

BOUNDS FOR THE PRICE OF A EUROPEAN-STYLE ASIAN OPTION IN A BINARY TREE MODEL

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Abstract– Inspired by the ideas of Rogers and Shi (1995), Chalasani, Jha & Varikooty (1998) derived accurate lower and upper bounds for the price of a European-style Asian option with continuous averaging over the full lifetime of the option, using a discrete-time binary tree model. In this paper, we consider arithmetic Asian options with discrete sampling and we generalize their method to the case of forward starting Asian options. In this case with daily time steps, the method of Chalasani et al. is still very accurate but the computation can take a very long time on a PC when the number of steps in the binomial tree is high. We derive analytical lower and upper bounds based on the approach of Kaas, Dhaene & Goovaerts (2000) for bounds for stop-loss premiums of sums of dependent random variables, and by conditioning on the value of the underlying at the exercise date. The comonotonic upper bound corresponds to an optimal superhedging strategy. By putting in less information than Chalasani et al. the bounds lose some accuracy but are still very good and they are easily computable and moreover the computation on a PC is fast. We illustrate our results by different numerical experiments and compare with bounds for the Black & Scholes model (1973) found in another paper Vanmaele et al. (2002). We notice that the intervals of Chalasani et al. do not always lie within the Black & Scholes intervals. We have proved that our bounds converge to the corresponding bounds in the Black & Scholes model. Our numerical illustrations also show that the hedging error is small if the Asian option is in the money. If the option is out of the money, the price of the superhedging strategy is not as adequate, but still lower than the straightforward hedge of buying one European option with the same exercise price.

Keywords– Comonotonicity, Asian options, superhedging strategy.

1. INTRODUCTION

The binomial tree model of the Cox-Ross-Rubenstein (CRR) (1979) model can be considered as a discrete-time version of the Black & Scholes (B&S) (1973) model. The European option prices have well-known formulae in this model. For example when the life time T of the option is divided into N time steps of length T/N , we have for a European call option:

$$EC(K, T) = e^{-rT} \sum_{j=0}^N \binom{N}{j} p^j (1-p)^{N-j} (S(0)u^j d^{N-j} - K)_+, \quad (1)$$

where r is the risk-free rate of interest, $S(t)$ is the price of the underlying asset at time t , p the risk-neutral probability that the price goes up with a factor $u = 1/d = \exp(\sigma\sqrt{T/N})$ with σ the volatility of the underlying. It is known that $p = \frac{e^{rT/N} - d}{u - d}$. Although in the

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CRR-model, American option pricing turns out easily numerically (see e.g. Hull (1989)), the pricing of Asian options remains an open question. Chalasani et al. (1998) have worked out a numerical recipe to obtain a very accurate price-interval for the Asian option price in the CRR-model. Their method is based upon grouping paths with the same geometric stock-price average that end at the same stock price. This is realised by conditioning on some well chosen vector $Z = (X(T), \chi(T))$, where $X(T)$ denotes the number of increases until time T and $\chi(T)$ is defined by $\chi(T) = \sum_{j=0}^N \sum_{\ell=1}^j X(\ell T/N)$. Denoting by $AV(T)$ the average stock price at time T in the CRR-model with N time steps, i.e.

$$AV(T) = \frac{1}{N} \sum_{j=0}^N S(0) u^{X(jT/N)} d^{N-X(jT/N)} \quad (2)$$

it follows by Jensen's inequality and from Rogers and Shi (1995) that:

$$E[(E[AV(T) | Z] - K)_+] \leq E[(AV(T) - K)_+] \leq E[(E[AV(T) | Z] - K)_+] + \varepsilon \quad (3)$$

where

$$\varepsilon = \frac{1}{2} E \left[\sqrt{\text{Var} [AV(T) | Z]} \right]. \quad (4)$$

Multiplying by the discount factor e^{-rT} relation (3) prevails a lower and an upper bound for the Asian option price. Chalasani et al. stipulate that for N large enough (say 30 to 40) those bounds give a precise price-interval for the Asian option in the Black and Scholes framework with a continuous averaging over the whole lifetime of the option.

In what follows we focus on European-style Asian options with discrete sampling which are forward starting, i.e. the averaging has not yet started at the beginning of the lifetime of the option and takes only the daily prices into account during the time interval $[T - n + 1, T]$. Recalling that $X(T - i)$ stands for the number of increases until time $T - i$, $i = 0, 1, \dots, n - 1$, we denote this arithmetic average in the CRR-model with time step one day as

$$\frac{1}{n} \mathbb{S} = \frac{1}{n} \sum_{i=0}^{n-1} S(0) u^{X(T-i)} d^{T-X(T-i)} \quad (5)$$

and the price of the Asian option is given by

$$AC(n, K, T) = e^{-rT} E \left[\left(\frac{1}{n} \mathbb{S} - K \right)_+ \right], \quad (6)$$

where we work under the risk-neutral probability $p = \frac{e^r - d}{u - d}$. The method of Chalasani et al. (1998) can easily be generalized to this case of forward starting Asian options and is very accurate for the CRR-model. However, the computations of the bounds are very time consuming on a PC in particular when the number of steps in the binomial tree is high. Note that the number of time steps cannot be chosen freely since it is at least equal to the number of sampling days of the lifetime of the option, for example 60 or 120 days. Thus when we average say for example over the last 10 days, we have to keep track of the whole tree of the underlying asset for the whole lifetime of the option.

The paper is composed as follows. In section 2 lower and upper bounds for the price of arithmetic Asian options with discrete sampling are derived. In section 3 the convergence of those bounds to the corresponding bounds in the Black & Scholes setting is considered. In section 4 we comment a numerical example, with special attention to the speed of the calculations on PC. Section 5 concludes.

2. BOUNDS FOR THE PRICE OF ARITHMETIC ASIAN OPTIONS WITH DISCRETE SAMPLING

2.1 COMONOTONIC BOUNDS AND BOUNDS WITH CONDITIONING ON THE FINAL VALUE OF THE UNDERLYING

2.1.1 LOWER BOUND

In order to obtain a lower bound in the binomial case, we condition on the value of the asset price at the final time T , that is equivalent to conditioning on the number $X(T)$ of increases of the price process until maturity date T :

$$\begin{aligned} AC(n, K, T) &\geq \frac{e^{-rT}}{n} E[(E[\mathbb{S} \mid X(T)] - nK)_+] \\ &= \frac{e^{-rT}}{n} \sum_{j=0}^T \left(\sum_{i=0}^{n-1} E[S(T-i) \mid X(T) = j] - nK \right)_+ \cdot P[X(T) = j]. \end{aligned} \quad (7)$$

Since $\ln S(T-i)/S(0) = X(T-i) \ln(\frac{u}{d}) + (T-i) \ln d$ with $X(T-i)$ a binomial random variable with parameters $(T-i)$ and p ,

$$E[S(T-i) \mid X(T) = j] = \sum_{\ell=\max(0, j-i)}^{\min(j, T-i)} S(0) u^\ell d^{(T-i)-\ell} \frac{\binom{T-i}{\ell} \binom{i}{j-\ell}}{\binom{T}{j}} \quad (8)$$

and the lower bound is found to be:

$$\frac{e^{-rT}}{n} \sum_{j=0}^T \binom{T}{j} p^j (1-p)^{T-j} \left(\sum_{i=0}^{n-1} \sum_{\ell=\max(0, j-i)}^{\min(j, T-i)} S(0) u^\ell d^{T-i-\ell} \frac{\binom{T-i}{\ell} \binom{i}{j-\ell}}{\binom{T}{j}} - nK \right)_+. \quad (9)$$

2.1.2 ROGERS AND SHI UPPER BOUND

An upper bound can be deduced by applying the approximation error (4) to the case of the average $\frac{1}{n}\mathbb{S}$ with $X(T)$ as conditioning variable:

$$E[\text{Var}^{1/2}[\frac{1}{n}\mathbb{S} \mid X(T)]] = \sum_{j=0}^T \text{Var}^{1/2}[\frac{1}{n}\mathbb{S} \mid X(T) = j] P[X(T) = j] \quad (10)$$

where

$$\text{Var}[\frac{1}{n}\mathbb{S} \mid X(T) = j] = E\left[\left(\frac{1}{n}\mathbb{S}\right)^2 \mid X(T) = j\right] - E[\frac{1}{n}\mathbb{S} \mid X(T) = j]^2. \quad (11)$$

The second term on the right-hand side can be obtained from (8) while the first term can be expressed as

$$\begin{aligned} E\left[\left(\frac{1}{n}\mathbb{S}\right)^2 \mid X(T) = j\right] &= \frac{1}{n^2} \sum_{i=0}^{n-1} E[S^2(T-i) \mid X(T) = j] \\ &\quad + \frac{2}{n^2} \sum_{i=0}^{n-1} \sum_{k=i+1}^{n-1} E[S(T-i)S(T-k) \mid X(T) = j], \end{aligned} \quad (12)$$

where the first term in the last expression is obtained analogously as in (8):

$$E[S^2(T-i) | X(T) = j] = \sum_{\ell=\max(0, j-i)}^{\min(j, T-i)} S(0)^2 u^{2\ell} d^{2(T-i)-2\ell} \frac{\binom{T-i}{\ell} \binom{i}{j-\ell}}{\binom{T}{j}} \quad (13)$$

and where the second term equals

$$E[S(T-i)S(T-k) | X(T) = j] \\ = \sum_{s=\max(j-k, 0)}^{\min(j, T-k)} \sum_{t=\max(j-i-s, 0)}^{\min(T-i-s, j-s, k-i)} S(0)^2 u^{2s+t} d^{2T-(i+k)-(2s+t)} \frac{\binom{k}{s} \binom{k-i}{t} \binom{j-(s+t)}{i}}{\binom{T}{j}}. \quad (14)$$

Due to the fact that for some values of the conditioning variable $X(T)$, some terms vanish in the difference

$$E\left[\left(\frac{1}{n}\mathbb{S} - K\right)_+ | X(T)\right] - (E\left[\frac{1}{n}\mathbb{S} | X(T)\right] - K)_+,$$

and thus can be omitted, one obtains an improved approximation error dependent on K :

$$\varepsilon(K) = \sum_{\substack{j=0 \\ S_{min} < K \wedge S_{max, j} > K}}^T \text{Var}^{1/2}\left[\frac{1}{n}\mathbb{S} | X(T) = j\right] P[X(T) = j], \quad (15)$$

where

$$S_{min} = \frac{S(0)}{n} \sum_{i=0}^{n-1} d^{T-i} \quad \text{and} \quad S_{max, j} = S_{min} u^{2j}, \quad (16)$$

leading to an improved upper bound for $AC(n, K, T)$ consisting of the lower bound (9) plus the approximating error $\varepsilon(K)$ in (15). This idea is in fact a natural one and Nielsen and Sandmann (2003) used a similar idea in the B&S-setting.

2.1.3 INVERSE DISTRIBUTION FUNCTIONS

Let X be a random vector with cumulative distribution function (cdf hereafter) F_X . Then the usual inverse function of F_X is the non-decreasing and left-continuous function $F_X^{-1}(q)$, defined by

$$F_X^{-1}(q) = \inf \{x \in \mathbb{R} \mid F_X(x) \geq q\}, \quad q \in [0, 1], \quad (17)$$

with $\inf \emptyset = +\infty$ by convention; for any $\alpha \in [0, 1]$, the (non-decreasing) α -mixed inverse function of F_X is defined as follows:

$$F_X^{-1(\alpha)}(q) = \alpha F_X^{-1}(q) + (1 - \alpha) F_X^{-1+}(q), \quad q \in (0, 1), \quad (18)$$

with the non-decreasing and right-continuous function $F_X^{-1+}(p)$ defined as

$$F_X^{-1+}(q) = \sup \{x \in \mathbb{R} \mid F_X(x) \leq q\}, \quad q \in [0, 1]. \quad (19)$$

2.1.4 COMONOTONIC UPPER BOUND

In financial and actuarial situations one encounters quite often random variables of the type $\mathbb{S} = \sum_{i=0}^{n-1} Y_i$ where the terms Y_i are not mutually independent, but the multivariate distribution function of the random vector $\underline{Y} = (Y_0, Y_1, \dots, Y_{n-1})$ is not completely specified because one only knows the marginal distribution functions of the random variables Y_i . Kaas et al. (2000) use comonotonic counterparts to derive bounds for such sums.

The comonotonic counterpart \mathbb{S}^c of \mathbb{S} with $\mathbb{S} = \sum_{i=0}^{n-1} S(T-i)$, is the random variable

$$\mathbb{S}^c = \sum_{i=0}^{n-1} F_{S(T-i)}^{-1}(U) \quad U \stackrel{d}{=} (0, 1)\text{-uniform.} \quad (20)$$

The reasoning of Simon et al. (2000) leads to the fact that the price of an Asian option can be bounded above by an optimal linear combination of European vanilla call options:

$$AC(n, K, T) \leq \frac{1}{n} \sum_{i=0}^{n-1} e^{-ri} EC \left[F_{S(T-i)}^{-1(\alpha)}(F_{\mathbb{S}^c}(nK)), T-i \right] \quad (21)$$

where $EC(\cdot, \cdot)$ is defined in (1) and where α is determined by

$$F_{\mathbb{S}^c}^{-1(\alpha)}(F_{\mathbb{S}^c}(nK)) = nK. \quad (22)$$

From (22) and the geometric interpretation (see Dhaene et al. (2001)) of stop-loss premiums, one obtains that for any stochastic process $(S(t); t \geq 0)$ for the risky asset, the following inequality holds

$$\begin{aligned} AC(n, K, T) &\leq \frac{e^{-rT}}{n} \sum_{i=0}^{n-1} E \left[\left(S(T-i) - F_{S(T-i)}^{-1}(F_{\mathbb{S}^c}(nK)) \right)_+ \right] \\ &\quad - \frac{e^{-rT}}{n} [nK - F_{\mathbb{S}^c}^{-1}(F_{\mathbb{S}^c}(nK))] (1 - F_{\mathbb{S}^c}(nK)). \end{aligned} \quad (23)$$

We apply this inequality in the case of the binary tree model where

$$F(j, T-i) = P[S(T-i) \leq S(0)u^j d^{T-i-j}] = \sum_{\ell=0}^j \binom{T-i}{j} p^\ell (1-p)^{T-i-\ell}, \quad (24)$$

for $i = 0, \dots, n-1$. Hence, for $0 < q \leq 1$, we find that

$$F_{S(T-i)}^{-1}(q) = \sum_{j=0}^{T-i} S(0)u^j d^{T-i-j} I_{F(j-1, T-i) < q \leq F(j, T-i)}, \quad (25)$$

where I denotes the indicator function, and therefore

$$\begin{aligned} F_{\mathbb{S}^c}(nK) &= \sup \left\{ q \in (0, 1] \mid \sum_{i=0}^{n-1} F_{S(T-i)}^{-1}(q) \leq nK \right\} \\ &= \sup \left\{ q \in (0, 1] \mid \sum_{i=0}^{n-1} \sum_{j=0}^{T-i} u^j d^{T-i-j} I_{F(j-1, T-i) < q \leq F(j, T-i)} \leq \frac{nK}{S(0)} \right\}. \end{aligned} \quad (26)$$

Further, we remark that due to comonotonicity (see Kaas et al. (2000))

$$\begin{aligned} F_{\mathbb{S}^c}^{-1}(F_{\mathbb{S}^c}(nK)) &= \sum_{i=0}^{n-1} F_{S(T-i)}^{-1}(F_{\mathbb{S}^c}(nK)) \\ &= \sum_{i=0}^{n-1} \sum_{j=0}^{T-i} S(0)u^j d^{T-i-j} I_{F(j-1, T-i) < F_{\mathbb{S}^c}(nK) \leq F(j, T-i)}. \end{aligned} \quad (27)$$

Clearly, $F_{\mathbb{S}^c}(nK)$ will be one of the cumulative distribution functions $F(j, T-i)$. Therefore, we first concentrate on the ordering among them for different j and i .

We consider the cdf-values $F(j, T-i)$ as elements F_{ji} , $j = 0, \dots, T-i$, $i = 0, \dots, n-1$, in a $((T+1) \times n)$ -matrix \mathbf{F} . Then only the elements F_{ji} with $j \leq T-i$ are taken into account. It is obvious that $F_{ji} = 1$ for $j = T-i$ and that $F_{j'i} > F_{ji}$ for $j' > j$. One can show that also $F_{ji'} > F_{ji}$ for $i' > i$ and that

$$F_{ji} = (1-p)F_{j,i+1} + pF_{j-1,i+1} \implies F_{j-1,i+1} < F_{ji} < F_{j,i+1}. \quad (28)$$

In other words, walking through rows in one column of the matrix \mathbf{F} , the value of the cdf increases. The same happens when walking through columns in one row. This implies that given a cdf-value $F(j, T-i)$ there are precisely two elements in each column such that $F(j, T-i)$ lies between these elements. Therefore, the double sum in (26) reduces to a sum with at most n terms. In fact the problem in (26) turns down to determine by an algorithm the largest $q = F(j(i), T-i)$ such that

$$\sum_{i=0}^{n-1} u^{j(i)} d^{T-i-j(i)} \leq nK/S(0). \quad (29)$$

Note that at the same time, the sum in (25) reduces to one term $u^{j(i)} d^{T-i-j(i)}$. Substituting the computed relations (25)-(26)-(27) into (23), we obtain a value for the upper bound.

As mentioned in Albrecher et al. (2003), this comonotonic upper bound (21) corresponds to the price of a static superhedging strategy consisting of buying and holding European call options on the same underlying asset, and that mature at the respective exercise dates.

Our numerical illustration will show that the hedging error is small if the Asian option is in the money. If the option is out of the money, the price of the superhedging strategy is not adequate, but still lower than the straightforward hedge of buying one European option with the same exercise date. The comonotonic hedge is useful because European call options are typically available on the market.

Moreover, among all superhedging strategies which are constructed as linear combinations of European calls on the same underlying asset, the comonotonic superhedge is optimal in the sense that it is the one with the smallest cost.

2.2 IMPROVED COMONOTONIC UPPER BOUND

We assume from now on that $\sum_{\ell=a}^b x_\ell = 0$ for $a > b$. Let

$$\mathbb{S}^u = \sum_{i=1}^n F_{S(T-i)|X(T)}^{-1}(U) \quad U \stackrel{d}{=} (0,1)\text{-uniform}. \quad (30)$$

then we obtain an improved comonotonic upper bound, analogously as in (21)-(23):

$$\begin{aligned}
& AC(n, K, T) \\
& \leq \frac{e^{-rT}}{n} \sum_{j=0}^T P[X(T) = j] \cdot \left\{ \sum_{i=0}^{n-1} E\left[\left(S(T-i) - F_{S(T-i)|X(T)=j}^{-1} \left(F_{\mathbb{S}^u|X(T)=j}(nK) \right) \right)_+ \mid X(T) = j \right] \right. \\
& \quad \left. - \left(nK - F_{\mathbb{S}^u|X(T)=j}^{-1} \left(F_{\mathbb{S}^u|X(T)=j}(nK) \right) \right) \left(1 - F_{\mathbb{S}^u|X(T)=j}(nK) \right) \right\} \\
& = \frac{e^{-rT}}{n} \sum_{j=0}^T \binom{T}{j} p^j (1-p)^{T-j} \times \\
& \quad \times \left\{ \sum_{i=0}^{n-1} \sum_{\ell=\max(0, j-i)}^{\min(j, T-i)} \frac{\binom{T-i}{\ell} \binom{i}{j-\ell}}{\binom{T}{j}} \left(S(0) u^\ell d^{T-i-\ell} - F_{S(T-i)|X(T)=j}^{-1} \left(F_{\mathbb{S}^u|X(T)=j}(nK) \right) \right)_+ \right. \\
& \quad \left. - \left(nK - F_{\mathbb{S}^u|X(T)=j}^{-1} \left(F_{\mathbb{S}^u|X(T)=j}(nK) \right) \right) \left(1 - F_{\mathbb{S}^u|X(T)=j}(nK) \right) \right\} \tag{31}
\end{aligned}$$

By the comonotonicity property, we have that

$$F_{\mathbb{S}^u|X(T)=j}^{-1}(q) = \sum_{i=0}^{n-1} F_{S(T-i)|X(T)=j}^{-1}(q) \quad 0 < q \leq 1 \tag{32}$$

with

$$F_{S(T-i)|X(T)=j}^{-1}(q) = \sum_{s=\max(0, j-i)}^{\min(j, T-i)} S(0) u^s d^{T-i-s} I_{F_j(s-1, T-i) < q \leq F_j(s, T-i)}, \tag{33}$$

where

$$F_j(s, T-i) = \begin{cases} 0 & s < \max(0, j-i) \\ \sum_{\ell=\max(0, j-i)}^s \frac{\binom{T-i}{\ell} \binom{i}{j-\ell}}{\binom{T}{j}} & \max(0, j-i) \leq s < \min(j, T-i) \\ 1 & s \geq \min(j, T-i). \end{cases} \tag{34}$$

Hence, the cdf of \mathbb{S}^u given the event $X(T) = j$ can be found from

$$\begin{aligned}
& F_{\mathbb{S}^u|X(T)=j}(nK) \\
& = \sup \left\{ q \in (0, 1] \mid \sum_{i=0}^{n-1} \sum_{s=\max(0, j-1)}^{\min(j, T-i)} u^s d^{T-i-s} I_{F_j(s-1, T-i) < q \leq F_j(s, T-i)} \leq \frac{nK}{S(0)} \right\}. \tag{35}
\end{aligned}$$

As for the comonotonic upper bound, $F_{\mathbb{S}^u|X(T)=j}$ will be one of the distribution functions $F_j(s, T-i)$ or will be zero in case of an empty set. The cdf's $F_j(s, T-i)$ can also be ordered in a matrix which has similar properties as for the comonotonic upper bound. Thus, a cdf-value $F_j(s, T-i)$ lies precisely between two elements in each column and the sum (33) reduces to one term and hence the double sum (32) to at most n terms. Once we found for each fixed j , the largest $q = F_j(s, T-i)$ by a search algorithm, we can easily compute (33), (32) and hence obtain the value of the improved comonotonic upper bound.

3. CONVERGENCE

When we consider steps with length equal to a fraction of one day, say $1/m$ day, and let m go to infinity, we can prove that the lower bound and the comonotonic upper and improved comonotonic upper bound converge to the corresponding bounds for a European-style arithmetic forward starting Asian option with discrete sampling in the Black & Scholes setting, i.e. the underlying asset follows a geometric Brownian motion, see Vanmaele et al. (2002). The proof follows the ideas of convergence of the CRR-model to the B&S-model.

4. A NUMERICAL EXAMPLE

We illustrate our results by some numerical examples. We consider the case of an European-style arithmetic forward starting Asian option with discrete sampling that in practice is used for protection against price manipulation. For this purpose, averaging over the last 10 days of the life time of the option is sufficient. Intuitively it is clear that this is a case where conditioning on the final value of the underlying asset will perform well. When one takes a longer averaging period into account, the weight of the final value diminishes and hence also the quality of the bounds. This can be seen by comparing the results for an expiration time $T = 120$ (days) but with different averaging periods n , namely $n = 10$ in table 1 and $n = 30$ in table 3. In table 2 we report for the case of an expiration time $T = 60$ (days) and $n = 10$ averaging days. In all cases, the risk-free interest rate r equals 9% yearly. The initial stock price $S(0)$ is fixed at 100. Further, we consider several exercise prices K in the range of 90 to 110 by steps of 5 or 10. We also study the influence of different volatilities σ ($=0.2, 0.3$ and 0.4) on the quality of the bounds. From tables 1-3 we observe the following:

1. The method of Chalasani is accurate for the CRR-model since the lower (LBC) and upper bound (UBC) coincide.
2. The lower bound (LB) with conditioning on the final value of the underlying asset together with the comonotonic upper bound (CUB) or with the improved comonotonic upper bound (ICUB) gives a price-interval which contains the bound $LBC=UBC$.
3. The Rogers and Shi approach for the upper bound ($LB + \varepsilon(K)$) is of very bad quality in the sense that it leads to very large upper bounds that in all cases are larger than CUB and ICUB.
4. The relative error $(ICUB-LB)/LB$ increases with increasing exercise price K which implies that the bounds are better for options in-the-money than out-of-the-money.
5. The absolute difference as well as the relative difference between the value LBC and LB increases with volatility or with , increasing number of averaging days for a fixed T or with decreasing life time of the option for a fixed n .
6. Chalasani et al. (1998) remark that 'For $N = 30$ (resp. $N = 40$) our algorithm computes the lower and upper bounds in about 5 (resp. 20) seconds on a Sun UltraSPARC workstation'. Speed is indeed very important for traders who have to make their decisions in a few seconds. But normally they don't use a Sun UltraSPARC but a PC. Note moreover that for the forward starting case the number of steps N in the binomial tree equals T which is 60 or 120 in our example (while n , the number of averaging dates, equals 10 or 30). This implies a higher CPU-time with their approach than for the case $T = N = 30$ or 40 they reported. It turns out that in the forward starting case the calculation time of our bounds on a Pentium III with 128Mb Ram is much shorter. For example the lower bound is very fast, it needed around 0.35 CPU-time seconds when $T = 60$ and $n = 10$, up to 4.6 CPU-time seconds when $T = 120$ and $n = 30$. The calculation time of the superhedging strategy equals 0.6 ($T = 60$ and $n = 10$) to 2.9 ($T = 120$ and $n = 30$) CPU-time seconds.

Table 1: Bounds when $T = 120$ and $n = 10$.

T	n	σ	K	LB	LBC = UBC	LBBS	UBBS	ICUB	CUB	LB + $\varepsilon(K)$
120	10	0.2	90	13.077	13.081	13.100	13.101	13.106	13.113	13.894
			95	9.120	9.123	9.135	9.154	9.163	9.176	9.866
			100	5.872	5.891	5.931	5.932	5.940	5.960	6.538
			105	3.498	3.502	3.544	3.546	3.566	3.571	4.000
			110	1.906	1.911	1.950	1.953	1.958	1.972	2.283
120	10	0.3	90	14.350	14.353	14.395	14.397	14.408	14.424	15.500
			95	10.894	10.902	10.954	10.967	10.972	10.992	11.939
			100	7.984	8.012	8.072	8.076	8.087	8.118	8.960
			105	5.695	5.701	5.762	5.767	5.782	5.807	6.508
			110	3.909	3.927	3.988	3.994	4.007	4.029	4.632
120	10	0.4	90	15.956	15.963	16.027	16.031	16.048	16.074	17.421
			95	12.825	12.836	12.909	12.915	12.934	12.959	14.211
			100	10.115	10.152	10.232	10.239	10.252	10.294	11.409
			105	7.885	7.905	7.987	7.995	8.016	8.045	9.074
			110	6.054	6.064	6.146	6.155	6.171	6.198	7.129

Table 2: Bounds when $T = 60$ and $n = 10$.

T	n	σ	K	LB	LBC = UBC	LBBS	UBBS	ICUB	CUB	LB + $\varepsilon(K)$
60	10	0.2	95	7.064	7.076	7.107	7.108	7.120	7.135	7.824
			100	3.685	3.723	3.767	3.768	3.780	3.810	4.353
			105	1.593	1.607	1.650	1.653	1.666	1.685	2.035
60	10	0.3	95	8.191	8.219	8.275	8.277	8.299	8.326	9.260
			100	5.146	5.203	5.269	5.273	5.290	5.336	6.134
			105	3.010	3.035	3.102	3.107	3.129	3.163	3.778
60	10	0.4	95	9.476	9.508	9.587	9.592	9.622	9.657	10.898
			100	6.615	6.690	6.778	6.785	6.806	6.868	7.925
			105	4.469	4.514	4.603	4.612	4.643	4.685	5.643

Table 3: Bounds when $T = 120$ and $n = 30$.

T	n	σ	K	LB	LBC = UBC	LBBS	UBBS	ICUB	CUB	LB + $\varepsilon(K)$
120	30	0.2	90	12.693	12.725	12.760	12.762	12.784	12.801	14.137
			100	5.359	5.443	5.522	5.526	5.574	5.615	6.742
			110	1.514	1.581	1.653	1.662	1.697	1.728	2.639
120	30	0.3	90	13.762	13.843	13.925	13.930	13.981	14.019	15.907
			100	7.288	7.414	7.535	7.546	7.614	7.678	9.349
			110	3.300	3.400	3.518	3.535	3.595	3.649	5.158
120	30	0.4	90	15.166	15.297	15.424	15.436	15.512	15.571	18.024
			100	9.236	9.403	9.564	9.584	9.671	9.756	11.978
			110	5.214	5.355	5.518	5.546	5.627	5.701	7.795

In the tables, we included also a price interval $[LBBS, UBBS]$ for the corresponding Asian option in the Black & Scholes setting, see Vanmaele et al. (2002). We may conclude that:

1. The price interval $[LB, ICUB]$ in the CRR-model contains the price interval $[LBBS, UBBS]$ and hence contains not only the price of the Asian option in the CRR-model but also in the B&S-model.
2. The method of Chalasani leads to an accurate price in the CRR-model but not in the B&S-model. It gives only a lower bound in the latter case.

5. CONCLUSIONS

In this paper, we have concentrated on the CRR-model with daily time step and we have derived price intervals for arithmetic Asian options with discrete sampling which are forward starting. Our method is based on comonotonicity and on conditioning on the final value of the underlying asset. Moreover, it can be shown that our upper and lower bounds in the CRR-model converge to those of the B&S-model.

We also generalized the method of Chalasani et al. (1998) to this setting. Their method turns out to be still very accurate but the computations are very time consuming. We further noticed that the Chalasani et al. results form a lower bound for the price of an arithmetic Asian option with discrete sampling in the B&S-model, but that this lower bound is of a lesser quality than the B&S lower bounds found in Vanmaele et al. (2002). Moreover, among all superhedging strategies which are constructed as linear combinations of European calls on the same underlying asset and with exercise dates corresponding to the averaging dates, the comonotonic superhedge is optimal in the sense that it is the one with the smallest cost.

As a conclusion, we can suggest that if the continuous B&S-model is an acceptable model, then one could better work with the B&S-bounds, see Vanmaele et al. (2002). If one prefers to work in the CRR-model itself for pricing arithmetic Asian options with discrete sampling which are forward starting, then it turns out that on a PC the calculation time of our bounds is much shorter than the one of the bounds based on the (generalized) method of Chalasani et al. and is therefore recommended.

An interesting question is to look at American-style Asian options in the CRR-model.

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