Default probabilities of a holding company, with complete and partial information.

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Abstract

This paper studies the valuation of credit risk for firms that own several subsidiaries or business lines. We provide simple analytical approximating expressions for probabilities of default, and for equity-debt market values, both in the case when the information is available in continuous time as well as in the case that it is not instantaneously available. The total firm's asset value being modeled as a sum of lognormal random variables, we use convex upper and lower approximations to infer these analytical approximating expressions. We extend the model to firms financed by multiple stochastic liabilities and conclude by numerical illustrations.

Keywords. default risk, structural model, incomplete information, convex ordering, comonotonicity.

1 Introduction.

The treatment of default is a crucial issue in determining the value of corporate securities and the firm's financing decisions. This task is particularly complex when the corporation is itself a group of subsidiaries that have dependent activities. Structural models such as developed by Merton (1974) and Black and Cox (1976) represent an elegant framework for the valuation of risky debts, when assets are modeled by a single Brownian motion. Since, many alternatives have been developed to replace the Brownian motion by more complex dynamics. Recently Fiorani (2010), Ballotta and Fusai (2013) and Hainaut and Colwell (2013) used Lévy, multivariate Lévy and switching Lévy processes. But, unto our knowledge, there are only very few extensions to multi-industry firms.

Moreover, most of the existing models assume that the dynamics of firm's assets are continuously observed while in practice, the information needed to assess efficiently the financial health is for most of the companies only released at discrete times. As emphasized by Duffie and Lando (2001), ignoring this aspect leads to an underestimation of short-term credit spreads.

The purpose of this work is hence twofold. Firstly, this paper proposes simple approximating formulas to appraise default probabilities, risky debts and equity for multi-industry firms. The total firm's asset is a sum of lognormal processes, each one corresponding to a firm's subsidiary. The statistical distribution of firm's total asset value exhibits then more leptokurticity and asymmetry than a single lognormal variable. Secondly, it studies the impact of a lack of information on these quantities. The framework of our model is partly inspired from papers of Leland (1994, 1998) and of Leland and Toft (1996). We assume that the default or simply the restructuring events are triggered when the total market value of all subsidiaries falls below a certain threshold. Two cases are considered. In the first one, this threshold is constant. It can eventually be regulatory imposed, or chosen by the firm's management. In the second case, the threshold is random and the sum of several liabilities. This approach is particularly well adapted for insurance companies that finance their investments by e.g. life or non life provisions.

The solution that we propose is based on the concept of convex orders and comonotonicity, which were introduced by Hoefdding (1940) and Frechet (1951) who studied lower and upper bounds for multivariate cumulative distributions. This theory became popular amongst researchers in actuarial sciences over the last two decades and has been applied successfully to various fields of research. Dhaene et al. (2002 (a), (b)) proposed in their review comonotone upper and lower approximations in the convex order sense for the sum of a finite number of random variables. The work of Vanduffel et al. (2003) reveals that the lower convex bound approximation is extremely accurate for an appropriate choice of parameters. We refer the interested reader to Denuit et al. (2005) for characterizations of convex orders. We further notice that convex orders and stop loss premiums are closely related (see e.g. Dhaene and Goovaerts (1996)). Comonotone bounds have been applied from derivatives pricing (Vanmaele et al. (2006)) to insurance (Ahcan et al. (2006)), including risk management, as in Van Weert et al. (2012). For a recent survey of applications in finance and insurance, we refer e.g. to Deelstra et al. (2011). But we did not encounter any applications of this theory to the valuation of credit risk. Our work tries to fill this gap.

The outline of this paper is as follows. The first section introduces the framework that we adopt to model a multi activity firm. In section 3, we build the convex bounds of the total firm's asset and infer in section 4 approximating formulae for the probabilities of default. In section 5 and 6, we respectively appraise the value of debts with complete and incomplete information. In section 7, the model is adapted to stochastic liabilities. Section 8 contains several numerical applications and we conclude our work in section 9.

2 The model.

We consider a holding company, composed of N subsidiaries or various business lines. Each subsidiary generates a stream of dividends or cash-flows distributed in its entirety to the parent company. The investment in the subsidiary is assumed irreversible, at least till an eventual re-structuring of the holding. Dividends are defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{F}$ is the filtration generated by M independent Brownian motions, denoted by $\tilde{W}^j$ for $j = 1...M$. The dividend provided by the i^{th} subsidiary is assumed to be a stochastic process denoted by F_t^i and has the following dynamics under the real measure $\mathbb{P}$:

$$\frac{dF_t^i}{F_t^i} = \mu_i dt + \sum_{j=1}^{M} \sigma_{i,j} d\tilde{W}_t^j \quad \forall i = 1, \ldots, N. \tag{2.1}$$

We denote Σ the $N \times M$ matrix of $(\sigma_{i,j})_{i=1...N,\, j=1...M}$ which are such that $rank(\Sigma) = M$. The covariance matrix containing the covariances between the flows of dividends is then equal to $\Sigma\Sigma^\top$. We also assume that there exists a risk free asset, such as a bank account, that provides a constant rate of return r. The total flow of dividends paid to the holding company is denoted by

$$F_t = \sum_{i=1}^{N} F_t^i.$$

Obviously, cash-flow processes are not tradeable assets. However, as the subsidiary is a separate entity, the entire value of this subsidiary can be seen as a traded asset. In this case, the value of the i^{th} subsidiary is equal to the expected sum of cash-flows discounted at the cost of the equity

r_E with $r_E > \mu_i$:

$$\begin{aligned} S_t^i &= \mathbb{E}\left(\int_t^{\infty} e^{-r_E(s-t)} F_s^i ds \,|\, \mathcal{F}_t\right) \\ &= \int_t^{\infty} e^{-r_E(s-t)} \mathbb{E}\left(F_t^i e^{\left(\mu_i - \sum_{j=1}^M \frac{\sigma_{i,j}^2}{2}\right)(s-t) + \sum_{j=1}^M \sigma_{i,j}(W_s^j - W_t^j)} ds \,|\, \mathcal{F}_t\right) \\ &= \frac{F_t^i}{r_E - \mu_i}. \end{aligned} \tag{2.2}$$

This last formula is similar to the Gordon-Shapiro formula as first exposed by Gordon and Myron (1959). The rate μ_i can be seen as the growth rate of dividends distributed by the i^{th} business line. Note that the expectation in equation (2.2) is calculated under the real measure $\mathbb{P}$, and this is the reason why the discount rate is given by the cost of equity and not by the risk free rate. The dynamics of S_t^i can be rewritten as

$$dS_t^i = \frac{dF_t^i}{r_E - \mu_i} = \mu_i S_t^i dt + S_t^i \sum_{j=1}^M \sigma_{i,j} d\tilde{W}_t^j. \tag{2.3}$$

Let us denote by μ, the vector of μ_i and by κ, the vector of market risk premiums κ_j for $\tilde{W}_t^i$. If $\mathbf{1}_N$ is a N vector of ones, κ is a solution (not necessary unique) of $\mu = \mathbf{1_N} r + \Sigma\kappa$. Then the risk neutral $\mathbb{Q}$ is defined by the following Radon Nikodym derivative

$$\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_t = \exp\left(-\frac{1}{2}\int_0^t \kappa'\kappa ds - \int_0^t \kappa' d\tilde{W}_s\right). \tag{2.4}$$

Under this risk neutral measure $\mathbb{Q}$ chosen by market participants, the drifts of the dynamics (2.3), are equal to the risk free rate, r, to ensure the absence of arbitrage. More precisely, we have that

$$dS_t^i = r S_t^i dt + S_t^i \sum_{j=1}^M \sigma_{i,j} dW_t^j \tag{2.5}$$

where $dW_t^j = d\tilde{W}_t^j + \kappa_j dt$ are here M independent Brownian motions defined on $(\Omega, \mathcal{F}, \mathbb{Q})$. As mentioned in e.g. Musiela and Rutkowski (1998) or Dhaene et al. (2013), the risk neutral measure $\mathbb{Q}$ is unique if $M = N = rank(\Sigma)$. Note that all following developments are done under the measure $\mathbb{Q}$. The market value of the holding at time t is denoted by $S_t = \sum_{i=1}^N S_t^i$ and its initial value is equal to the sum of S_0^i for $i = 1, \ldots, N$. In our framework, the total market value S_t of all subsidiaries is a sum of lognormal random variables and is therefore no more distributed as a lognormal. The investor partly finances his investment by its capital and issues debts to profit from the tax shield offered for interest expenses. The tax rate $\theta \in (0, 1)$ is assumed constant over time. In the first part of this work, the debt is modeled as a consol bond. This approach is well suited to fit the liabilities structure of most non financial corporations, that systematically renew their loans for tax purposes. The investor pays continuously and perpetually a constant coupon C. The tax benefit is then θC. The debt is issued at time 0 for some amount D. In section 7, we will assume that the holding is financed by stochastic liabilities. This may be used to model financial conglomerates such as insurance companies or banks that have several different uncertain liabilities.

We assume that the equity owner liquidates or restructures the holding when the total value of assets falls below a predetermined value denoted α, usually less than the accounting value of debts or a floor imposed by the regulator. As we will discuss later, α can also be chosen by the management so as to maximize the market value of the equity. In this case, α is a parameter of control. The default time is an $\mathcal{F}_t$ stopping time, denoted by τ. The liquidation value of the holding is $S_\tau = \sum_{i=1}^N S_\tau^i$ and is assigned to the debt holders. In this framework, the market value of the equity, denoted by E_0^α, is equal to the sum of the expected discounted cash-flows under the risk neutral measure, decreased by the cost of debts:

$$E_0^\alpha = \mathbb{E}^{\mathbb{Q}} \left(\int_0^\tau e^{-rt} \left(F_t - (1-\theta)C \right) dt \right). \tag{2.6}$$

We assume that in case of bankruptcy, holding shareholders do not receive any income from the sale of the assets. The difference between the total cash-flow and the coupon paid may be seen as a dividend, that can be positive or negative if the cash-flows are insufficient to pay debts. According to the relation (2.2) and given that $S_\tau = \alpha$, the first term in this last expression is equivalent to

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}} \left(\int_0^\tau e^{-rt} F_t dt \right) &= \sum_{i=1}^N \mathbb{E}^{\mathbb{Q}} \left(\int_0^\tau e^{-rt} F_t^i dt \right) \\ &= \sum_{i=1}^N \left(\mathbb{E}^{\mathbb{Q}} \left(\int_0^\infty e^{-rt} F_t^i dt \right) - \mathbb{E}^{\mathbb{Q}} \left(e^{-r\tau} \int_\tau^\infty e^{-r(t-\tau)} F_t^i dt \right) \right) \\ &= S_0 - \mathbb{E}^{\mathbb{Q}} \left(e^{-r\tau} \alpha \right). \end{aligned} \tag{2.7}$$

The market value of the equity can be rewritten as

$$\begin{aligned} E_0^\alpha &= \mathbb{E}^{\mathbb{Q}} \left(\int_0^\tau e^{-rt} F_t dt \right) - \mathbb{E}^{\mathbb{Q}} \left(\int_0^\tau e^{-rt} (1-\theta) C dt \right) \\ &= S_0 - \mathbb{E}^{\mathbb{Q}} \left(e^{-r\tau} \alpha + \int_0^\tau e^{-rt} (1-\theta) C dt \right), \end{aligned} \tag{2.8}$$

while the market value of the debt at time 0 is equal to the following expectation:

$$\begin{aligned} D_0^\alpha &= \mathbb{E}^{\mathbb{Q}} \left(\int_0^\tau e^{-rt} (1-\theta) C \; dt + e^{-r\tau} S_\tau \right) \\ &= \mathbb{E}^{\mathbb{Q}} \left(\int_0^\tau e^{-rt} (1-\theta) C \; dt + e^{-r\tau} \alpha \right). \end{aligned} \tag{2.9}$$

To our knowledge, these expressions of equity and debt market values do not admit any analytical solution in this framework, except if there is only one subsidiary. In the following sections, we develop lower and upper convex approximations for the sum $\sum_{i=1}^N S_t^i$, and infer lower and upper estimates of default probabilities of the holding company.

In reality, the financial information needed to assess the financial health of the holding is not necessarily available in continuous time. In particular, if the holding is not listed, the financial statements that are issued quarterly form the only available information. This remark motivates the second part of this work, in which we explore the impact of this lack of information on the estimates of default probabilities.

3 Convex bounds of S_t.

In this section, we briefly review results related to comonotone convex upper and lower bounds for a sum of lognormal variables. For details and proofs, we refer to Dhaene et al. (2002 a,b). As mentioned earlier, the sum S_t of market values of subsidiaries at time t is the sum of N lognormal variables:

$$S_t \stackrel{d}{=} \sum_{i=1}^N S_0^i e^{Z_t^i}. \tag{3.1}$$

Under the risk neutral measure, $Z_t = \left(Z_t^1, \dots, Z_t^N\right)$ is a Gaussian random vector with distribution $N(\mu^Z t\,,\, \Sigma\Sigma^\top\, t)$ where the mean vector is given by

$$\mu^Z = \begin{pmatrix} r - \frac{1}{2} e_1^\top \Sigma\Sigma^\top e_1 \\ \vdots \\ r - \frac{1}{2} e_N^\top \Sigma\Sigma^\top e_N \end{pmatrix} \tag{3.2}$$

where e_i is the i^{th} unit root vector of $\mathbb{R}^N$. To simplify further calculations, the variance of Z_t^i is defined as

$$Var(Z_t^i) = \left(\sigma_i^Z\right)^2 t = e_i^\top \Sigma\Sigma^\top e_i\, t. \tag{3.3}$$

The theory of comonotonicity is closely related to the concept of convex order. A r.v. X is said to precede a r.v. Y in the convex order sense if and only if for all convex functions $u(.)$, we have $\mathbb{E}(u(X)) \leq \mathbb{E}(u(Y))$, provided the expectations exist. This relation is denoted by $X \leq_{cx} Y$. It has been proven that $X \leq_{cx} Y$ if and only if the stop loss premiums satisfy the relation $\mathbb{E}\left(X-d\right)_+ \leq \mathbb{E}\left(Y-d\right)_+$, for all levels of retention d, and if $\mathbb{E}[X] = \mathbb{E}[Y]$. The following proposition allows us to build convex bounds for the total market value of subsidiaries.

Proposition 3.1. *Consider the conditioning process Λ_t defined as the weighted sum of processes Z_t^i*

$$\Lambda_t \;=\; \sum_{i=1}^{N} \gamma_i Z_t^i \tag{3.4}$$

where γ_i for $i = 1, ..., N$ are constant. Also consider processes defined by

$$S_t^{i,l} \;=\; S_0^i \exp\left(\left(\mu_i^Z + \frac{1}{2}(1-r_i^2)\left(\sigma_i^Z\right)^2\right) t + r_i\, \sigma_i^Z\, W_t^l\right) \tag{3.5}$$

$$S_t^{i,c} \;=\; S_0^i \exp\left(\mu_i^Z t + \sigma_i^Z\, W_t^c\right) \tag{3.6}$$

and their sums

$$S_t^l = \sum_{i=1}^{N} S_t^{i,l} \qquad\qquad S_t^c = \sum_{i=1}^{N} S_t^{i,c}$$

where W_t^l and W_t^c are independent Brownian motions such that $W_0^l = W_0^c = 0$. The coefficients r_i are constant and defined as follows

$$\begin{aligned} r_i \;&=\; \frac{cov(Z_t^i, \Lambda_t)}{\sqrt{Var(Z_t^i)}\,\sqrt{Var(\Lambda_t)}} \\ &=\; \frac{e_i^\top \Sigma\Sigma^\top \gamma}{\sqrt{e_i^\top \Sigma\Sigma^\top e_i}\sqrt{\gamma^\top \Sigma\Sigma^\top \gamma}}. \end{aligned} \tag{3.7}$$

Then, we have the following convex order relations:

$$S_t^l \leq_{cx} S_t \leq_{cx} S_t^c \tag{3.8}$$

In most applications, as mentioned in the work of Vanduffel et al. (2003), the lower bound approximation S_t^l is extremely accurate and can be used as a good proxy for S_t. Note that if $X \leq_{cx} Y$ then $Var(X) < Var(Y)$ must hold unless $X \stackrel{d}{=} Y$. The variances of S_t, S_t^i and S_t^c are given by the following expressions:

$$Var(S_t) \;=\; \sum_{i=1}^{N}\sum_{j=1}^{N} S_0^i S_0^j e^{\left(\mu_i^Z + \mu_j^Z + \frac{1}{2}\left(\left(\sigma_i^Z\right)^2 + \left(\sigma_j^Z\right)^2\right)\right)t}\left(e^{cov(Z_t^i, Z_t^j)} - 1\right), \tag{3.9}$$

$$Var(S_t^l) = \sum_{i=1}^{N}\sum_{j=1}^{N} S_0^i S_0^j e^{\left(\mu_i^Z+\mu_j^Z+\frac{1}{2}\left((\sigma_i^Z)^2+(\sigma_j^Z)^2\right)\right)t}\left(e^{r_i r_j \sigma_i^Z \sigma_j^Z t}-1\right), \tag{3.10}$$

$$Var(S_t^c) = \sum_{i=1}^{N}\sum_{j=1}^{N} S_0^i S_0^j e^{\left(\mu_i^Z+\mu_j^Z+\frac{1}{2}\left((\sigma_i^Z)^2+(\sigma_j^Z)^2\right)\right)t}\left(e^{\sigma_i^Z \sigma_j^Z t}-1\right). \tag{3.11}$$

Vanduffel et al. (2008) recommend weights $\gamma_i = S_0^i \mathbb{E}\left(e^{Z_t^i}\right)$ that maximize the variance $Var(S_t^l)$ over t years. But a finite time horizon and time-dependent γ_i's being not adapted to later developments, we look for γ_i such that S_t^l approximates well S_t at any time, and according a stop-loss distance. More precisely, as $S_t^l \leq_{cx} S_t$, we know from Kaas et al. (1994, p. 68) that

$$\int_{-\infty}^{+\infty} \mathbb{E}\left[(S_t-k)_+\right] - \mathbb{E}\left[\left(S_t^l-k\right)_+\right] dk = \frac{1}{2}\left(Var(S_t) - Var(S_t^l)\right).$$

Then $\frac{1}{2}\left(Var(S_t) - Var(S_t^l)\right)$ can be interpreted as a measure for the total error made when approximating the stop-loss premiums of S_t by those of the convex smaller S_t^l. If we adopt this measure, the best γ_i should then minimize the gap between variances

$$\gamma_i = argmin\left(Var(S_t) - Var(S_t^l)\right) \quad \forall t > 0.$$

But given that S_t and S_t^c are independent from γ_i, it is equivalent to

$$\begin{aligned}\gamma_i &= argmin\left(Var(S_t) + Var(S_t^c) - Var(S_t^c) - Var(S_t^l)\right)\\ &= argmin\left(Var(S_t^c) - Var(S_t^l)\right) \quad \forall t > 0.\end{aligned}$$

By definition (3.10) and (3.11) of $Var(S_t^c)$ and $Var(S_t^l)$

$$\gamma_i = argmin\sum_{i=1}^{N}\sum_{j=1}^{N} S_0^i S_0^j e^{\left(\mu_i^Z+\mu_j^Z+\frac{1}{2}\left((\sigma_i^Z)^2+(\sigma_j^Z)^2\right)\right)t}\left(e^{\sigma_i^Z \sigma_j^Z t}-e^{r_i r_j \sigma_i^Z \sigma_j^Z t}\right) \quad \forall t > 0,$$

and a first order Taylor approximation of exponential functions leads to

$$\gamma_i \approx argmin\sum_{i=1}^{N}\sum_{j=1}^{N} S_0^i S_0^j e^{\left(\mu_i^Z+\mu_j^Z+\frac{1}{2}\left((\sigma_i^Z)^2+(\sigma_j^Z)^2\right)\right)t}\left(1-r_i r_j\right)\sigma_i^Z \sigma_j^Z t \quad \forall t > 0.$$

As γ_i are time-independent, they ideally should cancel all spreads $(1 - r_i r_j)$. For this reason, we use in later developments the γ_i that minimizes the quadratic gap between $r_i r_j$ and 1:

$$\gamma_i = argmin\sum_{i=1}^{N}\sum_{j=1}^{N}\left(1-\frac{e_i^\top \Sigma\Sigma^\top \gamma\, e_j^\top \Sigma\Sigma^\top \gamma}{\sqrt{e_i^\top \Sigma\Sigma^\top e_i}\sqrt{e_j^\top \Sigma\Sigma^\top e_j}\,\gamma^\top \Sigma\Sigma^\top \gamma}\right)^2. \tag{3.12}$$

The accuracy of the convex approximations is tested in a numerical applications section concluding this work, see section 8.

4 Approximation of default probabilities.

Before assessing the market value of debt and equity, we build estimates of probabilities of bankruptcy. When the number of subsidiaries is small, the probability that the total asset breaches the floor α can always be computed by Monte-Carlo simulations. But once that this number is important, Monte-Carlo simulations require heavy calculations and furthermore, they have not the advantage of analytical tractability of closed or semi-closed formulas. In the following, we will always work (even without further mentioning) under the chosen risk-neutral probability $\mathbb{Q}$ and we will denote the expectations by $\mathbb{E}^{\mathbb{Q}}[.]$. However in order to avoid confusion with the notation for quantile functions $Q_p[.]$, we still will refer to the probability of an event A by $\mathbb{P}(A)$ although the probability used is the risk-neutral one.

In the remainder of this section, lower and upper bounds of S_t are respectively rewritten as functions $g^l(.,.)$ and $g^c(.,.)$ of time and of Brownian motions W_t^l and W_t^c:

$$\begin{aligned} S_t^l &= \sum_{i=1}^N S_0^i \exp\left(\left(\mu_i^Z + \frac{1}{2}(1-r_i^2)\left(\sigma_i^Z\right)^2\right)t + r_i\,\sigma_i^Z\,W_t^l\right) \\ &= g^l(t, W_t^l) \end{aligned} \tag{4.1}$$

$$\begin{aligned} S_t^c &= \sum_{i=1}^N S_0^i \exp\left(\mu_i^Z t + \sigma_i^Z\,W_t^c\right) \\ &= g^c(t, W_t^c). \end{aligned} \tag{4.2}$$

Functions $g^l(t,w)$ and $g^c(t,w)$ are differentiable with respect to w and then continuous. Furthermore as $\sigma_i^Z > 0$ for $i = 1...n$, the first order derivative of $g^c(t,w)$ with respect to w,

$$\frac{\partial g^c(t,w)}{\partial w} = \sum_{i=1}^N S_0^i \sigma_i^Z \exp\left(\mu_i^Z t + \sigma_i^Z\,w\right)$$

is positive and $g^c(t,w)$ is strictly increasing in w. Under the condition that all $r_i \geq 0$, the first order derivative of $g^l(t,w)$

$$\frac{\partial g^l(t,w)}{\partial w} = \sum_{i=1}^N S_0^i r_i \sigma_i^Z \exp\left(\left(\mu_i^Z + \frac{1}{2}(1-r_i^2)(\sigma_i^Z)^2\right)t + r_i\sigma_i^Z\,w\right)$$

is also positive and $g^l(t,w)$ is strictly increasing. If some r_i are negative, a positivity constraint ($r_i \geq 0$ for $i = 1...n$) should be added in the optimisation criterion (3.12). If this constraint is satisfied, $\left(\frac{\partial g^{c,l}}{\partial w}(t,w_1) - \frac{\partial g^{c,l}}{\partial w}(t,w_2)\right)(w_1 - w_2) \geq 0$, $\forall w_1, w_2 \in \mathbb{R}$, $g^l(t,w)$ and $g^c(t,w)$ are also convex in w. And their inverse functions are well defined. Furthermore, the p quantiles of S_t^l and S_t^c, respectively denoted by $Q_p[S_t^l]$ and $Q_p[S_t^c]$, are given by the following expressions (for a proof see Theorem 1 of Dhaene et al. (2002a)):

$$\begin{aligned} Q_p[S_t^l] &= g^l(t, Q_p[W_t^l]) \\ Q_p[S_t^c] &= g^c(t, Q_p[W_t^c]). \end{aligned}$$

If we use the notation Φ for the distribution function of a standard normal random variable, we can retrieve the distributions of the convex bounds as follows:

$$\begin{aligned} F_{S_t^l}(x) &= \Phi\left(\frac{\left(g^l\right)^{-1}(t,x)}{\sqrt{t}}\right) \\ F_{S_t^c}(x) &= \Phi\left(\frac{\left(g^c\right)^{-1}(t,x)}{\sqrt{t}}\right). \end{aligned}$$

Note that the inverses of the functions $g^{k=l,c}$ do not have a simple analytical expression. The holding company will go to bankruptcy when S_t hits the level α. The distribution of this hitting time is unknown. We would like to estimate the hitting times by using convex lower and upper bounds S_t^l and S_t^c:

$$\begin{aligned}\tau^{l,g^l} &= \inf\left\{t \geq 0 \,|\, S_t^l \leq \alpha,\, S_s^l \geq \alpha\, \forall s \leq t\right\} \\ &= \inf\left\{t \geq 0 \,|\, W_t^l \leq \left(g^l\right)^{-1}(t,\alpha),\, W_s^l \geq \left(g^l\right)^{-1}(s,\alpha)\, \forall s \leq t\right\} \end{aligned} \tag{4.3}$$

$$\begin{aligned}\tau^{c,g^c} &= \inf\left\{t \geq 0 \,|\, S_t^c \leq \alpha,\, S_s^c \geq \alpha\, \forall s \leq t\right\} \\ &= \inf\left\{t \geq 0 \,|\, W_t^c \leq \left(g^c\right)^{-1}(t,\alpha),\, W_s^c \geq \left(g^c\right)^{-1}(s,\alpha)\, \forall s \leq t\right\} \end{aligned} \tag{4.4}$$

As shown by Vanduffel (2005) in his PhD thesis, the lower comonotonic bound S_t^l is an excellent proxy for the original process S_t and we expect that default probabilities computed with S_t^l are also close to real ones. Numerical tests presented in section 8 (right graph of figure 8.1) tend to confirm this.

However, even if convex bounds allow us to reduce the dimension, we still face the problem that the hitting time of a non linear function of time by a Brownian motion does not admit any analytical expression, except in very few exceptions. A solution to preserve analytical tractability consists in approximating the function by a linear approximation with respect to time, of functions $\left(g^l\right)^{-1}(t,\alpha)$ and $\left(g^c\right)^{-1}(t,\alpha)$, as done by Schmidt and Nokinov (2008). We opt for this approach and use a first order Taylor development with respect to time as approximation method. Our choice is motivated by the fact that if $N = 1$, $\left(g^l\right)^{-1}(t,\alpha)$ and $\left(g^c\right)^{-1}(t,\alpha)$ are precisely linear function of time e.g. :

$$\left(g^c\right)^{-1}(t,\alpha) = \frac{1}{\sigma_1^Z}\ln\left(\frac{\alpha}{S_0^1}\right) - \frac{\mu_1^Z}{\sigma_1^Z}t \qquad N = 1$$

On another hand, numerical tests presented in section 8 (left graph of figure 8.1) confirm that a linear function of time is close to the original one. To lighten further calculations we denote:

$$f^l(t) = \left(g^l\right)^{-1}(t,\alpha).$$

Given that $g^l\left(t, f^l(t)\right) = \alpha$, it is easy to calculate that

$$\begin{aligned}\frac{\partial}{\partial t} g^l\left(t, f^l(t)\right) &= \sum_{i=1}^N S_0^i\left(\mu_i^Z + \frac{1}{2}(1-r_i^2)\left(\sigma_i^Z\right)^2\right)\exp\left(\left(\mu_i^Z + \frac{1}{2}(1-r_i^2)\left(\sigma_i^Z\right)^2\right)t + r_i\,\sigma_i^Z\, f^l(t)\right) \\ &\quad + \sum_{i=1}^N S_0^i\, r_i\,\sigma_i^Z \exp\left(\left(\mu_i^Z + \frac{1}{2}(1-r_i^2)\left(\sigma_i^Z\right)^2\right)t + r_i\,\sigma_i^Z\, f^l(t)\right)\frac{\partial}{\partial t} f^l(t) \\ &= 0\end{aligned}$$

and therefore we infer that the derivative of $f^l(t)$ with respect to time is equal to

$$\frac{\partial}{\partial t} f^l(t) = -\frac{\sum_{i=1}^N S_0^i\left(\mu_i^Z + \frac{1}{2}(1-r_i^2)\left(\sigma_i^Z\right)^2\right)\exp\left(\left(\mu_i^Z + \frac{1}{2}(1-r_i^2)\left(\sigma_i^Z\right)^2\right)t + r_i\,\sigma_i^Z\, f^l(t)\right)}{\sum_{i=1}^N S_0^i\, r_i\,\sigma_i^Z \exp\left(\left(\mu_i^Z + \frac{1}{2}(1-r_i^2)\left(\sigma_i^Z\right)^2\right)t + r_i\,\sigma_i^Z\, f^l(t)\right)}.$$

We choose a time t_0, calculate numerically the value $f^l(t_0)$ and develop $f(t)$ linearly around t_0:

$$f^l(t) = f^l(t_0) + (t - t_0)\frac{\partial}{\partial t} f^l(t)_{|t=t_0} + O(t^2).$$

The accuracy of this approximation is tested in the numerical applications section. We infer from this relation that

$$\left(g^l\right)^{-1}(t,\alpha) \approx \beta_1^l - \beta_2^l t, \tag{4.5}$$

where β_1^l and β_2^l are defined by:

$$\beta_2^l = \frac{\sum_{i=1}^N S_0^i \left(\mu_i^Z + \frac{1}{2}(1-r_i^2)\left(\sigma_i^Z\right)^2\right) \exp\left(\left(\mu_i^Z + \frac{1}{2}(1-r_i^2)\left(\sigma_i^Z\right)^2\right) t_0 + r_i\, \sigma_i^Z \, \left(g^l\right)^{-1}(t_0,\alpha)\right)}{\sum_{i=1}^N S_0^i r_i\, \sigma_i^Z \exp\left(\left(\mu_i^Z + \frac{1}{2}(1-r_i^2)\left(\sigma_i^Z\right)^2\right) t_0 + r_i\, \sigma_i^Z \, \left(g^l\right)^{-1}(t_0,\alpha)\right)}, \tag{4.6}$$

$$\beta_1^l = \left(g^l\right)^{-1}(t_0,\alpha) + \beta_2^l\, t_0. \tag{4.7}$$

In the same way, we get that

$$\left(g^c\right)^{-1}(t,\alpha) \approx \beta_1^c - \beta_2^c t, \tag{4.8}$$

where β_1^c and β_2^c are defined by:

$$\beta_2^c = \frac{\sum_{i=1}^N S_0^i \mu_i^Z \exp\left(\mu_i^Z t_0 + \sigma_i^Z \, \left(g^c\right)^{-1}(t_0,\alpha)\right)}{\sum_{i=1}^N S_0^i\, \sigma_i^Z \exp\left(\mu_i^Z t_0 + \sigma_i^Z \, \left(g^c\right)^{-1}(t_0,\alpha)\right)}, \tag{4.9}$$

$$\beta_1^c = \left(g^c\right)^{-1}(t_0,\alpha) + \beta_2^c\, t_0. \tag{4.10}$$

Note that β_1^l and β_1^c are negative by construction. Indeed, if this was not the case, the company would necessarily default at time 0. In most of the cases $\beta_2^{c,l}$ are positive but their signs depend on the signs of μ_i^Z. We will see at the end of this section, that the sign of $\beta_2^{c,l}$ has a serious impact on the asymptotic probability of ruin. But first, we define approximate default times of the convex bounds as follows:

$$\tau^l = \inf\left\{t \geq 0 \,|\, -\beta_1^l + \beta_2^l t + W_t^l \leq 0\,,\, -\beta_1^l + \beta_2^l s + W_s^l > 0\, \forall s \leq t\right\} \tag{4.11}$$

$$\tau^c = \inf\left\{t \geq 0 \,|\, -\beta_1^c + \beta_2^c t + W_t^c \leq 0\,,\, -\beta_1^c + \beta_2^c s + W_s^c > 0\, \forall s \leq t\right\}. \tag{4.12}$$

Given that we replace the frontiers $\left(g^l\right)^{-1}$ and $\left(g^c\right)^{-1}$ by their linear approximations, we get the following relations between approximate and exact hitting times of convex estimates:

$$\begin{aligned} &\mathbb{P}(\tau^{l,g^l} \leq t) \leq \mathbb{P}(\tau^l \leq t) \\ &\mathbb{P}(\tau^{c,g^c} \leq t) \leq \mathbb{P}(\tau^c \leq t). \end{aligned} \tag{4.13}$$

As mentioned earlier and in the work of Vanduffel et al. (2003), the lower bound approximation S_t^l is an accurate proxy for S_t. We could then expect that $\mathbb{P}(\tau^l \leq t)$ is a good proxy for the real probability of default $\mathbb{P}(\tau \leq t)$. Numerical applications concluding this work tend to confirm this.

It is well known that the hitting time of a Brownian motion with drift has an Inverse Gaussian (IG) distribution. From Bielecki and Rutkowski (2004 page 66), the probability that the holding goes to bankruptcy before time t is then approximated by the following expressions:

$$\mathbb{P}(\tau^k \leq t) = \Phi\left(h_1^k(t)\right) + e^{2\beta_2^k \beta_1^k}\, \Phi\left(h_2^k(t)\right) \quad \text{for } k = l \text{ or } c \tag{4.14}$$

$$h_1^k(t) = \frac{\beta_1^k - \beta_2^k t}{\sqrt{t}} \qquad h_2^k(t) = \frac{\beta_1^k + \beta_2^k t}{\sqrt{t}}$$

and the density of the default times can consequently by derived as follows for $k = l$ or c

$$\begin{aligned} d\mathbb{P}(\tau^k \leq t) &= \varphi\left(h_1^k(t)\right)\left(-\frac{1}{2}\beta_1^k t^{-\frac{3}{2}} - \frac{1}{2}\beta_2^k t^{-\frac{1}{2}}\right) \\ &\quad + e^{2\beta_2^k \beta_1^k}\, \varphi\left(h_2^k(t)\right)\left(-\frac{1}{2}\beta_1^k t^{-\frac{3}{2}} + \frac{1}{2}\beta_2^k t^{-\frac{1}{2}}\right), \end{aligned}$$

where $\varphi(.)$ denotes the density function of a standard normal distribution. Given that $\varphi\left(h_1^k(t)\right) = e^{2\beta_2^k\beta_1^k}\varphi\left(h_2^k(t)\right)$, the density of the default times is rewritten as

$$\begin{aligned} d\mathbb{P}(\tau^k \leq t) &= -\beta_1^k t^{-\frac{3}{2}}\varphi\left(h_1^k(t)\right) \\ &= \frac{-\beta_1^k}{\sqrt{2\pi t^3}}\exp\left(-\frac{1}{2}\frac{\left(\beta_1^k-\beta_2^k t\right)^2}{t}\right). \end{aligned} \tag{4.15}$$

As mentioned earlier, the sign of $\beta_2^{l,c}$ influences the long term ruin probability. Indeed, according to equation (4.14), if $\beta_2^{l,c}$ is positive (e.g. when all μ_i^Z are positive), the approximate probability of default over an infinite horizon is given by

$$\lim_{t\to\infty}\mathbb{P}(\tau^k \leq t) = e^{2\beta_2^k\beta_1^k} \quad \text{for } \ k=l \ \text{ or } \ c.$$

This asymptotic probability can be used as a measure of risk, instead of the Value at Risk (VaR) or Tail Value at Risk (TVaR), as done in actuarial sciences for insurance companies. If $\beta_2^{k=l,c}$ is negative, the asymptotic probability of ruin is equal to one and the holding will go to bankruptcy with certainty but the timing is unknown.

5 Valuation of debt and equity.

Approximate market values of debts are provided by the next proposition. These are obtained as the difference between a default free perpetuity and a payoff paid in case of default, weighted by a factor combining discount rate and default probabilities.

Proposition 5.1. *Under the assumption that default times are defined as in (4.11) and (4.12), the convex estimates of the market value of the debt are provided by the following expression:*

$$\begin{aligned} D_0^{\alpha,k} &= \mathbb{E}^{\mathbb{Q}}\left(\int_0^{\tau^k} e^{-rs}(1-\theta)C\,ds + e^{-r\tau^k}\alpha\right) \\ &= (1-\theta)C\frac{1}{r} + \left(\alpha-(1-\theta)C\frac{1}{r}\right)e^{\beta_1^k\left(\beta_2^k+\sqrt{2r+\left(\beta_2^k\right)^2}\right)} \quad \textit{for } \ k=l \ \textit{ or } \ c \end{aligned}$$

Proof. We prove this result for $k=l$. The market value of the debt is given by the following sum:

$$D_0^{\alpha,l} = \mathbb{E}^{\mathbb{Q}}\left(\int_0^{\tau^l} e^{-rs}(1-\theta)Cds\right) + \mathbb{E}^{\mathbb{Q}}\left(e^{-r\tau^l}\alpha\right). \tag{5.1}$$

Using Fubini's theorem, the first expectation can be rewritten as follows:

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}\left(\int_0^{\tau^l} e^{-rs}(1-\theta)Cds\right) &= \int_0^{+\infty}\int_0^t e^{-rs}(1-\theta)C\,ds\,d\mathbb{P}(\tau^l \leq t) \\ &= (1-\theta)C\int_0^{+\infty}\frac{1}{r}\left(1-e^{-rt}\right)d\mathbb{P}(\tau^l \leq t) \\ &= (1-\theta)C\frac{1}{r}\left(1-\int_0^{+\infty}e^{-rt}\,d\mathbb{P}(\tau^l \leq t)\right) \end{aligned} \tag{5.2}$$

and the second expectation in (5.1) can be developped as

$$\mathbb{E}^{\mathbb{Q}}\left(e^{-r\tau^l}\alpha\right) = \alpha\int_0^{+\infty}e^{-rt}\,d\mathbb{P}(\tau^l \leq t) \tag{5.3}$$

Equations (5.2) and (5.3) both depend on the integral $\int_0^{+\infty} e^{-rt}\, d\mathbb{P}(\tau^l \leq t)$, which is the moment generating function of τ^l:

$$\int_0^{+\infty} e^{-rt}\, d\mathbb{P}(\tau^l \leq t) = \int_0^{+\infty} \frac{-\beta_1^l}{\sqrt{2\pi t^3}} \exp\left(-\frac{1}{2}\frac{2rt^2 + \left(\beta_1^l - \beta_2^l t\right)^2}{t}\right) dt$$

which can be reformulated as

$$\int_0^{+\infty} e^{-rt}\, d\mathbb{P}(\tau^l \leq t) = e^{-\beta_1^l\left(\sqrt{2r+\left(\beta_2^l\right)^2}-\beta_2^l\right)} \int_0^{+\infty} \frac{-\beta_1^l}{\sqrt{2\pi t^3}} \exp\left(-\frac{1}{2}\frac{\left(\beta_1^l - \sqrt{2r + \left(\beta_2^l\right)^2} t\right)^2}{t}\right) dt.$$

But in view of (4.15), the integrand is the density of a hitting time, that we denote by $\tau^{m,l}$, such that $\mathbb{P}(\tau^{m,l} \leq \infty) = e^{2\beta_1^l\sqrt{2r+\left(\beta_2^l\right)^2}}$ and $\mathbb{P}(\tau^{m,l} \leq 0) = 0$. Consequently, we immediately infer that:

$$\begin{aligned}\int_0^{+\infty} e^{-rt}\, d\mathbb{P}(\tau^l \leq t) &= e^{-\beta_1^l\left(\sqrt{2r+\left(\beta_2^l\right)^2}-\beta_2^l\right)} e^{2\beta_1^l\sqrt{2r+\left(\beta_2^l\right)^2}}. \\ &= e^{\beta_1^l\left(\beta_2^l+\sqrt{2r+\left(\beta_2^l\right)^2}\right)}.\end{aligned}$$

□

Remark that if the holding company is incorporated at time $t = 0$, the market value of debts must in theory be equal to its accounting value D_0, that is the principal lent to the firm. If this is not the case, the transaction is not free of arbitrage. In this case, the (approximate) fair value of α that ensures the equality between market and accounting values of the debt is given by:

$$\alpha^k_{fair} = (1-\theta)C\frac{1}{r} + \left(D_0 - (1-\theta)C\frac{1}{r}\right) e^{-\beta_1^k\left(\beta_2^k+\sqrt{2r+\left(\beta_2^k\right)^2}\right)} \quad \text{for } k = l \text{ or } c.$$

This is the sum of a perpetual annuity and of a kind of option price related to the loss in case of default. In numerical applications ending this work, we illustrate the influence of the floor α on the debt market value.

Corollary 5.2. *Under the assumption that default times are defined as (4.11) and (4.12), the convex estimates of the market value of the equity are approximated by the difference between market values of assets and debts:*

$$\begin{aligned}E_0^{\alpha,k} &= S_0 - \mathbb{E}^{\mathbb{Q}}\left(e^{-r\tau}\alpha + \int_0^{\tau} e^{-rt}(1-\theta)C dt\right) \\ &= S_0 - (1-\theta)C\frac{1}{r} - \left(\alpha - (1-\theta)C\frac{1}{r}\right) e^{\beta_1^k\left(\beta_2^k+\sqrt{2r+\left(\beta_2^l\right)^2}\right)} \quad k = l \text{ or } c.\end{aligned}$$

Remark that our results can be used to infer approximations for credit default swap (CDS) premiums. A credit default swap is an insurance protecting the owner of a corporate bond issued by the holding, in case of default. CDS's are used for hedging and speculation purposes. In exchange of regular payments (named the premium leg), the buyer of the CDS receives the part of the bond principal which is not repaid in case of bankruptcy of the bond issuer. The payment done in case of default, is called the default leg. The premium paid for this insurance is usually expressed as a percentage of a unit bond principal. This percentage is called the CDS spread and we denote it by p. Premiums are paid at regular intervals of time, Δt, ranging from t_1 to t_n. The premium leg is equal to

$$
\begin{aligned}
Premium\,leg &= p\,\Delta t \sum_{t_i=t_1}^{t_n} e^{-r\,t_i}\mathbb{E}^{\mathbb{Q}}\left(I_{t_i<\tau}\,|\,\mathcal{F}_{t_0}\right) \\
&= p\,\Delta t \sum_{t_i=t_1}^{t_n} e^{-r\,t_i}\mathbb{P}(t_i<\tau) \qquad (5.4)
\end{aligned}
$$

If the bond issuer goes to bankrupcty, the CDS pays the difference between the principal and a constant recovery rate percentage, denoted by R. The default leg is then

$$
\begin{aligned}
Default\,leg &= (1-R)\sum_{t_i=t_1}^{t_n} e^{-r\,t_i}\mathbb{E}^{\mathbb{Q}}\left(I_{t_{i-1}<\tau<t_i}\,|\,\mathcal{F}_{t_0}\right) \\
&= (1-R)\sum_{t_i=t_1}^{t_n} e^{-r\,t_i}\mathbb{P}(t_{i-1}<\tau\le t_i). \qquad (5.5)
\end{aligned}
$$

By equating equations (5.4) and (5.5) and replacing the probabilities of default by their approximations, we get approximate estimates for the CDS spread:

$$
p^k = \frac{(1-R)}{\Delta t}\frac{\sum_{t_i=t_1}^{t_n} e^{-r\,t_i}\left(\mathbb{P}(\tau^k\le t_i)-\mathbb{P}(\tau^k\le t_{i-1})\right)}{\sum_{t_i=t_1}^{t_n} e^{-r\,t_i}\left(1-\mathbb{P}(\tau^k\le t_i)\right)} \qquad k=l \text{ or } c, \qquad (5.6)
$$

where $\mathbb{P}(\tau^k \le t_i)$ is provided by equation (4.14). The weakness of our model is the same as the one of Merton's structural model. Given that trajectories of assets are continuous and default is the first hitting time of a barrier, the default is a predictable stopping time. This leads to an underestimation of short term probabilities of default. This major flaw of the structural model has given rise to other approaches in credit risk modelling. One possibility consists in introducing jumps in the assets dynamics. Another approach is to introduce incompleteness about the available information. We have explored this alternative in the following section and inferred modified expressions for the probabilities of default, debts and equity.

6 Valuation of the debt with incomplete information.

If the holding is not listed or if the volume of stocks traded are too low to be considered as reliable, the information about the holding is only available at discrete times. Mainly when the financial statements are published or when the holding goes to bankruptcy. Even when the company is listed, the information is reported to the market with a certain delay. In this section, we assess the impact of the lack of information on default probabilities and on market values of debt and equity. We assume in the remainder of this section that the only information available at time t has been communicated at time 0 and that the holding is still active. The information carried by the filtration $(\mathcal{F}_t)_t$ is not accessible in continuous time. Let us denote by τ the time of default and by H_t the indicator variable $1_{t<\tau}$, equal to one if the issuer is still solvent. The information available to the market is represented by the filtration $\mathcal{G}_t = \sigma\left\{S_0^i, i=1,...,N;\, H_u, u\le t\right\}$. As before, we will denote the approximate default times defined in (4.11) and (4.12) by τ^k with $k=l$ or c, and then let denote H_t^k the indicator variable $1_{t<\tau^k}$ for $k=l$ or c. We define the corresponding filtrations by $\mathcal{G}_t^k = \sigma\left\{S_0^i, i=1,...,N;\, H_u^k, u\le t\right\}$.

We are interested in the probability (still under the risk neutral measure) at time $t>0$ that the holding company is still in activity at time T, which is given by

$$
\mathbb{P}\left(\tau>T\,|\,S_0^i, i=1,...,N,\, t<\tau\right) = \mathbb{P}\left(\tau^k>T\,|\,\mathcal{G}_t\right).
$$

The convex estimates of these survival probabilities are respectively provided by the following expressions:

$$\mathbb{P}\left(\tau^k > T \,|\, \mathcal{G}_t^k\right) = \frac{1 - \Phi\left(h_1^k(T)\right) - e^{2\beta_2^k \beta_1^k}\Phi\left(h_2^k(T)\right)}{1 - \Phi\left(h_1^k(t)\right) - e^{2\beta_2^k \beta_1^k}\Phi\left(h_2^k(t)\right)} \quad \text{for } k = l \text{ or } c \tag{6.1}$$

where

$$h_1^k(s) = \frac{\beta_1^k - \beta_2^k s}{\sqrt{s}} \qquad h_2^k(s) = \frac{\beta_1^k + \beta_2^k s}{\sqrt{s}} \quad \text{for } k = l \text{ or } c. \tag{6.2}$$

The density function of the survival time is then obtained by differentiating with respect to T:

$$\begin{aligned} d\mathbb{P}\left(\tau^k \leq T \,|\, \mathcal{G}_t^k\right) &= \frac{\frac{\partial}{\partial T}\Phi\left(h_1^k(T)\right) + e^{2\beta_2^k \beta_1^k}\frac{\partial}{\partial T}\Phi\left(h_2^k(T)\right)}{1 - \Phi\left(h_1^k(t)\right) - e^{2\beta_2^k \beta_1^k}\Phi\left(h_2^k(t)\right)} \\ &= \frac{\frac{-\beta_1^k}{\sqrt{2\pi T^3}}\exp\left(-\frac{1}{2}\frac{\left(-\beta_1^k + \beta_2^k T\right)^2}{T}\right)}{1 - \Phi\left(h_1^k(t)\right) - e^{2\beta_2^k \beta_1^k}\Phi\left(h_2^k(t)\right)} \quad \text{for } k = l \text{ or } c. \end{aligned} \tag{6.3}$$

As we will see in numerical applications, the delay in the information disclosure allows us to model non negligible probabilities of default over a short term period of time. Note that if β_1^k and β_2^k for $k = l$, c are respectively negative and positive (e.g. if all μ_i^Z are positive), the approximate probabilities of default over an infinite horizon are given by

$$\lim_{T\to\infty} \mathbb{P}(\tau^k \leq T \,|\, \mathcal{G}_t^k) = \frac{e^{2\beta_2^k \beta_1^k} - \Phi\left(h_1^k(t)\right) - e^{2\beta_2^k \beta_1^k}\Phi\left(h_2^k(t)\right)}{1 - \Phi\left(h_1^k(t)\right) - e^{2\beta_2^k \beta_1^k}\Phi\left(h_2^k(t)\right)} \quad \text{for } k = l \text{ or } c.$$

As mentioned earlier, this asymptotic measure of default can be used as substitute to other risk measures such as e.g. VaR or TVaR.

The approximate market values of debts are provided by the next proposition and may still be seen as the difference between a default free perpetuity and a modified cash-flow paid in case of bankruptcy.

Proposition 6.1. *The approximate values of the debt are equal to the following expressions, for $k = l$ or c :*

$$\begin{aligned} D_t^{\alpha,k} = (1-\theta)C\frac{1}{r} - \left((1-\theta)C\frac{1}{r} - \alpha\right) \times \\ e^{\beta_1^k\left(\sqrt{2r + \left(\beta_2^k\right)^2} + \beta_2^k\right)} \frac{e^{rt}\left(1 - \Phi\left(h_1^{D,k}(t)\right) - e^{2\beta_1^k\sqrt{2r+\left(\beta_2^k\right)^2}}\Phi\left(h_2^{D,k}(t)\right)\right)}{1 - \Phi\left(h_1^k(t)\right) - e^{2\beta_2^k \beta_1^k}\Phi\left(h_2^k(t)\right)} \end{aligned} \tag{6.4}$$

where $h_1^k(t)$ and $h_2^k(t)$ are defined by equations (6.2), while $h_1^{D,k}(.)$ and $h_2^{D,k}(.)$ are given by

$$h_1^{D,k}(s) = \frac{\beta_1^k - \sqrt{2r + \left(\beta_2^k\right)^2}\, s}{\sqrt{s}} \tag{6.5}$$

$$h_2^{D,k}(s) = \frac{\beta_1^k + \sqrt{2r + \left(\beta_2^k\right)^2}\, s}{\sqrt{s}}. \tag{6.6}$$

Proof. Let us consider the case that $k = l$.

If the holding is still active at time t, the value of the debt is equal to the following expectation:

$$
\begin{aligned}
D_t^{\alpha,l} &= \mathbb{E}^{\mathbb{Q}}\left(e^{-r(\tau^l - t)}\alpha \,|\, \mathcal{G}_t^l\right) + \mathbb{E}^{\mathbb{Q}}\left(\int_t^{\tau^l} e^{-r(s-t)}(1-\theta)C ds \,|\, \mathcal{G}_t^l\right) \\
&= \alpha \int_t^{+\infty} e^{-r(s-t)}\, d\mathbb{P}(\tau^l \leq s|\mathcal{G}_t^l) + (1-\theta)C\frac{1}{r}\left(1 - \int_t^{+\infty} e^{-r(s-t)}\, d\mathbb{P}(\tau^l \leq s|\mathcal{G}_t^l)\right) \\
&= (1-\theta)C\frac{1}{r} - \left((1-\theta)C\frac{1}{r} - \alpha\right)\int_t^{+\infty} e^{-r(s-t)}\, d\mathbb{P}(\tau^l \leq s|\mathcal{G}_t^l) \qquad (6.7)
\end{aligned}
$$

If we introduce the notation

$$
v(t) \quad = \quad \frac{e^{rt}}{1 - \Phi\left(h_1^l(t)\right) - e^{2\beta_2^l\beta_1^l}\Phi\left(h_2^l(t)\right)},
$$

the integral in (6.7) can be rewritten as follows:

$$
\begin{aligned}
&\int_t^{+\infty} e^{-r(s-t)}\, d\mathbb{P}(\tau^l \leq s \,|\, \mathcal{G}_t^l) \\
&\quad = v(t)\frac{-\beta_1^l}{\sqrt{2\pi s^3}}\int_t^{+\infty} \exp\left(-\frac{\left(-\beta_1^l + \beta_2^l s\right)^2 + 2rs^2}{2s}\right) ds \\
&\quad = v(t)e^{-\beta_1^l\left(\sqrt{2r+\left(\beta_2^l\right)^2} - \beta_2^l\right)}\int_t^{+\infty} \frac{-\beta_1^l}{\sqrt{2\pi s^3}}\exp\left(-\frac{\left(\beta_1^l - \sqrt{2r + \left(\beta_2^l\right)^2}s\right)^2}{2s}\right) ds.
\end{aligned}
$$

We notice that the integrand is the density of a hitting time, that we denote $\tau^{D,l}$, and which is such that

$$
\mathbb{P}(\tau^{D,l} \leq s) = \Phi\left(h_1^{D,l}(s)\right) + e^{2\beta_1^l\sqrt{2r+\beta_2^{l\,2}}}\Phi\left(h_2^{D,l}(s)\right)
$$

$$
h_1^{D,l}(s) = \frac{\beta_1^l - \sqrt{2r + \beta_2^{l\,2}}s}{\sqrt{s}} \qquad\qquad h_2^{D,l}(s) = \frac{\beta_1^l + \sqrt{2r + \left(\beta_2^l\right)^2}s}{\sqrt{s}}.
$$

As $\mathbb{P}(\tau^{D,l} \leq \infty) = e^{2\beta_1^l\sqrt{2r+\left(\beta_2^l\right)^2}}$ and $\mathbb{P}(\tau^{D,l} \leq 0) = 0$, we immediately infer that:

$$
\begin{aligned}
\int_t^{+\infty} e^{-r(s-t)}\, d\mathbb{P}(\tau^l \leq s \,|\, \mathcal{G}_t^l) &= e^{-\beta_1^l\left(\sqrt{2r+\left(\beta_2^l\right)^2} - \beta_2^l\right)} e^{2\beta_1^l\sqrt{2r+\left(\beta_2^l\right)^2}} \\
&\quad \times \frac{e^{rt}\left(1 - \Phi\left(h_1^{D,l}(t)\right) - e^{2\beta_1^l\sqrt{2r+\left(\beta_2^l\right)^2}}\Phi\left(h_2^{D,l}(t)\right)\right)}{1 - \Phi\left(h_1^l(t)\right) - e^{2\beta_2^l\beta_1^l}\Phi\left(h_2^l(t)\right)}.
\end{aligned}
$$

□

As stated by the following proposition, the market value of equity is the difference between the conditional expectation of the total value of the assets, conditionally upon the available information, and the value of debts:

Proposition 6.2. *The approximate values of the equity are equal to the following differences, for* $k = l$ *or* c *:*

$$
E_t^{\alpha,k} = \mathbb{E}^{\mathbb{Q}}\left(S_t^k|\mathcal{G}_t^k\right) - D_t^{\alpha,k} \qquad (6.8)
$$

where $D^{\alpha,k}$ is provided by proposition 6.1. The conditional expectations of S_t^l and S_t^c are given by the following expressions:

$$\mathbb{E}^{\mathbb{Q}}\left(S_t^l|\mathcal{G}_t^l\right) = \sum_{i=1}^{N} S_0^i e^{r\,t}\frac{1-\Phi\left(h_1^{S,l}(i,t)\right)-e^{2\beta_1^l(\beta_2^l+r_i\sigma_i^Z)}\Phi\left(h_2^{S,l}(i,t)\right)}{1-\Phi\left(h_1^l(t)\right)-e^{2\beta_2^l\beta_1^l}\Phi\left(h_2^l(t)\right)} \tag{6.9}$$

$$\mathbb{E}^{\mathbb{Q}}\left(S_t^c|\mathcal{G}_t^c\right) = \sum_{i=1}^{N} S_0^i e^{r\,t}\frac{1-\Phi\left(h_1^{S,c}(i,t)\right)-e^{2\beta_1^c(\beta_2^c+\sigma_i^Z)}\Phi\left(h_2^{S,c}(i,t)\right)}{1-\Phi\left(h_1^c(t)\right)-e^{2\beta_2^c\beta_1^c}\Phi\left(h_2^c(t)\right)} \tag{6.10}$$

where

$$h_1^{S,l}(i,t) = \frac{\beta_1^l-\beta_2^l t-r_i\,\sigma_i^Z\,t}{\sqrt{t}} \qquad h_2^{S,l}(i,t)=\frac{\beta_1^l+\beta_2^l t+r_i\,\sigma_i^Z\,t}{\sqrt{t}} \tag{6.11}$$

$$h_2^{S,c}(i,t) = \frac{\beta_1^c+\beta_2^c t+\sigma_i^Z\,t}{\sqrt{t}} \qquad h_2^{S,c}(i,t)=\frac{\beta_1^c+\beta_2^c t+\sigma_i^Z\,t}{\sqrt{t}} \tag{6.12}$$

and where $h_1^k(t)$ and $h_2^k(t)$ for $k=l$ or c are defined by equations (6.2).

Proof. First, we calculate the expected total value of the assets namely $\mathbb{E}^{\mathbb{Q}}\left(S_t^l|\mathcal{G}_t^l\right)=\mathbb{E}^{\mathbb{Q}}\left(g^l(W_t^l)|\mathcal{G}_t^l\right)$. This expectation can be expressed as

$$\mathbb{E}^{\mathbb{Q}}\left(S_t^l|\mathcal{G}_t^l\right) = \int_{\beta_1^l-\beta_2^l t}^{+\infty} g^l(t,w)d\mathbb{P}(W_t^l\le w\,|\inf_{0\le s\le t}\beta_2^l s+W_s^l\ge\beta_1^l).$$

Let us define $w'=w+\beta_2^l t$

$$\mathbb{E}^{\mathbb{Q}}\left(S_t^l|\mathcal{G}_t^l\right) = \int_{\beta_1^l}^{+\infty} g^l(t,w'-\beta_2^l t)d\mathbb{P}(\beta_2^l t+W_t^l\le w'\,|\inf_{0\le s\le t}\beta_2^l s+W_s^l\ge\beta_1^l). \tag{6.13}$$

Then, by denoting $X_t=\beta_2^l t+W_t^l$ and $m_t^l=\inf_{0\le s\le t}X_s$, we get that

$$\begin{aligned}\mathbb{P}(\beta_2^l t+W_t^l\le w'\,|\inf_{0\le s\le t}\beta_2^l s+W_s^l\ge\beta_1^l) &= \mathbb{P}(X_t\le w'\,|\,m_t^l\ge\beta_1^l)\\ &= \frac{\mathbb{P}(X_t\le w'\,,\,m_t^l\ge\beta_1^l)}{\mathbb{P}(\,m_t^l\ge\beta_1^l)}.\end{aligned}$$

The denominator is equal to the approximate probability that the company does not go to bankruptcy before time t:

$$\begin{aligned}\mathbb{P}(\,m_t^l\ge\beta_1^l) &= \mathbb{P}\left(\tau^l\ge t\right)\\ &= 1-\Phi\left(h_1^l(t)\right)-e^{2\beta_2^l\beta_1^l}\Phi\left(h_2^l(t)\right)\end{aligned} \tag{6.14}$$

where $h_1^k(t)$ and $h_2^k(t)$ are defined by equations (6.2). Furthermore:

$$\begin{aligned}\mathbb{P}(X_t\le w'\,,\,m_t^l\ge\beta_1^l) &= 1-\mathbb{P}(m_t^l\le\beta_1^l)-\mathbb{P}(X_t\ge w'\,,\,m_t^l\ge\beta_1^l)\\ &= \mathbb{P}(\tau^l\ge t)-\mathbb{P}(X_t\ge w'\,,\,m_t^l\ge\beta_1^l)\end{aligned}$$

where $\mathbb{P}(\tau^l\ge t)$ is given by (6.14). From Bielecki and Rutkowski (2004 page 68), we have that

$$\mathbb{P}(X_t\ge w'\,,\,m_t^l\ge\beta_1^l) = \Phi\left(\frac{-w'+\beta_2^l t}{\sqrt{t}}\right)-e^{2\beta_2^l\beta_1^l}\Phi\left(\frac{2\beta_1^l-w'+\beta_2^l t}{\sqrt{t}}\right)$$

such that

$$d\mathbb{P}(X_t\ge w'\,,\,m_t^l\ge\beta_1^l) = -\varphi\left(\frac{-w'+\beta_2^l t}{\sqrt{t}}\right)\frac{1}{\sqrt{t}}+e^{2\beta_2^l\beta_1^l}\varphi\left(\frac{2\beta_1^l-w'+\beta_2^l t}{\sqrt{t}}\right)\frac{1}{\sqrt{t}} \tag{6.15}$$

Hence, the density of X_t conditionally to the survival of the holding equals

$$d\mathbb{P}(X_t \leq w' \,|\, m_t^l \geq \beta_1^l) \quad = \quad -\frac{1}{1-\Phi\left(h_1^l(t)\right) - e^{2\beta_2^l\beta_1^l}\Phi\left(h_2^l(t)\right)} d\mathbb{P}(X_t \geq w' \,,\, m_t^l \geq \beta_1^l) \tag{6.16}$$

Introducing the notation

$$v^l(t) \quad = \quad \frac{1}{1-\Phi\left(h_1^l(t)\right) - e^{2\beta_2^l\beta_1^l}\Phi\left(h_2^l(t)\right)}$$

and combining equations (6.13), (6.15) and (6.16), leads to the following expressions of the conditional expectation of the total value of the assets at t:

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left(S_t^l|\mathcal{G}_t^l\right) &= -v^l(t)\int_{\beta_1^l}^{+\infty} g^l(t, w'-\beta_2^l t) d\mathbb{P}(X_t \geq w' \,,\, m_t^l \geq \beta_1^l) \\
&= v^l(t)\frac{1}{\sqrt{t}}\left[\int_{\beta_1^l}^{+\infty} g^l(t, w'-\beta_2^l t)\varphi\left(\frac{-w'+\beta_2^l t}{\sqrt{t}}\right)dw' \right. \\
&\quad \left. -e^{2\beta_2^l\beta_1^l}\int_{\beta_1^l}^{+\infty} g^l(t, w'-\beta_2^l t)\varphi\left(\frac{2\beta_1^l - w' + \beta_2^l t}{\sqrt{t}}\right)dw'\right].
\end{aligned}$$

By a change of variable $w = w' - \beta_2^l t$, we get that

$$\mathbb{E}^{\mathbb{Q}}\left(S_t^l|\mathcal{G}_t^l\right) \quad = \quad v^l(t)\frac{1}{\sqrt{t}}\left[\int_{\beta_1^l-\beta_2^l t}^{+\infty} g^l(t,w)\,\varphi\left(\frac{w}{\sqrt{t}}\right)dw - e^{2\beta_2^l\beta_1^l}\int_{\beta_1^l-\beta_2^l t}^{+\infty} g^l(t,w)\varphi\left(\frac{w-2\beta_1^l}{\sqrt{t}}\right)dw\right].$$

Given the definition of $g^l(w)$, we rewrite the last expression as follows:

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left(S_t^l|\mathcal{G}_t^l\right) &= v^l(t)\frac{1}{\sqrt{t}}\left[\sum_{i=1}^N S_0^i e^{\left(\mu_i^Z+\frac{1}{2}(1-r_i^2)\left(\sigma_i^Z\right)^2\right)t}\int_{\beta_1^l-\beta_2^l t}^{+\infty} e^{r_i\,\sigma_i^Z\,w}\,\varphi\left(\frac{w}{\sqrt{t}}\right)dw\right. \\
&\left.-\sum_{i=1}^N S_0^i e^{\left(\mu_i^Z+\frac{1}{2}(1-r_i^2)\left(\sigma_i^Z\right)^2\right)t} e^{2\beta_2^l\beta_1^l}\int_{\beta_1^l-\beta_2^l t}^{+\infty} e^{r_i\,\sigma_i^Z\,w}\varphi\left(\frac{w-2\beta_1^l}{\sqrt{t}}\right)dw\right].
\end{aligned} \tag{6.17}$$

As it is well-known that

$$e^{r_i\,\sigma_i^Z\,w}\,\varphi\left(\frac{w}{\sqrt{t}}\right) \quad = \quad e^{\frac{1}{2}\left(r_i\,\sigma_i^Z\right)^2 t}\varphi\left(\frac{w-r_i\,\sigma_i^Z\,t}{\sqrt{t}}\right)$$

and that

$$e^{r_i\,\sigma_i^Z\,w}\varphi\left(\frac{w-2\beta_1^l}{\sqrt{t}}\right) \quad = \quad e^{-\frac{1}{t}\left(2\left(\beta_1^l\right)^2-\frac{1}{2}\left(2\beta_1^l+r_i\,\sigma_i^Z\,t\right)^2\right)}\varphi\left(\frac{w-(2\beta_1^l+r_i\,\sigma_i^Z\,t)}{\sqrt{t}}\right),$$

the expectation (6.17) becomes:

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left(S_t^l|\mathcal{G}_t^l\right) &= v^l(t)\left[\sum_{i=1}^N S_0^i e^{\left(\mu_i^Z+\frac{1}{2}\left(\sigma_i^Z\right)^2\right)t}\left(1-\Phi\left(\frac{\beta_1^l-\beta_2^l t - r_i\,\sigma_i^Z\,t}{\sqrt{t}}\right)\right)\right. \\
&\left.-\sum_{i=1}^N S_0^i e^{\left(\mu_i^Z+\frac{1}{2}\left(\sigma_i^Z\right)^2\right)t+2\beta_1^l\left(\beta_2^l+r_i\sigma_i^Z\right)}\left(1-\Phi\left(\frac{-\beta_1^l-\beta_2^l t - r_i\,\sigma_i^Z\,t}{\sqrt{t}}\right)\right)\right]
\end{aligned}$$

and we infer the result from this last relation.

□

To end this paragraph, we say a word about the pricing of CDS in this incomplete framework. If premiums are paid at regular intervals of time, Δt, ranging from t_1 to t_n, the CDS spread at time t, given that the last information disclosure has been done at time zero, is provided for $k = c$ or $k = l$ by:

$$p^k(t) = \frac{(1-R)}{\Delta t} \frac{\sum_{t_i=t_1}^{t_n} e^{-r\,t_i} \left(\mathbb{P}\left(\tau^k > t_{i-1} \,|\, \mathcal{G}_t^k\right) - \mathbb{P}\left(\tau^k > t_i \,|\, \mathcal{G}_t \mathcal{G}_t^k\right)\right)}{\sum_{t_i=t_1}^{t_n} e^{-r\,t_i} \mathbb{P}\left(\tau^k > t_i \,|\, \mathcal{G}_t^k\right)}, \tag{6.18}$$

where $\mathbb{P}(\tau^k > t_i)$ is provided by equation (6.1).

7 Stochastic liabilities.

The model developped in the previous sections is well suited for non financial corporations that mainly finance their activities by debts and equity. This model does not fit so well financial holdings such as banks or insurance companies, which have most of the times random liabilities. To remedy to this problem, we assume in this section that the holding company is financed by N_L stochastic liabilities L_t^i having the following risk neutral dynamics:

$$dL_t^i = L_t^i r dt + L_t^i \sum_{j=1}^{M} \sigma_{i,j}^L dW_t^j \quad i = 1, \ldots, N_L. \tag{7.1}$$

The total value of the liabilities is given by $L_t = \sum_{i=1}^{M} L_t^i$. We denote by Σ^L the $N_L \times M$ matrix of $\sigma_{i,j}^L$. The holding invests in N_S activities or subsidiaries whose market values are solution of the following stochastic differential equations:

$$dS_t^i = rS_t^i dt + S_t^i \sum_{j=1}^{M} \sigma_{i,j}^S dW_t^j \quad i = 1, ..., N_S. \tag{7.2}$$

The total value of the investments is still denoted by $S_t = \sum_{i=1}^{N_S} S_t^i$. We denote by Σ^S the $N_S \times M$ matrix of $\sigma_{i,j}^S$. The market capitalization is the difference between the total values of investments and liabilities

$$E_t = S_t - L_t \stackrel{d}{=} \sum_{i=1}^{N_S} S_0^i e^{Z_t^i} - \sum_{i=N_S+1}^{N_S+N_L} L_0^i e^{Z_t^i}, \tag{7.3}$$

where the vector of Z_t counts $N_S + N_L$ components Z_t^i and is distributed as a multivariate normal $N(\mu^Z t\,,\, \Sigma\Sigma^\top\, t)$ where Σ is the $(N_S + N_L) \times M$ matrix of assets and liabilities covariances:

$$\Sigma = \begin{pmatrix} \Sigma^S \\ \Sigma^L \end{pmatrix} \tag{7.4}$$

and the $(N_S + N_L)$ mean vector is given by

$$\mu^Z = \begin{pmatrix} \mu^{ZS} \\ \mu^{ZL} \end{pmatrix} \tag{7.5}$$

with

$$\mu^{ZS} = \begin{pmatrix} r - \frac{1}{2} e_1^\top \Sigma^S \Sigma^{S\top} e_1 \\ \vdots \\ r - \frac{1}{2} e_{N_S}^\top \Sigma^S \Sigma^{S\top} e_{N_S} \end{pmatrix} \quad \text{and} \quad \mu^{ZL} = \begin{pmatrix} r - \frac{1}{2} e_1^\top \Sigma^L \Sigma^{L\top} e_1 \\ \vdots \\ r - \frac{1}{2} e_{N_L}^\top \Sigma^L \Sigma^{L\top} e_{N_L} \end{pmatrix} \tag{7.6}$$

where μ^{ZS} and μ^{ZL} are respectively drifts of the assets and the liabilities and where e_i denotes in this section the i^{th} unit root vector of $\mathbb{R}^{N_S+N_L}$. To simplify further calculations, the variance of Z_t^i is defined as

$$Var(Z_t^i) = \left(\sigma_i^{ZS}\right)^2 t = e_i^\top \Sigma\Sigma^\top e_i \, t \qquad i = 1, \ldots, N_S \tag{7.7}$$

$$Var(Z_t^i) = \left(\sigma_i^{ZL}\right)^2 t = e_i^\top \Sigma\Sigma^\top e_i \, t \qquad i = N_S + 1, \ldots, N_L. \tag{7.8}$$

As previously, we introduce a process $\Lambda_t = \sum_{i=1}^{N_S+N_L} \gamma_i Z_t^i$ that is a weighted sum of processes Z_t^i where the constant γ_i are chosen to minimize the quadratic spread between covariances of upper and lower convex approximations of assets/liabilities:

$$\gamma_i = argmin \sum_{i=1}^{N_S+N_L} \sum_{j=1}^{N_S+N_L} \left(1 - \frac{e_i^\top \Sigma\Sigma^\top \gamma \, e_j^\top \Sigma\Sigma^\top \gamma}{\sqrt{e_i^\top \Sigma\Sigma^\top e_i} \sqrt{e_j^\top \Sigma\Sigma^\top e_j} \, \gamma^\top \Sigma\Sigma^\top \gamma} \right)^2 . \tag{7.9}$$

We denote the correlations by r_i for $i = 1, ..., N_S$ as follows:

$$\begin{aligned} r_i &= \frac{cov(Z_t^i, \Lambda_t)}{\sqrt{Var(Z_t^i)} \sqrt{Var(\Lambda_t)}} \\ &= \frac{e_i^\top \Sigma\Sigma^\top \gamma}{\sqrt{e_i^\top \Sigma\Sigma^\top e_i} \sqrt{\gamma^\top \Sigma\Sigma^\top \gamma}}. \end{aligned} \tag{7.10}$$

We split the vector of correlations in two sub vectors, one related to assets and one to liabilities:

$$r_i^S = r_i \qquad i = 1, ..., N_s,$$
$$r_i^L = r_{N_S+i} \qquad i = 1, ..., N_L.$$

The processes defined hereafter are used as lower and upper approximations of asset and liability processes:

$$S_t^{i,l} = S_0^i \exp\left(\left(\mu_i^{ZS} + \frac{1}{2}(1 - \left(r_i^S\right)^2) \left(\sigma_i^{ZS}\right)^2 \right) t + r_i^S \, \sigma_i^{ZS} \, W_t^l \right) \tag{7.11}$$

$$L_t^{i,l} = L_0^i \exp\left(\left(\mu_i^{ZL} + \frac{1}{2}(1 - \left(r_i^L\right)^2) \left(\sigma_i^{ZL}\right)^2 \right) t + r_i^L \, \sigma_i^{ZL} \, W_t^l \right) \tag{7.12}$$

$$S_t^{i,c} = S_0^i \exp\left(\mu_i^{ZS} t + \sigma_i^{ZS} \, W_t^c \right) \tag{7.13}$$

$$L_t^{i,c} = L_0^i \exp\left(\mu_i^{ZL} t - \sigma_i^{ZL} \, W_t^c \right) \tag{7.14}$$

where W_t^l and W_t^c are independent Brownian motions such that $W_0^l = W_0^c = 0$. Note that in the definition of $L_t^{i,c}$, the sign of $\sigma_i^{ZL} \, W_t^c$ is negative because liabilities are substracted from assets (for details, we refer to Dhaene et al. 2002, paragraph 4). Estimates of the market value of equity are provided by the following differences:

$$E_t^l = \sum_{i=1}^{N_S} S_t^{i,l} - \sum_{i=1}^{N_L} L_t^{i,l} \qquad\qquad E_t^c = \sum_{i=1}^{N_S} S_t^{i,c} - \sum_{i=1}^{N_L} L_t^{i,c}.$$

By construction, the following convex order relations are satisfied[1]

$$E_t^l \leq_{cx} E_t^c. \tag{7.15}$$

The default is triggered when the equity falls below a certain level denoted by α, but the distribution of the hitting time is unknown. Convex estimates of E_t can be respectively rewritten as function $g^l(.)$ and $g^c(.)$ of time and of Brownian motions W_t^l and W_t^c :

$$\begin{aligned} E_t^l &= \sum_{i=1}^{N_S} S_0^i \exp\left(\left(\mu_i^{ZS} + \frac{1}{2}(1-\left(r_i^S\right)^2))\left(\sigma_i^{ZS}\right)^2\right)t + r_i^S\,\sigma_i^{ZS}\,W_t^l\right) \\ &\quad -\sum_{i=1}^{N_L} L_0^i \exp\left(\left(\mu_i^{ZL} + \frac{1}{2}(1-\left(r_i^L\right)^2))\left(\sigma_i^{ZL}\right)^2\right)t + r_i^L\,\sigma_i^{ZL}\,W_t^l\right) \\ &= g^l(t, W_t^l) \end{aligned} \tag{7.16}$$

$$\begin{aligned} E_t^c &= \sum_{i=1}^{N_S} S_0^i \exp\left(\mu_i^{ZS}t + \sigma_i^{ZS}\,W_t^c\right) - \sum_{i=1}^{N_L} L_0^i \exp\left(\mu_i^{ZL}t - \sigma_i^{ZL}\,W_t^l\right) \\ &= g^c(t, W_t^c). \end{aligned} \tag{7.17}$$

These functions $g^l(t,w)$ and $g^c(t,w)$ do not admit any simple analytical inverse functions. As in previous sections, we use linear approximations of functions $\left(g^l\right)^{-1}$ and $\left(g^c\right)^{-1}$ to keep closed form expressions for approximate default probabilities. We choose a time t_0, calculate numerically $\left(g^l\right)^{-1}(t_0,\alpha)$ and develop $\left(g^l\right)^{-1}$ linearly around t_0:

$$\left(g^l\right)^{-1}(t,\alpha) \quad \approx \quad \beta_1^l - \beta_2^l t, \tag{7.18}$$

where β_1^l and β_2^l are defined by:

$$\beta_2^l = \frac{\begin{array}{c}\sum_{i=1}^{N_S} S_0^i \left(\mu_i^{ZS} + \frac{1}{2}(1-r_i^{S\,2})\left(\sigma_i^{ZS}\right)^2\right) e^{\left(\mu_i^{ZS}+\frac{1}{2}(1-r_i^{S\,2})\left(\sigma_i^{ZS}\right)^2\right)t + r_i^S\,\sigma_i^{ZS}\,\left(g^l\right)^{-1}(t_0,\alpha)} \\ -\sum_{i=1}^{N_L} L_0^i \left(\mu_i^{ZL} + \frac{1}{2}(1-r_i^{L\,2})\left(\sigma_i^{ZL}\right)^2\right) e^{\left(\mu_i^{ZL}+\frac{1}{2}(1-r_i^{L\,2})\left(\sigma_i^{ZL}\right)^2\right)t + r_i^L\,\sigma_i^{ZL}\,\left(g^l\right)^{-1}(t_0,\alpha)}\end{array}}{\begin{array}{c}\sum_{i=1}^{N_S} r_i^S\,\sigma_i^{ZS} S_0^i e^{\left(\mu_i^{ZS}+\frac{1}{2}(1-r_i^{S\,2})\left(\sigma_i^{ZS}\right)^2\right)t + r_i^S\,\sigma_i^{ZS}\,\left(g^l\right)^{-1}(t_0,\alpha)} \\ -\sum_{i=1}^{N_L} r_i^L\,\sigma_i^{ZL} S_0^i e^{\left(\mu_i^{ZL}+\frac{1}{2}(1-r_i^{L\,2})\left(\sigma_i^{ZL}\right)^2\right)t + r_i^L\,\sigma_i^{ZL}\,\left(g^l\right)^{-1}(t_0,\alpha)}\end{array}}. \tag{7.19}$$

$$\beta_1^l \quad = \quad \left(g^l\right)^{-1}(t_0,\alpha) + \beta_2^l\,t_0 \tag{7.20}$$

In the same way, we get that,

$$\left(g^c\right)^{-1}(t,\alpha) \quad \approx \quad \beta_1^c - \beta_2^c t, \tag{7.21}$$

where β_1^c and β_2^c are defined as follows:

$$\beta_2^c = \frac{\sum_{i=1}^{N_S} S_0^i \mu_i^{ZS} e^{\mu_i^{ZS}t + \sigma_i^{ZS}\,(g^c)^{-1}(t_0,\alpha)} - \sum_{i=1}^{N_L} L_0^i \mu_i^{ZL} e^{\mu_i^{ZL}t + \sigma_i^{ZL}\,(g^c)^{-1}(t_0,\alpha)}}{\sum_{i=1}^{N_S} \sigma_i^{ZS} S_0^i e^{\mu_i^{ZS}t + \sigma_i^{ZS}\,(g^c)^{-1}(t_0,\alpha)} - \sum_{i=1}^{N_L} \sigma_i^{ZL} S_0^i e^{\mu_i^{ZL}t + \sigma_i^{ZL}\,(g^c)^{-1}(t_0,\alpha)}}. \tag{7.22}$$

[1] Despite what notations suggest, we don't necessary have $S_t^{i,l} \leq_{cx} S_t^{i,c}$ and $L_t^{i,l} \leq_{cx} L_t^{i,c}$. In general it is neither true that E_t^l is a comonotonic sum.

$$\beta_1^c = (g^c)^{-1}(t_0, \alpha) + \beta_2^c t_0. \tag{7.23}$$

Linear functions (7.18) and (7.21) delimit the continuation and bankruptcy regions, in function of W_t^l and W_t^c. But given that $g^l(t,w)$ and $g^c(t,w)$ are either increasing or decreasing functions of w, continuation regions can be above or below the linear approximations of $(g^c)^{-1}(t_0,\alpha)$ and $(g^l)^{-1}(t,\alpha)$. The (approximate) defaults can then occur when W_t^l and W_t^c hit an upper or a lower boundary. The easiest way to detect if boundaries (7.18) and (7.21) are upper of lower frontiers, is to check the sign of β_1^l and β_1^c. If $\beta_1^l \leq 0$ or $\beta_1^c \leq 0$, the continuation region is above them. Indeed, if it is not the case, the holding is in bankruptcy at time 0 given that $W_t^l = 0$. The approximate default times of convex bounds are then defined as hitting times of

$$\tau^l = \inf\left\{t \geq 0 \,|\, -\beta_1^l + \beta_2^l t + W_t^l \leq 0\,,\, -\beta_1^l + \beta_2^l s + W_s^l > 0 \,\forall s < t\right\} \tag{7.24}$$

$$\tau^c = \inf\left\{t \geq 0 \,|\, -\beta_1^c + \beta_2^c t + W_t^c \leq 0\,,\, -\beta_1^c + \beta_2^c s + W_s^c > 0 \,\forall s < t\right\}. \tag{7.25}$$

If $\beta_1^l > 0$ or $\beta_1^c > 0$, the continuation region is below the linear boundaries for the same reason. The approximate default times are defined then as follows:

$$\tau^l = \inf\left\{t \geq 0 \,|\, \beta_1^l - \beta_2^l t + W_t^l \leq 0\,,\, \beta_1^l - \beta_2^l s + W_s^l > 0 \,\forall s < t\right\} \tag{7.26}$$

$$\tau^c = \inf\left\{t \geq 0 \,|\, \beta_1^c - \beta_2^c t + W_t^c \leq 0\,,\, \beta_1^c - \beta_2^c s + W_s^c > 0 \,\forall s < t\right\}. \tag{7.27}$$

As these are all hitting times of a Brownian motion with drift, the approximate probabilities of default are then provided by similar expressions to those obtained in section 4, except that the sign of $\beta_1^{l,c}$ plays now an important role:

$$\mathbb{P}(\tau^k \leq t) = \Phi\left(h_1^k(t)\right) + e^{2\beta_2^k \beta_1^k}\Phi\left(h_2^k(t)\right) \quad \text{for } \; k = l \; \text{ or } \; c \tag{7.28}$$

$$h_1^k(t) = -sign(\beta_1^k)\left(\frac{\beta_1^k - \beta_2^k t}{\sqrt{t}}\right) \qquad h_2^k(t) = -sign(\beta_1^k)\frac{\beta_1^k + \beta_2^k t}{\sqrt{t}}.$$

The asymptotic probabilities of ruin when $\beta_1^k \beta_2^k > 0$ or $\beta_1^k \beta_2^k < 0$ are respectively 1 and $e^{2\beta_2^k \beta_1^k}$. The CDS premium in this setting can be easily computed by formula (5.6) in which we substitute the probabilities of default by these obtained for random liabilities. We now consider that the information about the holding is not continuously but as in section 6. We assume that the only information available at time t has been disclosed at time 0 and that the holding is still active. The information carried by the filtration $(\mathcal{F}_t)_t$ is not accessible in continuous time. As previously the information available to the market is represented by the filtration $\mathcal{G}_t$, and we will work with the filtrations $\mathcal{G}_t^k$ for $k = l$ or c for the approximations. Given the similarities between the approximate default times for models without and with stochastic liabilities, we can easily infer that the approximate default probabilities are still provided by the formula (6.1):

$$\mathbb{P}\left(\tau^k > T \,|\, \mathcal{G}_t^k\right) = \frac{1 - \Phi\left(h_1^k(T)\right) - e^{2\beta_2^k\beta_1^k}\Phi\left(h_2^k(T)\right)}{1 - \Phi\left(h_1^k(t)\right) - e^{2\beta_2^k\beta_1^k}\Phi\left(h_2^k(t)\right)} \quad \text{for } \; k = l \; \text{ or } \; c. \tag{7.29}$$

Furthermore, the estimates of the equity, provided in the next corollary, are obtained in the same way as in proposition 6.2, except that a particular care must be granted to the sign of $\beta_1^{l,c}$.

Corollary 7.1. *The expected value of S_t^l given the information $\mathcal{G}_t^l$, equals*

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left(S_t^l | \mathcal{G}_t^l\right) &= \sum_{i=1}^{N_S} S_0^i e^{\left(\mu_i^{ZS} + \frac{1}{2}\left(\sigma_i^{ZS}\right)^2\right)t} \frac{1 - \Phi\left(h_1^{S,l}(i,t)\right) - e^{2\beta_1^l\left(\beta_2^l + r_i^S \sigma_i^{ZS}\right)}\Phi\left(h_2^{S,l}(i,t)\right)}{1 - \Phi\left(h_1^l(t)\right) - e^{2\beta_2^l\beta_1^l}\Phi\left(h_2^l(t)\right)} \\
&\quad - \sum_{i=1}^{N_L} L_0^i e^{\left(\mu_i^{ZL} + \frac{1}{2}\left(\sigma_i^{ZL}\right)^2\right)t} \frac{1 - \Phi\left(h_1^{L,l}(i,t)\right) - e^{2\beta_1^l\left(\beta_2^l + r_i^L \sigma_i^{ZL}\right)}\Phi\left(h_2^{L,l}(i,t)\right)}{1 - \Phi\left(h_1^l(t)\right) - e^{2\beta_2^l\beta_1^l}\Phi\left(h_2^l(t)\right)}
\end{aligned} \tag{7.30}$$

where

$$h_1^{S,l}(i,t) = \frac{-sign(\beta_1^l)\left(\beta_1^l - \beta_2^l t\right) - r_i^S\,\sigma_i^{ZS}\,t}{\sqrt{t}} \qquad h_2^{S,l}(i,t) = \frac{-sign(\beta_1^l)\left(\beta_1^l + \beta_2^l t\right) + r_i^S\,\sigma_i^{ZS}\,t}{\sqrt{t}}$$

$$h_1^{L,l}(i,t) = \frac{-sign(\beta_1^l)\left(\beta_1^l + \beta_2^l t\right) - r_i^L\sigma_i^{ZL}\,t}{\sqrt{t}} \qquad h_2^{L,l}(i,t) = \frac{-sign(\beta_1^l)\left(\beta_1^l + \beta_2^l t\right) + r_i^L\,\sigma_i^{ZL}\,t}{\sqrt{t}}$$

The expected value of S_t^c given the information $\mathcal{G}_t^l$ is obtained by replacing β_1^l, β_2^l, r_i^S and r_i^L respectively by β_1^c, β_2^c, 1 and -1.

CDS premiums can also be calculated by substituting the right approximate probabilities of default in the formula (6.18).

8 Numerical Applications.

In a first example, we compare the exact and approximate default probabilities of a holding composed of five business lines that deliver correlated dividends. The initial value of investments, their volatilities and correlations are respectively set to:

$$S_0 = (20,\ 20,\ 20,\ 20,\ 20)' \tag{8.1}$$

$$\left(\sigma_i^Z\right) = (10\%,\ 20\%,\ 30\%,\ 40\%,\ 50\%)' \tag{8.2}$$

$$\rho = \begin{pmatrix} 1.0 & -0.3 & -0.6 & -0.2 & -0.1 \\ -0.3 & 1 & 0.5 & 0.3 & 0.1 \\ -0.6 & 0.5 & 1 & 0.7 & 0.2 \\ -0.2 & 0.3 & 0.7 & 1 & 0.3 \\ -0.1 & 0.1 & 0.2 & 0.3 & 1 \end{pmatrix}. \tag{8.3}$$

The risk free rate is equal to $r = 2\%$ and the floor triggering the default of the company is set equal to $\alpha = 90$. In a following series of tests, multiple levels of α are considered. The numerical minimization of the criterion (3.12) yields the following γ_i:

$$(\gamma_i)_i = (1.6497,\ 0.5774,\ 0.3840,\ 0.2427,\ 0.2318)'.$$

The coefficients r_i that define the lower convex bound of S_t are

$$(r_i)_i = (0.1050,\ 0.5714,\ 0.5742,\ 0.7448,\ 0.5673)'.$$

As the first business line is negatively correlated with others, the coefficient r_1 is the smallest. The left graph of figure (8.1) presents the exact inverse functions $\left(g^l\right)^{-1}(t,\alpha)$, $(g^c)^{-1}(t,\alpha)$ and their linear approximations (4.5) and (4.8), calculated with $t_0 = 0$. In this case, linear approximations offer a good fit. The right graph of the same figure shows the approximate default probabilities obtained with the lower and upper linear bounds (dotted and dot dash curves). The real probabilities of default have also been computed by Monte Carlo simulations (5000 runs) with discretization step $\Delta t = 0.0005$ (continuous curve). Using the same method, we also calculate the lower convex estimate of default probabilities, defined by (4.3) calculated with the exact inverse boundary $\left(g^l\right)^{-1}(t,\alpha)$ (see dashed line). Probabilities of default computed with the lower convex bound are not far from the real one. The upper convex estimate of default probabilities is relatively less accurate. As the driving risk processes used are continuous Brownian motions, default probabilities vanish over a short period. This feature is not necessary observed in reality.

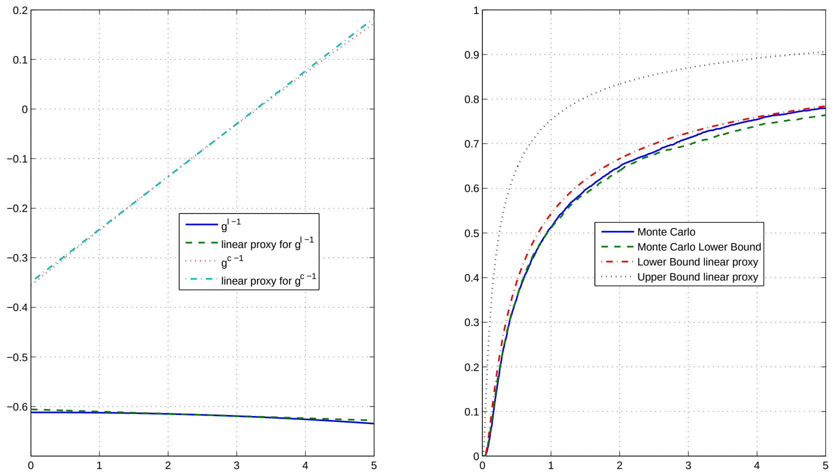

Figure 8.1: Example of function $\left(g^{l,c}\right)^{-1}(t,\alpha)$ and default probabilities.

Figure 8.2 exhibits estimates of the market value of debts, in function of the floor α. The accounting value of the debt is set to 90 and the tax rate is null. Two scenarios are considered. On the left side, the coupon rate is 2% while on the right side, the coupon rate is 2.5%. We remark that the market value of debts and equity can respectively be minimized and maximized by a judicious choice of α. The lower approximation being the most relevant, the holding's owner optimizes the equity market value by closing his activities when the market value of assets reaches 62.50 for a cost of debts of 2% or 70.00 for a cost of debts of 2.5% .

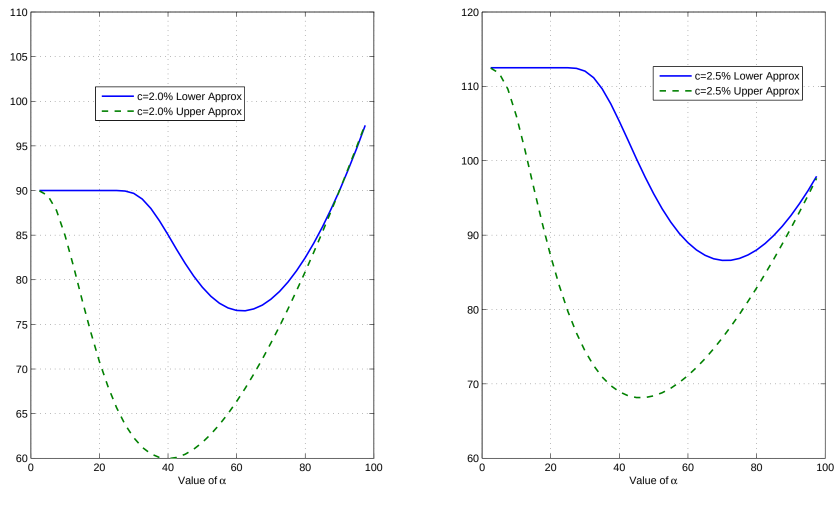

Figure 8.2: Market Value of debts as a function of α.

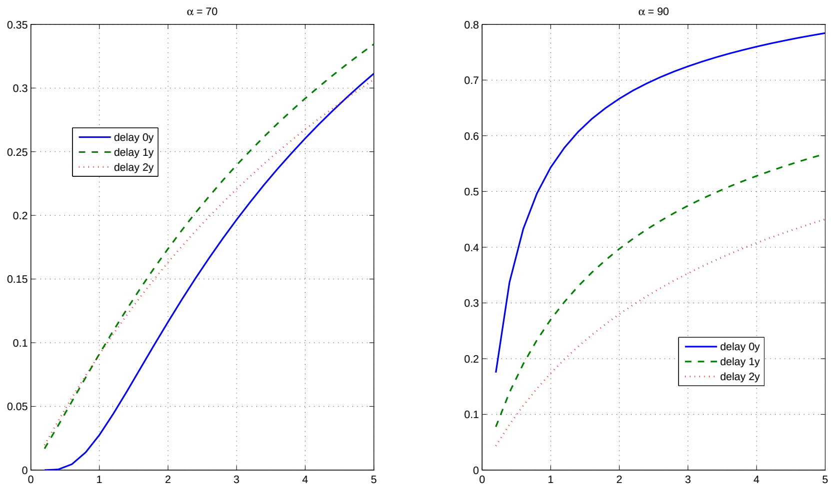

Figure 8.3: Influence of a delay on probabilities of default (lower approximation).

Figure 8.3 illustrates the influence of a delay in the disclosure of information on the probabilities of default for maturities, estimated with the lower comonotone bound. Two scenarios are considered. In the first one, the floor is set to $\alpha = 70$ and is then relatively far from the initial value value of assets ($S_0 = 100$). In this case, the higher is the time lag after the last publication of information, the higher are the estimated probabilities of default. In the second scenario, the floor triggering the default is close to the initial value of assets, $\alpha = 90$. Here, the influence of the delay is exactly the opposite and decreases estimated probabilities of default. We explain this as follows: if the holding is still alive after one or two years, the probability that the total assets value is far above α is high. After one or two years, the simple fact that the company did not go to bankruptcy, is sufficient to infer that the company is now in a less critical situation than in the past.

The lack of information also influences the appraisal of debts and equity. This point is illustrated in figure 8.4. It exhibits on the left side, the market value of debts (coupon 2.5%, debt accounting value 90) for different levels of the floor α, and for a delay of one or two years. The right graph presents the expected value of the total of assets. We observe that the higher is the delay, the lower is the debt market value and the higher is the expected value of assets, whatsoever the level of the floor. The difference between the expected market value of assets and debts gives the market value of the equity, which can also be maximized by a judicious choice of α.

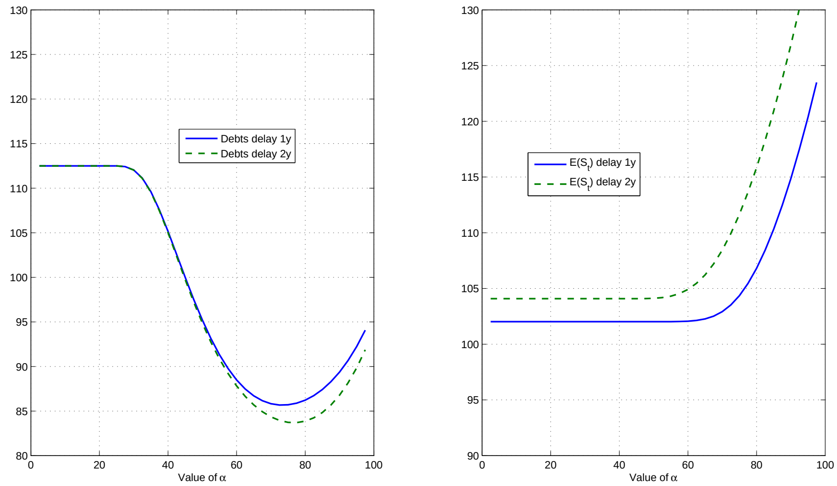

Figure 8.4: Market values of debts and equity, in function of α.

To end this section, we test the model of holdings financed by stochastic liabilities. We have considered a firm that invests in two business lines and financed by two random liabilities. The parameters chosen (initial values, volatilities and correlations) are the following:

$$S_0 = (50,\ 50)' \qquad L_0 = (40,\ 40)'$$
$$\left(\sigma_i^{ZS}\right) = (10\%,\ 10\%)' \qquad \left(\sigma_i^{ZL}\right) = (20\%,\ 20\%)'$$

$$\rho = \begin{pmatrix} 1 & 0.3 & 0.6 & 0.2 \\ 0.3 & 1 & 0.5 & 0.3 \\ 0.6 & 0.5 & 1 & 0.7 \\ 0.2 & 0.3 & 0.7 & 1 \end{pmatrix}$$

The initial market value of the equity is then 20. The risk free rate is set to $r = 2\%$ and the floor triggering the default of the company is $\alpha = 16$. The numerical minimization of the criterion (7.9) yields the following γ_i:

$$(\gamma_i)_i = (0.7724,\ 0.7621,\ 0.2560,\ 0.3756)^{'}\ .$$

The coefficients r_i that define the lower convex bound of S_t are

$$(r_i)_i = (0.7004,\ 0.7084,\ 0.9040,\ 0.7169)^{'}.$$

The left figure of exhibit 8.5 presents convex estimates of probabilities of default. Compared to the results that we get in previous examples, the lower bound convex estimates are relatively far from the real probabilities of default, obtained by Monte Carlo simulations. Our approximation seems less reliable. In this case, as done in Vyncke et al. (2004), we can still work with a weighted average of estimates to price the debt or the equity. The right graph of figure 8.5 reveals the influence of a delay in the disclosure of information on the expected market value of the equity, for different levels α. For the chosen parameters, the higher is the time lag, the lower is the price.

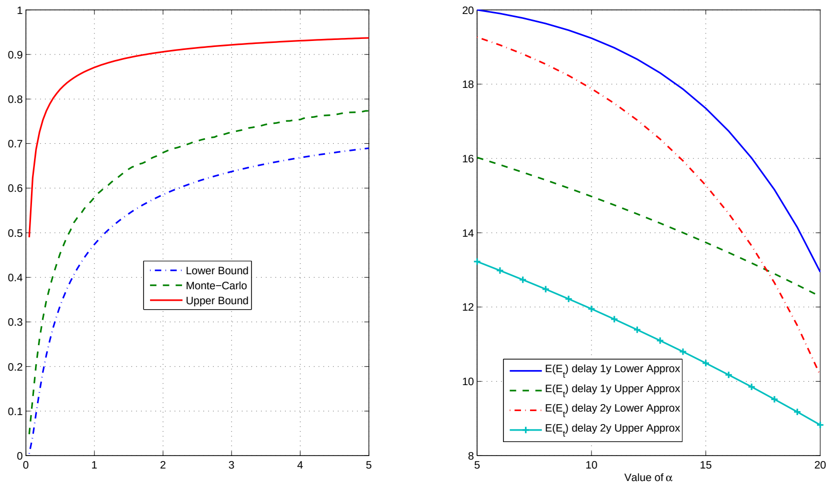

Figure 8.5: Probabilities of default and expected equity.

9 Conclusions.

This paper investigates an approximating method to assess the credit risk related to a multi-industry firm. If the market values of subsidiaries are lognormal, the distribution of the total asset's firm is not know. In this case, equity and debts cannot be appraised analytically. Convex bounds allow us to circumvent this drawback. It leads to simple analytical expressions for approximate probabilities of default, equity and debts, of a holding company, both when the complete information about the holding is released in a continuous way and when only incomplete information is available. Numerical applications tend to confirm that probabilities of default estimated from the lower convex bounds are close to these obtained by Monte Carlo simulations.

The main drawback of our approach is that as the driving risk processes used are Brownian motions, default probabilities vanish over a short period. This feature is not necessarily observed in reality. It is possible to remedy to this by using jump processes, but in this case, analytical tractability is lost. Our approach has however other advantages. It duplicates dependent business lines. And as the total firm's asset is a sum of lognormally distributed variables, its distribution presents more asymmetry and leptokurticity than a single lognormal variable. Furthermore the managerial implications of our model are multiple. Asymptotic ruin probabilities may be used as a risk indicator. The management can also use it to determine a threshold triggering the holding's default, so as to maximize the shareholder's interests. And an operator on CDS markets can eventually use our approach for pricing purposes. Finally, the model can be extended to holdings financed by stochastic liabilities. However, numerical results seem less persuasive in the case of stochastic liabilities.

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