3$)	LB1	LBopt	LB2	ICUBGA after LB$FA1$	LB1PECUBGA z^u	PECUBGA
83.26	9.78671(1.612)	9.67734	9.71631	9.78122	9.79000	9.72526	10.6577
92.51	4.37694(1.736)	4.30100	4.30109	4.37695	4.43781	4.36086	5.53664
101.76	1.71031(1.321)	1.65567	1.68154	1.71349	1.71978	1.77196	2.59509