Optimal timing for annuitization, based on jump diffusion fund and stochastic mortality.

Donatien Hainaut†, Griselda Deelstra*

March 26, 2014

† *ESC Rennes Business School and CREST, France.*
* *Department of Mathematics, Université Libre de Bruxelles, Belgium .*
Email: donatien.hainaut@esc-rennes.fr, griselda.deelstra@ulb.ac.be

Abstract

Optimal timing for annuitization is developped along three approaches. Firstly, the mutual fund in which the individual invests before annuitization is modeled by a jump diffusion process. Secondly, instead of maximizing an economic utility, the stopping time is used to maximize the market value of future cash-flows. Thirdly, a solution is proposed in terms of Expected Present Value operators: this shows that the non annuitization (or continuation) region is either delimited by a lower or upper boundary, in the domain time-assets return. The necessary conditions are given under which these mutually exclusive boundaries exist. Further, a method is proposed to compute the probability of annuitization. Finally, a case study is presented where the mutual fund is fitted to the S&P500 and mortality is modeled by a Gompertz Makeham law with several real scenarios being discussed.

Keywords : Annuity puzzle, Hitting time, Wiener-Hopf factorization, expected present value.

JEL Classification: J26; G11

1 Introduction.

Buying a fixed-payout life annuity is an efficient solution to preserve standards of living during retirement and it also protects individuals against poverty in old age. The main drawbacks of this type of insurance are its irreversibility and the fact that payments are contingent on the recipient's survival. On the other hand, insurance companies or banks distribute financial products based on mutual funds, designed for people willing to take more risk with their money in exchange for a larger growth potential of their investments. In this context, the literature provides a great deal of evidence that pre-retirement people should invest in such schemes rather than in life insurance products. The question then arises whether and when to switch from such a financial investment to a life annuity.

Numerous papers have covered the various aspects of the annuitization problem since the well-known paper of Yaari (1965), which showed that individuals with no bequest motive should annuitize all their wealth at retirement. By using a shortfall probability approach, Milevsky (1998) considers by the setting up of a Brownian motion fund and using CIR interest rates, the probability of successful deferral, i.e. to defer annuitization as long as investment returns guarantee an income at least equal to that provided by the annuity. Milevsky et al. (2006) derive the optimal investment and annuitization strategies for a retiree whose objective is to minimize the probability of lifetime ruin. Hainaut and Devolder (2006) present a numerical study on the optimal allocation between annuities and financial assets when considering a utility maximization problem. Stabile (2006) examined the optimal annuitization time for a retired individual who is subject to the constant force of mortality in an all-or-nothing framework (i.e. the individual invests all his wealth to buy the annuity) with different utility functions for consumption before and after annuitization. Milevsky and Young (2007) examined optimal annuitization strategies for time-dependent mortality functions based on maximizing the returns from the investment in the case of the all-or-nothing context compared to the case when the individual can annuitize fractions of his wealth at any time. Emms and Haberman (2008) discuss both the optimal annuitization timing and the income draw-down scheme by minimizing a loss function and by using the Gompertz mortality function and a fund based on Brownian motion. Purcal and Piggott (2008) explain the low annuity demand by the relative importance of pre-existing annuitization and by considering utility maximization, a geometric Brownian motion modelling the fund and mortality tables. Horneff et al. (2008) study, using a discrete time model, the optimal gradual annuitization for a retired individual applying Epstein-Zin preferences and quantifying the costs of switching to annuities. Gerrard et al. (2012) take the problem of maximizing the value of the investment to analyze (using a Brownian model and with constant force of mortality) the optimal time of annuitization for a retired individual managing his own investment and consumption strategy. Di Giacinto and Vigna (2012) consider a member of a defined contribution pension fund who has the option of taking programmed withdrawals at retirement. They then explore the sub-optimal cost of immediate annuitization, when minimizing a quadratic cost criterion in a Brownian motion setting and with a constant force of mortality. Huang et al. (2013) are also interested in the problem of optimal timing of annuitization, and especially in the optimal initiation of a Guaranteed Lifetime Withdrawal Benefit (GLWB) in a Variable Annuity. They focus on the problem from the perspective of the policyholder (i.e. when to begin withdrawals from the GLWB) and they adopt a No Arbitrage perspective, (i.e. they assume that the individual is trying to maximize the cost of the guarantee to the insurance company offering the GLWB). Huang et al. (2013) provides a detailed and relevant overview of the literature concerning Variable Annuities and their guarantees.

This paper looks at the optimal timing to switch from a financial investment to a life annuity. It differs from previous publications in several ways. Firstly, the financial asset into which the individual invests (before transferring to annuitization) is modeled by a jump diffusion process instead of a geometric Brownian motion. Numerical applications, by which the return from this asset is fitted to the S&P500 index, reveal that the presence of jumps modifies significantly the point of switching, when compared with the prediction from a Brownian model. Secondly, instead of maximizing an economic utility, the stopping time maximizes the market value of future cash-flows.

When the discount rate is equal to the risk free rate, the objective is the market value or price of future expected discounted payouts. Huang and al. (2013) use a similar criterion for GLWB annuities and interpret it as the cost to the insurance company that provides this service. The investor acts to maximize this cost. In this case and as detailed in the body of the paper, this cost is split into an immediate lifetime payout annuity and an option to defer this annuity. By analogy to a classical American option, the annuitization should only be exercised once the value from waiting is zero, at a point in time when the asset value or return cross a boundary. Stanton (2000) use a similar approach to estimate long-lived put option, embedded in 401(k) pension plans.

Since this problem has similarities with American option pricing, this paper proposes a semi-closed form solution in terms of Expected Present Value (EPV) operators, such as defined by Boyarchenko and Levendorskii (2007). However, for American options pricing, we know beforehand if the boundary delimiting the exercise region is an upper (call) or a lower (put) barrier, in the domain time-accrued return. However, in the current approach, this aspect would not be known at the beginning. On the one hand, a basic reasoning suggests that one should consider switching to annuity if the financial asset performs poorly due to the fear of subsequent erosion of wealth. In this respect the non annuitization (or continuation) region should be delimited by a lower boundary, in the space time versus realized returns. On the other hand, another reasoning leads to consider changing to annuitization when the realized financial return is high enough to receive a reasonable annuity. In this case, the continuation region should be delimited by an upper boundary. The originality of the current study is to present necessary conditions under which these mutually exclusive boundaries exist and a method to compute them.

This reasoning is sustained by empirical observations. Stanton (2000) mentions that in September and October, 1998, more than three times as many pilots of American Airlines retired as during an average month. According to the Wall Street Journal, this surge in retirements was occurring because pilots retiring at this date can take away retirement distributions based on July's high stock-market prices. Similar accelerated retirements occurred after the stock market crash of 1987. On Monday November 2, 1987, over 600 Lockheed Corp. employees had submitted early retirement papers the previous Friday, October 30 (approximately three times the usual monthly figure). Stanton (2000) determines in a Brownian framework, that the investor optimally exercises the option to time their retirement or rollovers to another plan if the asset value cross a boundary.

A third contribution is the assumption of a time dependent current force of mortality, which is contrary to many existing papers (e.g. Stabile 2006 , Gerrard et al. 2012). Finally, this article proposes a method to estimate numerically the probability of annuitization. Of special note is that the solution based on expected value operators can be extended to constant and time dependent consumption/contribution rates, or to planned lump sum payments before annuitization. However, the proposed method does not allow one to dynamically manage the consumption.

Section 2 of this paper presents the dynamics of the financial asset into which the individual invests his savings, before annuitization. Section 3 discusses the current assumptions related to the mortality process. Section 4 introduces the maximisation problem and in particular the objective function. Section 5 reviews the basic working of the Wiener-Hopf factorization that is used in Section 6 to locate the optimal annuitization time. Section 7 presents the Laplace transform of the hitting time of the asset return to reach the boundary that triggers the annuitization. Its numerical inversion provides the probabilities relating to annuitization. This article is concluded by a numerical illustration in which the mutual fund is calibrated to daily returns of S&P500 and with Gompertz Makeham mortality rates. The calibration is done by loglikelihood maximization and the density of the fund return is computed by a Discrete Fourier Transform (see Appendix C). A comparison with the pure Brownian motion case (see Appendix B) as well as several scenarios are then discussed.

2 The wealth process.

A life annuity can preserve the standard of living during retirement but it is an irreversible transaction. Financial advisors propose a wide variety of mutual funds designed for people looking for larger growth potential, and most papers recommend pre-retirement people to invest in this category of product. The question that arises is whether and when to switch from a financial investment to a life annuity. In order to answer this question, this paper considers the situation of an individual who invests all his wealth into a mutual fund and expects to make reasonable profit before converting his investment into a life annuity. The value and return of the fund are respectively modelled by the processes $(W_t)_t$ and $(X_t)_t$. They are stochastic processes defined in a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_t, P)$ and are related in the following way:

$$W_t = W_0 e^{X_t}. \tag{2.1}$$

The return X_t is modelled by a double exponential jump diffusion. This type of process allows a better fit to the actual returns of investment than for models based on Brownian motion. Furthermore, jump diffusion processes include asymmetric and leptokurtic features in modelling asset dynamics. In the numerical applications reported here, this is fitted by loglikelihood maximization to daily figures of the S&P 500 index, observed between June 2003 and June 2013. Some of the main features of the jump diffusion process are firstly considered ahead of proceeding to the calibration method. Lipton (2002), Kou and Wang (2003, 2004) used this process to price options. They define its dynamics by:

$$dX_t = (\theta - \alpha)dt + \sigma d\tilde{W}_t + Y dN_t \quad \text{with} \quad X_0 = 0, \tag{2.2}$$

where θ is the average continuous return from the fund, σ is the constant volatility of the Brownian motion component $\tilde{W}_t$ and α is the constant dividend rate. If α is high compared with the average fund return, it can be interpreted as the withdrawal rate of an immediate variable annuity. Such financial products pay an income equal to a percentage of the fund market value and this income varies depending on the performance of the managed portfolio. A combination of withdrawals and market declines could reduce a variable annuity's account value to zero, in which case the contract would terminate. Huang et al. (2013) give a more complete description of the variable annuity product and its guarantees. If α is negative, it should be interpreted as a contribution rate, paid during the accumulation phase. Note that the contribution/withdrawal rate can possibly be time dependent, $\alpha(t)$. Also some planned lump sums, increasing W_t at discrete times before annuitization, may be considered. Both of these cases are discussed later in this paper under the heading Remark 6.1, but such generalizations do not require any modification of the following developments.

The jump part is modelled by a Poisson process N_t with a constant intensity λ which is independent of the Brownian motion $\tilde{W}_t$. The step increase is distributed as a double exponential variable Y with the following density:

$$f_Y(y) = p\lambda^+ e^{-\lambda^+ y} 1_{\{y\geq 0\}} - (1-p)\lambda^- e^{-\lambda^- y} 1_{\{y<0\}} \tag{2.3}$$

where p and λ^+ are positive constants and λ^- is a negative constant. They represent the probability of observing respectively upward and downward exponential jumps. The expectation of Y is then equal to a weighted sum of expected average jumps:

$$\mathbb{E}(Y) = p\frac{1}{\lambda^+} + (1-p)\frac{1}{\lambda^-}. \tag{2.4}$$

The dynamics of the individual's wealth can be rewritten as:

$$W_t = W_0 e^{X_t} = W_0 e^{(\theta-\alpha)t+\sigma\tilde{W}_t+\sum_{j=1}^{N_t} Y_j}. \tag{2.5}$$

As the jump and diffusion processes are independent, the Laplace transform of X_t is the product of Laplace transforms of the diffusion and jump components. Shreve (2004) gives the Laplace transform of a compound Poisson process as equal to the following expression:

$$\mathbb{E}\left(\exp\left(z\sum_{j=1}^{N_t} Y_j\right)\right) = \exp\left(\lambda t\left(\phi_Y(z)-1\right)\right) \tag{2.6}$$

where $\phi_Y(u)$ is the Laplace transform of Y. If ξ^+ and ξ^- are respectively exponential random variables of intensities λ^+ and λ^-, the function $\phi_Y(z)$ for $\lambda^- < z < \lambda^+$ is given by:

$$\begin{aligned}
\phi_Y(z) &= \mathbb{E}\left(\exp(zY)\right) \\
&= p\mathbb{E}\left(\exp(z\xi^+)\right) + (1-p)\mathbb{E}\left(\exp(-z\xi^-)\right) \\
&= p\frac{\lambda^+}{\lambda^+-z} - (1-p)\frac{\lambda^-}{z-\lambda^-}.
\end{aligned} \tag{2.7}$$

The Laplace transform of X_t is then defined in terms of its related characteristic exponent $\psi(z)$:

$$\mathbb{E}\left(e^{zX_t}\right) = e^{t\psi(z)}$$

where $\psi(z)$ is such that:

$$\begin{aligned}
\psi(z) &= (\theta-\alpha)z + \frac{1}{2}z^2\sigma^2 + \int_{\mathbb{R}} \left(e^{zy}-1\right)\lambda f_Y(dy) \\
&= (\theta-\alpha)z + \frac{1}{2}z^2\sigma^2 + \lambda\left(\phi_Y(z)-1\right).
\end{aligned} \tag{2.8}$$

It has already been noted that the jump diffusion process will be fitted in by loglikelihood maximization to daily returns of the S&P 500 for some numerical applications (section 8). However, the probability density function of returns which is required for such an operation has no closed form expression. This is resolved by computing the discrete Fourier's Transform of its characteristic function and approaching it by a discrete sum, as detailed in Proposition 9.3 in Appendix C.

3 The mortality risk.

This work considers the case for which the investor is required to annuitize all her wealth at one point in time. The optimal age is linked to the actuarial force of mortality and obviously gender specific. But it also depends on the individual's health status, which is unknown from the insurer. Since the development of the theoretical model of Rothschild and Stiglitz (1976), the role of asymmetric information in insurance markets is well identified. Annuitants have more information about their life expectancy than insurance companies and adjust their demand in accordance. To formalize implications of this asymmetric information between the insurance company and annuitants, mortality assumptions used by the insurer differ from these defining the individual's mortality.

The time of the individual's death, denoted by τ_d, is modeled by an inhomogeneous Poisson process in $(\Omega, \mathcal{F}, \{\mathcal{F}\}_t, P)$. The death process is assumed to be independent from N_t and $\tilde{W}_t$. Its intensity, also called mortality rate, is a deterministic function of time, denoted by $\mu(t)$. In this framework, the probability that a person of age η years at time 0 survives the next u years is provided by the following formula:

$$\begin{aligned} {}_u p_\eta &= P(\tau_d > u) \\ &= e^{-\int_0^u \mu(\eta+s)ds}, \end{aligned} \tag{3.1}$$

and the probability that the same person dies during the next u years is ${}_u q_\eta = 1 - {}_u p_\eta$. Moreover, the instantaneous probability of death at time u, is defined by the derivative of ${}_u q_\eta$ with respect to u. This should be understood as the probability that an individual of age η dies between times u and $u + du$:

$$\frac{\partial}{\partial u} {}_u q_\eta = \mu(\eta+u) e^{-\int_0^u \mu(\eta+s)ds} du. \tag{3.2}$$

For a constant discount rate ρ, the expected present value of a lifetime annuity, paying one monetary unit from the point t on until death of the individual is defined as follows,

$$\overline{a}_{\eta+t} = \int_t^{T_m} e^{-\rho(s-t)} {}_{s-t}p_{\eta+t} ds, \tag{3.3}$$

where T_m denotes the maximum lifespan of a human being.

On another hand, the insurance company works with mortality rates and survival probabilities that are respectively denoted by $\mu^{tf}(t)$ and ${}_u p_\eta^{tf} = e^{-\int_0^u \mu^{tf}(\eta+z)dz}$. They are inferred from the observation of a reference population and differ from these of the individual. If the interest rate guaranteed by the insurer is denoted by ρ^{tf}, the annuity coefficient is equal to

$$\overline{a}_{\eta+t}^{tf} = \int_t^{T_m} e^{-\rho^{tf}(s-t)} {}_{s-t}p_{\eta+t}^{tf} ds \tag{3.4}$$

This coefficient determines the annuity payout: if the person purchases the annuity at time t, the cash flow paid by the insurer, noted $B_{\eta+t}$, is calculated by:

$$B_{\eta+t} = \frac{W_t - K}{\overline{a}_{\eta+t}^{tf}} \frac{1}{1-\epsilon}, \tag{3.5}$$

where ϵ is a commercial loading and K is either a fixed acquisition fee ($K > 0$) or a tax incentive ($K < 0$). In later developments, the following ratio

$$f(s) \quad = \quad \frac{1}{1-\epsilon} \frac{\overline{a}_{\eta+s}}{\overline{a}_{\eta+s}^{tf}} \tag{3.6}$$

is used to compare the expected present value of annuity payments with the price paid for the annuity. This conventional measure in actuarial sciences, called the money's worth (Mitchell et al, 1999), is directly related to the gap between individual's mortality rates and these used by the insurer to price the annuity. For individuals who are more healthy on average than the reference population, this function is greater than 100% and the annuity is underpriced. Such persons are also more likely to purchase an annuity as shown further on in numerical illustrations. On the other hand, for the less healthy individuals, the function $f(s)$ is below 100%. The annuity being in this case overpriced by the insurer, early annuitization is less attractive as illustrated later.

4 The objective function.

An investment policy comprises two stages. During the first, the investor both capitalizes on his savings and consumes dividends. In the event of the investor dying, during this period, beneficiaries inherit the accrued capital. When a sufficient profit has been taken or when losses are too great, the individual may then switch and purchase a life annuity. During this second phase, the annuity is consumed. The stopping time is chosen so as to maximize the market value of individual's investment portfolio. Most of the existing publications on annuitization focus on the optimization of expected economic utility of cash-flows. Utility functions measure both preference and risk aversion. However determining the risk aversion parameter of an individual is often a tedious exercise and yet its influence on the annuitization timing is huge, as illustrated by Milevski and Young (2007). Huang et al. (2013) adopt a "no-arbitrage" perspective. In particular, these authors assume that the individual is trying to maximize the cost of the GLWB guarantee to the insurance company offering this service.

Based on a purely financial point of view, this paper uses the market value as the optimization criterion. This value is the sum of expected discounted future payments. The discount rate used in the calculation is assumed constant in this paper and is henceforth denoted by ρ. Exponential discounting factors have been chosen for the ease of the calculations, but further study might be necessary to select a model that is more suitable for addressing aspects of the interest risk associated with the valuation of long-term issues, (such as pension matters), which have a social dimension. Brody et al. (2013) discusses this in greater detail.

The moment at which the person purchases the annuity, denoted by τ, depends both on his age and on his available wealth. A first constraint comes from practical commercial observations. Indeed, in practice, insurers refuse to sell annuities to the elderly in order to limit the risk of anti-selection. Let us denote this age by $\tilde{T}_m + \eta$, so that a person aged η years at time 0 will reach the maximal age in $\tilde{T}_m$ years. Before reaching this age, the annuitization is triggered when the accrued financial return crosses an unknown boundary, in the domain time-assets return. This limit is denoted by b_t and $\mathcal{C}$ denotes the region of the domain $[0,\tilde{T}_m]\times\mathbb{R}$ on which it is optimal to postpone the purchase of the annuity (also called continuation region). In the following discussion, its complementary is denoted by $\bar{\mathcal{C}}$.

A first basic reasoning suggests that the invididual should switch to an annuity if the financial asset performs poorly, due to the fear of subsequent erosion of wealth. In this respect, the continuation region should be delimited by a lower boundary,

$$\mathcal{C}=\left\{(t,x)\,|\,0\leq t\leq\tilde{T}_m\,,\,W_0e^x\geq b_t\right\}.$$

The purchase time τ is then defined as $\inf\{s\,|\,W_s\leq b_s, s\geq t\}\wedge\tilde{T}_m$. However an alternative reasoning leads to considering annuitization only when the financial return achieved is high enough to provide a reasonable annuity. In this case, the continuation region should be delimited by an upper limit,

$$\mathcal{C}=\left\{(t,x)\,|\,0\leq t\leq\tilde{T}_m\,,\,W_0e^x\leq b_t\right\}.$$

The purchase time τ is then equal to $\inf\{s\,|\,W_s\geq b_s, s\geq t\}\wedge\tilde{T}_m$. At this stage, it is not possible to determine whether $\mathcal{C}$ is the upper part or the lower part of the domain $[0,\tilde{T}_m]\times\mathbb{R}$. One can only guess that they are mutually exclusive. The necessary conditions (such that they are indeed mutually exclusive) is given later (section 6) along with specifying the type of boundary linked to the actuarial and financial parameters.

The objective pursued by the investor at a time $t\leq\tilde{T}_m$, is to determine the boundary maximizing the market value of his portfolio. This value of future discounted cash-flows is denoted by $V(t,X_t)$ and is defined for an elapsed time $t\leq\tilde{T}_m$ as

$$\begin{aligned}V(t,X_t) &= \max_\tau\mathbb{E}\left(\int_t^{\tau\wedge\tau_d\wedge\tilde{T}_m}e^{-\rho(s-t)}\alpha W_s ds+e^{-\rho(\tau_d-t)}1_{\tau_d\leq(\tau\wedge\tilde{T}_m)}W_{\tau_d}\right. \\ &\qquad\left.+\int_{\tau\wedge\tilde{T}_m\wedge\tau_d}^{\tau_d}e^{-\rho(s-t)}B_{\eta+(\tau\wedge\tilde{T}_m)}ds\,|\,\mathcal{F}_t\right),\end{aligned}\tag{4.1}$$

whereas $V(\tilde{T}_m,X_{\tilde{T}_m})=\mathbb{E}\left(\int_{\tilde{T}_m\wedge\tau_d}^{\tau_d}e^{-\rho(s-\tilde{T}_m)}B_{\eta+\tilde{T}_m}ds\,|\,\mathcal{F}_{\tilde{T}_m}\right)$ if there was no conversion of funds before reaching $\tilde{T}_m$. Given that the time of death is independent from the filtration of financial returns X_t, the value function for $t\leq\tilde{T}_m$ is rewritten as follows

$$\begin{aligned}V(t,X_t) &= \max_\tau\mathbb{E}\left(\int_t^{\tau\wedge\tilde{T}_m}e^{-\rho(s-t)}\left({}_{s-t}p_{\eta+t}\alpha+\frac{\partial}{\partial s}{}_{s-t}q_{\eta+t}\right)W_s ds\right. \\ &\qquad\left.+\int_{\tau\wedge\tilde{T}_m}^{T_m}e^{-\rho(s-t)}{}_{s-t}p_{\eta+t}\,B_{\eta+\tau\wedge\tilde{T}_m}ds\,|\,\mathcal{F}_t\right) \\ &= \max_\tau\mathbb{E}\left(\int_t^{\tau\wedge\tilde{T}_m}e^{-\int_t^s(\rho+\mu(\eta+u))du}\left(\alpha+\mu(\eta+s)\right)W_s ds\right. \\ &\qquad\left.+\int_{\tau\wedge\tilde{T}_m}^{T_m}e^{-\int_t^s(\rho+\mu(\eta+u))du}B_{\eta+\tau\wedge\tilde{T}_m}ds\,|\,\mathcal{F}_t\right).\end{aligned}\tag{4.2}$$

In view of equations (3.3) and (3.5), the second term of this last expectation is equal to

$$\int_{\tau\wedge\tilde{T}_m}^{T_m}e^{-\int_t^s(\rho+\mu(\eta+u))du}B_{\eta+\tau\wedge\tilde{T}_m}ds = e^{-\int_t^{\tau\wedge\tilde{T}_m}(\rho+\mu(\eta+u))du}\left(W_{\tau\wedge\tilde{T}_m}-K\right)\frac{1}{1-\epsilon}\frac{\overline{a}_{\eta+\tau\wedge\tilde{T}_m}}{\overline{a}^{tf}_{\eta+\tau\wedge\tilde{T}_m}}$$

where $\frac{1}{1-\epsilon}\frac{\overline{a}_{\eta+\tau\wedge\tilde{T}_m}}{\overline{a}^{tf}_{\eta+\tau\wedge\tilde{T}_m}} = f(\tau\wedge\tilde{T}_m)$ is the money's worth, as defined by equation (3.6), that compares the expected present value of annuity payments with the price paid for the annuity. This function is directly related to the gap between real mortality rates and those used by the insurer to price the annuity. For persons who are more healthy on average than those used as a reference population by the insurer, this function will be greater than 100%. On the other hand, for the less healthy individuals, the function $f(\tau)$ will be below 100%. The value function can then be rewritten as follows for the range $t\leq\tilde{T}_m$:

$$\begin{aligned}V(t,X_t) &= \max_{\tau}\mathbb{E}\left(\int_t^{\tau\wedge\tilde{T}_m} e^{-\int_t^s(\rho+\mu(\eta+u))du}\left(\alpha+\mu(\eta+s)\right)W_s ds+\right.\\&\qquad\left.+e^{-\int_t^{\tau\wedge\tilde{T}_m}(\rho+\mu(\eta+u))du}\left(W_{\tau\wedge\tilde{T}_m}-K\right)f(\tau\wedge\tilde{T}_m)\,|\,\mathcal{F}_t\right)\end{aligned}\tag{4.3}$$

and similarly $V(\tilde{T}_m,X_{\tilde{T}_m})=\left(W_{\tilde{T}_m}-K\right)f(\tilde{T}_m)$ (if there is no conversion before reaching $\tilde{T}_m$).

From the theory of stochastic control (e.g. Fleming and Rishel 1975), for a given boundary, the value function is the solution of the following system of equations for $(t\leq s\leq\tilde{T}_m)$:

$$\begin{cases}\frac{\partial V(s,x)}{\partial s}-(\rho+\mu(\eta+s))\,V(s,x)+\mathcal{L}V(s,x) & \\ \qquad =-(\alpha+\mu(\eta+s))\,W_t e^{(x-X_t)} & \text{for } (s,x)\in\mathcal{C}\\ V(s,x)=\left(W_t e^{(x-X_t)}-K\right)f(s) & \text{for } (s,x)\in\bar{\mathcal{C}},\end{cases}\tag{4.4}$$

where $\mathcal{L}u(x)$ is the infinitesimal generator of the process X_t, as defined by:

$$\mathcal{L}u(x) = (\theta-\alpha)\frac{\partial u}{\partial x}+\frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2}+\lambda\mathbb{E}\left(u(x+Y)-u(x)\right),\tag{4.5}$$

and with the following terminal condition $V(\tilde{T}_m,x)=\left(W_{\tilde{T}_m}-K\right)f(\tilde{T}_m)$ (if no conversion before $\tilde{T}_m$). The continuation region is delimited by an optimal boundary $h_s:=\ln\left(\frac{b_s}{W_0}\right)$ and is set so to guarantee the continuity of the value function on the boundary:

$$V(s,h_s) = \left(W_t e^{(h_s-X_t)}-K\right)f(s).$$

At the time of writing, the authors were unaware of a closed form solution for systems as represented by equation (4.4). Thus, trying to solve it directly by a finite difference method is far from straightforward. For this reason, another approach, combining the Wiener-Hopf factorization and time stepping, was used.

5 Wiener-Hopf factorization.

The fundamental principles of the Wiener-Hopf factorization are now considered along with the expected present value operators (EPV-operator) such as defined by Boyarchenko and Levendorskii (2007). Let $q>0$ be defined as a riskless rate. The expected present value operator EPV of a stream $g(X_t)$ is defined as follows:

$$\left(\mathcal{E}_q g\right)(x) = q\mathbb{E}^x\left(\int_0^\infty e^{-qt}g(X_t)\,dt\right),$$

where in general $\mathbb{E}^x\left(g(X_t)\right) = \mathbb{E}\left(g(X_t) \mid X_0 = x\right)$. The following result holds for an exponential function $g(x) = e^{zx}$ by the definition of the Lévy exponent and by direct integration:

$$(\mathcal{E}_q g)(x) = q\mathbb{E}^x\left(\int_0^\infty e^{-qt} g(X_t)\, dt\right) = \frac{qe^{zx}}{q - \psi(z)}, \tag{5.1}$$

which applies under the condition $q > \psi(z)$ where z is real and under the condition $q > \psi(\mathrm{Re} z)$, where z is complex.
Let the two functions $\bar{X}_t = \sup_{0 \le s \le t} X_s$ and $\underline{X}_t = \inf_{0 \le s \le t} X_s$ be respectively the supremum and the infimum of the process X_s on the time interval $[0, t]$. If a random exponential time Γ is introduced, having an intensity equal to q, the Wiener-Hopf factorization is in the case that $X_0 = 0$ for $z \in i\mathbb{R}$:

$$\mathbb{E}^0\left(e^{zX_\Gamma}\right) = \mathbb{E}^0\left(e^{z\bar{X}_\Gamma}\right)\mathbb{E}^0\left(e^{z\underline{X}_\Gamma}\right). \tag{5.2}$$

This relation comes from the observation that $X_\Gamma = \bar{X}_\Gamma + X_\Gamma - \bar{X}_\Gamma$ and the fact that $\bar{X}_\Gamma$ and $X_\Gamma - \bar{X}_\Gamma$ are independent from each other and that $X_\Gamma - \bar{X}_\Gamma$ is distributed like $\underline{X}_\Gamma$. Introducing the notation

$$\kappa_q^+(z) = q\mathbb{E}^0\left(\int_0^\infty e^{-qs} e^{z\bar{X}_s} ds\right) = \mathbb{E}^0\left(e^{z\bar{X}_\Gamma}\right) \tag{5.3}$$

$$\kappa_q^-(z) = q\mathbb{E}^0\left(\int_0^\infty e^{-qs} e^{z\underline{X}_s} ds\right) = \mathbb{E}^0\left(e^{z\underline{X}_\Gamma}\right). \tag{5.4}$$

Since $\mathbb{E}^0\left(e^{zX_\Gamma}\right) = \frac{q}{q-\psi(z)}$, the Wiener-Hopf factorization formula (5.2) can be represented as:

$$\frac{q}{q - \psi(z)} = \kappa_q^+(z)\kappa_q^-(z). \tag{5.5}$$

For any function $g(.)$ defined on $\mathbb{C}$, three EPV operators are defined as follows

$$\begin{aligned} (\mathcal{E}_q g)(x) &= q\mathbb{E}^x\left(\int_0^\infty e^{-qs} g(X_s)\, ds\right) \\ \left(\mathcal{E}_q^+ g\right)(x) &= q\mathbb{E}^x\left(\int_0^\infty e^{-qs} g(\bar{X}_s)\, ds\right) \\ \left(\mathcal{E}_q^- g\right)(x) &= q\mathbb{E}^x\left(\int_0^\infty e^{-qs} g(\underline{X}_s)\, ds\right). \end{aligned} \tag{5.6}$$

The Wiener-Hopf factors $\kappa_q^+(z)$ and $\kappa_q^-(z)$ defined in equation (5.3 and 5.4) are closely related to these EPV operators. Indeed, if $g(.) = e^{z\cdot}$, then

$$\begin{aligned} (\mathcal{E}_q e^{z\cdot})(x) &= \frac{q}{q - \psi(z)} e^{zx} \\ \left(\mathcal{E}_q^+ e^{z\cdot}\right)(x) &= e^{zx}\kappa_q^+(z) \\ \left(\mathcal{E}_q^- e^{z\cdot}\right)(x) &= e^{zx}\kappa_q^-(z) \end{aligned} \tag{5.7}$$

which with equation (5.1) leads to $(\mathcal{E}_q e^{z\cdot}) = \left(\mathcal{E}_q^+ \mathcal{E}_q^- e^{z\cdot}\right)$. It is well-known that the Wiener-Hopf factorization of a given function is unique under weak conditions, in particular, it is unique in case of a rational function that does not vanish on the imaginary line. Boyarchenko and Levendorskii (2007) give a proof of this result for all functions $g \in \mathcal{L}_\infty(\mathbb{R})$. The operator $\mathcal{E}_q$ is the inverse of the operator $q^{-1}(q-\mathcal{L})$ where $\mathcal{L}$ is the infinitesimal generator of the process X_t. Furthermore, $\mathcal{E}_q^{-1} = \left(\mathcal{E}_q^+\right)^{-1}\left(\mathcal{E}_q^-\right)^{-1}$ and $\mathcal{E}_q^{-1} = \left(\mathcal{E}_q^-\right)^{-1}\left(\mathcal{E}_q^+\right)^{-1}$. These results are used in the next section.

Generally, the Wiener-Hopf factors do not have closed form formulae. However, given that $q-\psi(z)$ is the ratio of two polynomials $P(z)$ and $Q(z)$, namely

$$q-\psi(z) = \frac{P(z)}{Q(z)}, \tag{5.8}$$

Boyarchenko and Levendorskii (2007) have proven the uniqueness of the Wiener-Hopf factors and found their expressions. The numerator $P(z)$ is a polynomial of degree 4:

$$\begin{aligned} P(z) &= -\left((\theta-\alpha)z + \frac{1}{2}z^2\sigma^2 - \lambda - q\right)\left(\lambda^+ - z\right)\left(z-\lambda^-\right) \\ &\quad -\lambda p \lambda^+\left(z-\lambda^-\right) + \lambda(1-p)\lambda^-\left(\lambda^+ - z\right), \end{aligned}$$

whereas the denominator $Q(z)$ is the product

$$Q(z) = \left(\lambda^+ - z\right)\left(z-\lambda^-\right).$$

whose positive and negative roots are λ^+ and λ^-. An analysis of variation, reveals that the ratio $(P/Q)(z)$ has two asymptotes located at these roots of $Q(z)$, one thus being located in the left half-plane and the other one in the right half-plane. The polynomial $P(z)$ has 4 real roots. Indeed, it suffices to note that $q-\psi(0)>0$, $q-\psi(z)\to-\infty$ as $z\to\pm\infty$, $z\to\lambda^+-0$ and $z\to\lambda^-+0$, and $q-\psi(z)\to-\infty$ as $z\to\lambda^- -0$ and $z\to\lambda^++0$. Then $P(z)$ crosses four times the zero axis and has two positive and negative roots, denoted by β_k^+ and β_k^-, $k=1,2$ which can be set in the following order:

$$\beta_2^- < \lambda^- < \beta_1^- < 0 < \beta_1^+ < \lambda^+ < \beta_2^+.$$

In this context, the Wiener-Hopf factors are provided by:

$$\kappa_q^+(z) = \frac{\lambda^+ - z}{\lambda^+}\prod_{k=1}^{2}\frac{\beta_k^+}{\beta_k^+ - z} \tag{5.9}$$

$$\kappa_q^-(z) = \frac{\lambda^- - z}{\lambda^-}\prod_{k=1}^{2}\frac{\beta_k^-}{\beta_k^- - z}. \tag{5.10}$$

These Wiener-Hopf factors can also be rewritten as follows

$$\kappa_q^\pm(z) = a_1^\pm\frac{\beta_1^\pm}{\beta_1^\pm - z} + a_2^\pm\frac{\beta_2^\pm}{\beta_2^\pm - z} \tag{5.11}$$

where

$$a_1^\pm = \frac{\beta_2^\pm}{\lambda^\pm}\frac{(\beta_1^\pm - \lambda^\pm)}{(\beta_1^\pm - \beta_2^\pm)}; \qquad a_2^\pm = \frac{\beta_1^\pm(\beta_2^\pm - \lambda^\pm)}{\lambda^\pm(\beta_2^\pm - \beta_1^\pm)}. \tag{5.12}$$

And as shown by Boyarchenko and Levendorskii (2007, page 201), $\mathcal{E}_q^+$ and $\mathcal{E}_q^-$ act on bounded measurable functions $g(.)$ as the following integral operators:

$$\begin{aligned}(\mathcal{E}_q^+ g)(x) &= \sum_{j=1}^{2} a_j^+ \int_0^{+\infty} \beta_j^+ e^{-\beta_j^+ y} g(x+y)dy \\ &= \sum_{j=1}^{2} a_j^+ \int_x^{+\infty} \beta_j^+ e^{\beta_j^+(x-y)} g(y)dy,\end{aligned} \tag{5.13}$$

$$\begin{aligned}(\mathcal{E}_q^- g)(x) &= \sum_{j=1}^{2} a_j^- \int_{-\infty}^{0} (-\beta_j^-) e^{-\beta_j^- y} g(x+y)dy \\ &= \sum_{j=1}^{2} a_j^- \int_{-\infty}^{x} (-\beta_j^-) e^{\beta_j^-(x-y)} g(y)dy.\end{aligned} \tag{5.14}$$

It is also easy to check that these formulae are true for exponential functions $g(x) = e^{zx}$ or for any linear combination of exponential functions. Expressions (5.13) and (5.14) will be used later.

6 Time stepping.

The system (4.4) is solved using the method of Levendorskii (2004), which is a generalization of Carr's randomization to price American put options. Therefore, the time interval $[t, \tilde{T}_m]$ is split into n subperiods of time $t = t_0 < t_1 < ... < t_n = \tilde{T}_m$. Δ_j denotes the time interval between t_j and t_{j+1}. On these intervals of time, functions $(\rho + \mu(\eta + s))$ and b_s are assumed to be constant:

$$\begin{cases} (\rho + \mu(\eta + s)) = (\rho + \mu_j) & \text{if } \ s \in [t_j, t_{j+1}[\\ b_s = b_j & \text{if } \ s \in [t_j, t_{j+1}[\end{cases} \tag{6.1}$$

where $\mu_j = \mu(\eta + t_j)$ and $b_j = b_{t_j}$. The derivative with respect to time present in the system (4.4) is broken into time steps. If $V(t_j, x)$ is denoted by $v_j(x)$ and $f(t_j)$ by f_j, the following discrete version of the system (4.4) is obtained:

$$\begin{cases} v_{j+1}(x) - (1 + \Delta_j(\rho + \mu_j) - \Delta_j \mathcal{L}) v_j(x) = -(\alpha + \mu_j)\Delta_j W_t e^{(x - X_t)} & \text{for } \ (j, x) \in \mathcal{C} \\ v_j(x) = \left(W_t e^{(x-X_t)} - K\right) f_j & \text{for } \ (j, x) \in \bar{\mathcal{C}} \end{cases} \tag{6.2}$$

with $v_n(x) = \left(W_t e^{(x-X_t)} - K\right) f_n$. In order to build a solution in terms of EPV operators, a new function is defined:

$$\tilde{v}_j(x) = v_j(x) - (W_t e^{(x-X_t)} - K) f_j \tag{6.3}$$

which is the difference between the value of the investment policy and the value of purchasing immediately a life annuity. $\tilde{v}_j(x)$ is the value of the option to delay the annuitization and is strictly positive on $\mathcal{C}$. The first equation of (6.2) can be rewritten in terms of $\tilde{v}_j(x)$ as follows

$$\begin{aligned}(1 + \Delta_j(\rho + \mu_j) - \Delta_j \mathcal{L}) \tilde{v}_j(x) &= v_{j+1}(x) + (\alpha + \mu_j)\Delta_j W_t e^{(x-X_t)} \\ &- (1 + \Delta_j(\rho + \mu_j) - \Delta_j \mathcal{L})(W_t e^{(x-X_t)} - K) f_j \quad \text{for } \ (j, x) \in \mathcal{C}\end{aligned} \tag{6.4}$$

and the boundary condition becomes

$$\tilde{v}_j(x) = 0 \qquad \text{for } (j,x) \in \bar{\mathcal{C}}. \tag{6.5}$$

Given that the infinitesimal generator can be reformulated as a function of the characteristic exponent of Y (equation (2.7))

$$\mathcal{L}W_t e^{(x-X_t)} = W_t e^{(x-X_t)} \left((\theta - \alpha) + \frac{1}{2}\sigma^2 + \lambda\left(\phi_Y(1) - 1\right) \right), \tag{6.6}$$

equation (6.4) is rewritten as follows:

$$\begin{aligned}\left(\frac{1}{\Delta_j} + (\rho + \mu_j) - \mathcal{L}\right)\tilde{v}_j(x) = \frac{1}{\Delta_j}v_{j+1}(x) - \\ \left(-(\alpha+\mu_j) + f_j\left(\frac{1}{\Delta_j} + \rho + \mu_j - (\theta-\alpha) - \frac{1}{2}\sigma^2 - \lambda\left(\phi_Y(1)-1\right)\right)\right) W_t e^{(x-X_t)} \\ + \left(\frac{1}{\Delta_j} + (\rho+\mu_j)\right) f_j K \qquad \text{for } (j,x) \in \mathcal{C}.\end{aligned} \tag{6.7}$$

In order to simplify the notation in the following calculations, δ_j is defined as a constant on the interval of time $[t_j, t_{j+1})$:

$$\delta_j := -(\alpha + \mu_j) + f_j\left(\frac{1}{\Delta_j} + \rho + \mu_j - (\theta-\alpha) - \frac{1}{2}\sigma^2 - \lambda\left(\phi_Y(1)-1\right)\right). \tag{6.8}$$

It is now possible to present the solution in terms of EPV of successive functions. In the following propositions, the wealth appearing in the equations is expressed as a function of the individual's initial wealth W_0. Since the period $[t, \tilde{T}_m]$ is considered, the replacing of W_0 by $W_t e^{-X_t}$ in the equations, would better underline the fact that W_t and X_t are known at time t. However, since these formulae would then turn out to be quite long, it is better to work using W_0 for notational use.

Proposition 6.1. *Let us define the function $g_j(.)$ as follows*

$$g_j(x) = \frac{1}{\Delta_j}v_{j+1}(x) - \delta_j W_0 e^x + q_j f_j K \tag{6.9}$$

where

$$q_j = \frac{1}{\Delta_j} + (\rho + \mu_j). \tag{6.10}$$

1) If $g_j(x)$ is monotone decreasing, the value function at time t_j is equal to

$$v_j(x) = (W_0 e^x - K) f_j + q_j^{-1}\left(\mathcal{E}^+_{q_j}\mathbf{1}_{(-\infty, \ln\frac{b_j}{W_0}]}\mathcal{E}^-_{q_j} g_j\right)(x) \tag{6.11}$$

and the continuation region $\mathcal{C}$ is the half plane of $[0, \tilde{T}_m] \times \mathbb{R}$ below the boundary $\ln\frac{b_j}{W_0}$.
2) If $g_j(x)$ is monotone increasing, the value function at time t_j is equal to

$$v_j(x) = (W_0 e^x - K) f_j + q_j^{-1}\left(\mathcal{E}^-_{q_j}\mathbf{1}_{[\ln\frac{b_j}{W_0}, +\infty)}\mathcal{E}^+_{q_j} g_j\right)(x) \tag{6.12}$$

and the continuation region $\mathcal{C}$ is the half plane of $[0, \tilde{T}_m] \times \mathbb{R}$ above the boundary $\ln\frac{b_j}{W_0}$.

Proof. According to equation (6.7) and the expression δ_j, the function $\tilde{v}_j(x)$ is solution of the following system

$$\begin{cases} (q_j - \mathcal{L})\, \tilde{v}_j(x) = g_j(x) & \text{if } (j,x) \in \mathcal{C} \\ \tilde{v}_j(x) = 0 & \text{if } (j,x) \in \bar{\mathcal{C}}. \end{cases} \tag{6.13}$$

Given that $\mathcal{E}_{q_j}^{-1} = q_j^{-1}(q_j - \mathcal{L})$, the system (6.13) implies that

$$\mathcal{E}_{q_j}^{-1} \quad \tilde{v}_j(x) = \quad q_j^{-1} g_j(x) + g_j^{+}(x)$$

where $g_j^{+}(x) := \mathcal{E}_{q_j}^{-1}\tilde{v}_j(x) - q_j^{-1} g_j(x)$ is a function vanishing on $\mathcal{C}$. As $\mathcal{E}_{q_j}^{-1} = \left(\mathcal{E}_{q_j}^{-}\right)^{-1}\left(\mathcal{E}_{q_j}^{+}\right)^{-1}$ and $\mathcal{E}_{q_j}^{-1} = \left(\mathcal{E}_{q_j}^{+}\right)^{-1}\left(\mathcal{E}_{q_j}^{-}\right)^{-1}$, the last equation leads to:

$$\begin{aligned} \left(\mathcal{E}_{q_j}^{+}\right)^{-1} \tilde{v}_j(x) &= q_j^{-1}\mathcal{E}_{q_j}^{-} g_j(x) + \mathcal{E}_{q_j}^{-} g_j^{+}(x) \\ \left(\mathcal{E}_{q_j}^{-}\right)^{-1} \tilde{v}_j(x) &= q_j^{-1}\mathcal{E}_{q_j}^{+} g_j(x) + \mathcal{E}_{q_j}^{+} g_j^{+}(x)\,. \end{aligned} \tag{6.14}$$

In order to proof the statements in 1), it is assumed that the continuation region is defined by the half plane of $[0, \tilde{T}_m] \times \mathbb{R}$ above the given boundary $\ln \frac{b_j}{W_0}$. Then, by construction, $g_j^{+}(x) = 0$ and $\mathcal{E}_{q_j}^{+} g_j^{+}(x) = 0$ for $x \geq \ln\left(\frac{b_j}{W_0}\right)$. From equation (6.14), the price of the option to delay the annuitization should then be equal to:

$$\tilde{v}_j(x) = q_j^{-1}\left(\mathcal{E}_{q_j}^{-}\mathbf{1}_{[\ln \frac{b_j}{W_0}, +\infty)}\mathcal{E}_{q_j}^{+} g_j\right)(x).$$

As $g_j(x)$ is monotone decreasing, $\mathcal{E}_{q_j}^{+} g_j$ and $\mathcal{E}_{q_j}^{-}\mathbf{1}_{[\ln \frac{b_j}{W_0}, +\infty)}\mathcal{E}_{q_j}^{+} g_j$ are also monotone decreasing (see Proposition 10.2.1 given by Boyarchenko and Levendorskii 2007), but it is also a direct consequence of the definition of the EPV operators. Then $\tilde{v}_j(x)$ is monotone decreasing. As $\tilde{v}_j(\ln \frac{b_j}{W_0}) = 0$ to guarantee the continuity of the value function on the boundary, $\tilde{v}_j(\ln \frac{b_j}{W_0}) = 0$ is the maximum of $\tilde{v}_j(.)$ on $\mathcal{C}$. From this, $\tilde{v}_j(.)$ is negative on $\mathcal{C}$ which contradicts the fact that the option to annuitize is strictly positive everywhere on the continuation region.

The assumption is now made that the continuation region is defined by the half plane of $[0, \tilde{T}_m] \times \mathbb{R}$ below the given boundary $\ln \frac{b_j}{W_0}$. Then $g_j^{+}(x) = 0$ for $x \leq \ln\left(\frac{b_j}{W_0}\right)$. By construction, $\mathcal{E}_{q_j}^{-} g_j^{+}(x)$ is null below $\ln\left(\frac{b_j}{W_0}\right)$. From equation (6.14), the price of the option to delay the annuitization is equal to:

$$\tilde{v}_j(x) = q_j^{-1}\left(\mathcal{E}_{q_j}^{+}\mathbf{1}_{(-\infty, \ln \frac{b_j}{W_0}]}\mathcal{E}_{q_j}^{-} g_j\right)(x).$$

As $g_j(x)$ is monotone decreasing $\mathcal{E}_{q_j}^{-} g_j$ and $\mathcal{E}_{q_j}^{+}\mathbf{1}_{(-\infty, \ln \frac{b_j}{W_0}]}\mathcal{E}_{q_j}^{-} g_j$ are also monotone decreasing. In this case $\tilde{v}_j(\ln \frac{b_j}{W_0}) = 0$ is the minimum of $\tilde{v}(x)$ on $\mathcal{C}$ and ensures that $\tilde{v}_j(x)$ is strictly positive on $\mathcal{C}$.

The second statement 2) can be proven by a similar reasoning.

□

The calculation of the EPV operators is done numerically with the method to identify optimal boundaries being given later. Firstly, $\beta_k^{\pm}$ is denoted as the roots of the numerator of $q_j - \psi(z)$ and $a_k^{\pm}$ the related coefficients such as defined by equations (5.12). Then analytical expressions of EPV operators are provided in the next result:

Proposition 6.2. *The value of* $\left(\mathcal{E}_{q_j}^{-} g_j\right)(x)$ *and* $\left(\mathcal{E}_{q_j}^{+} g_j\right)(x)$ *are given by*

$$\left(\mathcal{E}_{q_j}^{-} g_j\right)(x) = -\frac{1}{\Delta_j}\sum_{k=1}^{2} a_k^{-}\beta_k^{-} w_{k,j+1}^{-}(x) - \delta_j W_0 \kappa_{q_j}^{-}(1)\, e^x + q_j f_j K \tag{6.15}$$

$$\left(\mathcal{E}_{q_j}^{+} g_j\right)(x) = \frac{1}{\Delta_j}\sum_{k=1}^{2} a_k^{+}\beta_k^{+} w_{k,j+1}^{+}(x) - \delta_j W_0 \kappa_{q_j}^{+}(1)\, e^x + q_j f_j K \tag{6.16}$$

where the functions $w_{k,j+1}^{-}(.)$ *and* $w_{k,j+1}^{+}(.)$ *are defined as follows*

$$w_{k,j+1}^{-}(.) = e^{\beta_k^{-} x}\int_{-\infty}^{x} e^{-\beta_k^{-} y} v_{j+1}(y)dy, \tag{6.17}$$

$$w_{k,j+1}^{+}(.) = e^{\beta_k^{+} x}\int_{x}^{+\infty} e^{-\beta_k^{+} y} v_{j+1}(y)dy. \tag{6.18}$$

Proof. The result is a direct consequence of equations (5.13) and (5.14) which state that

$$\begin{aligned}
(\mathcal{E}_{q_j}^{+} v_{j+1})(x) &= \sum_{k=1}^{2} a_k^{+}\beta_k^{+} e^{\beta_k^{+} x}\int_{x}^{+\infty} e^{-\beta_k^{+} y} v_{j+1}(y)dy, \\
(\mathcal{E}_{q_j}^{-} v_{j+1})(x) &= -\sum_{k=1}^{2} a_k^{-}\beta_k^{-} e^{\beta_k^{-} x}\int_{-\infty}^{x} e^{-\beta_k^{-} y} v_{j+1}(y)dy
\end{aligned}$$

and also of equation (5.7).

□

In applications, the integrals are computed numerically in order to calculate $w_{k,j+1}^{-}(.)$ and $w_{k,j+1}^{+}(.)$.

Remark 6.1. It is already noted that the distribution/contribution rate can be time dependent, denoted by $\alpha(t)$. In this case, $\alpha(t)$ is approached by a staircase function which is constant between t_j and t_{j+1}. All previous results can be applied by replacing α by α_j in the definition of δ_j. If some lump sum payments are planned before the annuitization, the arguments can be easily adapted. Thus if lump sum payments C are scheduled on the date t_j, then the value function in the definition of $g_j(x)$, equation (6.9), is equal to $v_{j+1}(x) = v_{j+1}(x_+) - C$, where $x_+ = \ln\left(e^x + \frac{C}{W_0}\right)$.

The optimal boundary is determined such that the continuity of the value function is guaranteed on the line delimiting the domain into continuation and annuitization regions (section 4). This means that if $g_j(x)$ is monotone decreasing, $\mathcal{E}_{q_j}^{-} g_j(x) = \mathcal{E}_{q_j}^{-}\tilde{v}_j(x) = 0$ for $x = \ln(b_j/W_0)$. Similarly if $g_j(x)$ is monotone increasing, $\mathcal{E}_{q_j}^{+} g_j(x) = \mathcal{E}_{q_j}^{+}\tilde{v}_j(x) = 0$ for $x = \ln(b_j/W_0)$. The optimal boundaries then easily follow on from the results of Boyarchenko and Levendorskii (2007), as explained in the following corollary.

Corollary 6.3. *When $g_j(.)$ are respectively monotone decreasing or monotone increasing functions with one root, the optimal boundaries $h_j^* = \ln\left(\frac{b_j}{W_0}\right)$ are respectively solutions of*

$$\left(\mathcal{E}_{q_j}^{-} g_j\right)(h_j^*) \quad = \quad -\frac{1}{\Delta_j}\sum_{k=1}^{2} a_k^{-}\beta_k^{-} w_{k,j+1}^{-}(h_j^*) - \delta_j W_0 \kappa_{q_j}^{-}(1)\, e^{h_j^*} + q_j f_j K = 0. \quad (6.19)$$

$$\left(\mathcal{E}_{q_j}^{+} g_j\right)(h_j^*) \quad = \quad \frac{1}{\Delta_j}\sum_{k=1}^{2} a_k^{+}\beta_k^{+} w_{k,j+1}^{+}(h_j^*) - \delta_j W_0 \kappa_{q_j}^{+}(1)\, e^{h_j^*} + q_j f_j K = 0. \quad (6.20)$$

Proof. The proof is a direct consequence of Proposition 10.2.4 of Boyarchenko and Levendorskii (2007).

□

The following proposition presents some necessary conditions satisfied when $g_j(x)$ are monotone increasing or decreasing with one root.

Proposition 6.4. *If the function $g_j(x)$ is monotone increasing with one root then*

$$\frac{1}{\Delta_j} f_{j+1} - \delta_j > 0 \quad \textit{and} \quad \left(q_j f_j - \frac{1}{\Delta_j} f_{j+1}\right) K < 0. \qquad (6.21)$$

If the function $g_j(x)$ is monotone decreasing with one root then

$$\frac{1}{\Delta_j} f_{j+1} - \delta_j < 0 \quad \textit{and} \quad \left(q_j f_j - \frac{1}{\Delta_j} f_{j+1}\right) K > 0. \qquad (6.22)$$

Proof. $g_j(x)$ is defined by equation (6.9)

$$g_j(x) = \frac{1}{\Delta_j} v_{j+1}(x) - \delta_j W_0 e^x + q_j f_j K$$

where $v_{j+1}(x)$ is an increasing function. Furthermore, $v_{j+1}(x) \geq (W_0 e^x - K) f_{j+1}$ in the continuation region $\mathcal{C}$ (if it is not the case, the investor should move directly to a life annuity) and $v_{j+1}(x) = (W_0 e^x - K) f_{j+1}$ in the annuitization region $\bar{\mathcal{C}}$. Given these facts, the following limits are inferred:

$$\begin{aligned} g_j(x) &= \left(\frac{1}{\Delta_j} f_{j+1} - \delta_j\right) W_0 e^x + \left(q_j f_j - \frac{1}{\Delta_j} f_{j+1}\right) K \quad \text{for } x \in \bar{\mathcal{C}} \qquad (6.23)\\ g_j(x) &\geq \left(\frac{1}{\Delta_j} f_{j+1} - \delta_j\right) W_0 e^x + \left(q_j f_j - \frac{1}{\Delta_j} f_{j+1}\right) K \quad \text{for } x \in \mathcal{C}. \end{aligned}$$

If $g_j(x)$ is monotone increasing then $\mathcal{C}$ is bounded from below. Taking equation (6.23), where $x \leq \ln\left(\frac{b_j}{W_0}\right)$ then $\frac{1}{\Delta_j} f_{j+1} - \delta_j$ must be positive. As $g_j(x)$ has one root, then $\lim_{x\to-\infty} g_j(x) < 0$ and $\left(q_j f_j - \frac{1}{\Delta_j} f_{j+1}\right) K < 0$. The same approach can be used to prove the conditions (6.22).

□

Conditions (6.21) or (6.22) are necessary but not sufficient. However, it has been observed in numerical tests that they seem to be sufficient, despite the lack of proof offered by the authors. In any case, these conditions are useful to detect problems for which there appears to be no solution such as developed in Proposition 6.1. When conditions (6.21) or (6.22) are not satisfied, $g_j(x)$ cannot be monotone increasing nor decreasing with one root. In this case, the optimization problem is not well-formulated. To understand what happens in this case, $\frac{1}{\Delta_j} f_{j+1} - \delta_j > 0$ is assumed. Then $g_j(x)$ is bounded from below by an increasing exponential function and if g_j is monotone increasing, $\mathcal{C}$ is delimited by a lower boundary. However, when $\left(q_j f_j - \frac{1}{\Delta_j} f_{j+1}\right) K > 0$, the function $g_j(x)$ is strictly positive everywhere on $\bar{\mathcal{C}}$ and cannot be null on its boundary. In this case, the price of the option to delay the annuitization is positive and increasing with $\ln\left(\frac{b_j}{W_0}\right)$. Choosing $b_j = +\infty$ optimizes then the value function and the recommandation for annuitization never occurs before $\tilde{T}_m$.

If $\frac{1}{\Delta_j} f_{j+1} - \delta_j < 0$, $g_j(x)$ is bounded from below by a decreasing exponential function and if g_j is monotone decreasing, $\mathcal{C}$ is delimited by an upper boundary. Nonetheless when $\left(q_j f_j - \frac{1}{\Delta_j} f_{j+1}\right) K < 0$, the function $g_j(x)$ is then strictly negative everywhere on $\bar{\mathcal{C}}$ and cannot be null on its boundary. The price of the option to delay the annuitization is here negative and decreasing with $\ln\left(\frac{b_j}{W_0}\right)$. Choosing $b_j = -\infty$ is optimal and annuitization should be done immediately.

The next proposition presents the value of $\left(\mathcal{E}^{+}_{q_j}\mathbf{1}_{(-\infty,\ln\frac{b_j}{W_0}]}\mathcal{E}^{-}_{q_j} g_j\right)(x)$.

Proposition 6.5. *The price at t_j of the option to delay annuitization is in the case of a monotone decreasing function $g_j(.)$ equal to*

$$q_j^{-1}\left(\mathcal{E}^{+}_{q_j}\mathbf{1}_{(-\infty,\ln\frac{b_j}{W_0}]}\mathcal{E}^{-}_{q_j} g_j\right)(x) = \Bigg[-\frac{1}{\Delta_j}\sum_{k=1}^{2}\sum_{l=1}^{2} a_k^{-}\beta_k^{-} a_l^{+}\beta_l^{+} z^{+}_{k,l,j+1}(x)$$
$$-\delta_j W_0 \kappa^{-}_{q_j}(1)\sum_{k=1}^{2} a_k^{+}\frac{\beta_k^{+}}{1-\beta_k^{+}} e^{x}\left(e^{(1-\beta_k^{+})(\ln\frac{b_j}{W_0}-x)} - 1\right)$$
$$-q_j f_j K \sum_{l=1}^{2} a_l^{+}\left(e^{\beta_l^{+}(x-\ln\frac{b_j}{W_0})} - 1\right)\Bigg] q_j^{-1} \tag{6.24}$$

and in the case of a monotone increasing function $g_j(.)$ equal to

$$q_j^{-1}\left(\mathcal{E}^{-}_{q_j}\mathbf{1}_{[\ln\frac{b_j}{W_0},+\infty)}\mathcal{E}^{+}_{q_j} g_j\right)(x) = \Bigg[-\frac{1}{\Delta_j}\sum_{k=1}^{2}\sum_{l=1}^{2} a_k^{-}\beta_k^{-} a_l^{+}\beta_l^{+} z^{-}_{k,l,j+1}(x)$$
$$-\delta_j W_0 \kappa^{+}_{q_j}(1)\sum_{l=1}^{2} a_l^{-}\frac{\beta_l^{-}}{(1-\beta_l^{-})} e^{x}\left(e^{(1-\beta_l^{-})(\ln\frac{b_j}{W_0}-x)} - 1\right)$$
$$-q_j f_j K \sum_{l=1}^{2} a_l^{-}\left(e^{\beta_l^{-}\left(x-\ln\frac{b_j}{W_0}\right)} - 1\right)\Bigg] q_j^{-1} \tag{6.25}$$

where the functions $z^{+}_{k,l,j+1}(.)$ *and* $z^{-}_{k,l,j+1}(.)$ *are defined as follows*

$$z^{+}_{k,l,j+1}(.) \quad = \quad e^{\beta_l^+ x}\int_x^{\ln\frac{b_j}{W_0}} e^{-\beta_l^+ y} w^{-}_{k,j+1}(y)dy \quad x \le \ln\frac{b_j}{W_0} \tag{6.26}$$

$$z^{-}_{k,l,j+1}(.) \quad = \quad e^{\beta_l^- x}\int_{\ln\frac{b_j}{W_0}}^{x} e^{-\beta_l^- y} w^{+}_{k,j+1}(y)dy \quad x \ge \ln\frac{b_j}{W_0} \tag{6.27}$$

and are null everywhere else.

Proof. The operator $\mathcal{E}^{+}_{q_j}\mathbf{1}_{(-\infty,\ln\frac{b_j}{W_0}]}\mathcal{E}^{-}_{q_j}g_j$ can be seen as the sum of three terms:

$$\begin{aligned}\left(\mathcal{E}^{+}_{q_j}\mathbf{1}_{(-\infty,\ln\frac{b_j}{W_0}]}\mathcal{E}^{-}_{q_j}g_j\right)(x) &= -\frac{1}{\Delta_j}\sum_{k=1}^{2}a_k^-\beta_k^-\left(\mathcal{E}^{+}_{q_j}\mathbf{1}_{(-\infty,\ln\frac{b_j}{W_0}]}w^{-}_{k,j+1}(x)\right)\\ &-\delta_j W_0\kappa^-_{q_j}(1)\left(\mathcal{E}^{+}_{q_j}\mathbf{1}_{(-\infty,\ln\frac{b_j}{W_0}]}e^x\right)+q_jf_jK\left(\mathcal{E}^{+}_{q_j}\mathbf{1}_{(-\infty,\ln\frac{b_j}{W_0}]}\right).\end{aligned} \tag{6.28}$$

In view of equation (5.13), the first term equals

$$\begin{aligned}\left(\mathcal{E}^{+}_{q_j}\mathbf{1}_{(-\infty,\ln\frac{b_j}{W_0}]}w^{-}_{k,j+1}(x)\right) &= \sum_{l=1}^{2}a_l^+\beta_l^+\int_x^{\ln\frac{b_j}{W_0}} e^{\beta_l^+(x-y)}w^{-}_{k,j+1}(y)dy\\ &= \sum_{l=1}^{2}a_l^+\beta_l^+e^{\beta_l^+ x}\int_x^{\ln\frac{b_j}{W_0}} e^{-\beta_l^+ y}w^{-}_{k,j+1}(y)dy.\end{aligned}$$

A direct calculation leads to the following expression for the second term:

$$\begin{aligned}\left(\mathcal{E}^{+}_{q_j}\mathbf{1}_{(-\infty,\ln\frac{b_j}{W_0}]}e^x\right) &= \sum_{l=1}^{2}a_l^+\beta_l^+\int_x^{\ln\frac{b_j}{W_0}} e^{\beta_l^+(x-y)}e^y dy\\ &= \sum_{l=1}^{2}a_l^+\frac{\beta_l^+}{(1-\beta_l^+)}e^x\left(e^{(1-\beta_l^+)(\ln\frac{b_j}{W_0}-x)}-1\right).\end{aligned}$$

Finally the last term of (6.28) can be rewritten as

$$\begin{aligned}\left(\mathcal{E}^{+}_{q_j}\mathbf{1}_{(-\infty,\ln\frac{b_j}{W_0}]}\right) &= \sum_{l=1}^{2}a_l^+\beta_l^+\int_x^{\ln\frac{b_j}{W_0}} e^{\beta_l^+(x-y)}dy\\ &= -\sum_{l=1}^{2}a_l^+\left(e^{\beta_l^+(x-\ln\frac{b_j}{W_0})}-1\right).\end{aligned}$$

The operator $\mathcal{E}^{-}_{q_j}\mathbf{1}_{[\ln\frac{b_j}{W_0},\infty)}\mathcal{E}^{+}_{q_j}g_j$ is obtained in a similar way.

□

The functions $z^{+}_{k,l,j+1}(.)$ and $z^{-}_{k,l,j+1}(.)$ are computed numerically. Section 8 presents some results in order to illustrate the feasibility of the method.

The algorithm 1 summarizes the backward procedure and main steps implemented to retrieve the optimal boundaries in numerical applications.

Algorithm 1 Backward calculation of upper or lower boundaries.

Initialize $v_n(x) = (W_0 e^x - K)\, f_n$ (compulsory annuitization at time $\tilde{T}_m$)
For $j = n-1$ to 0

1. Calculation of $g_j(x) = \frac{1}{\Delta_j} v_{j+1}(x) - \delta_j W_0 e^x + q_j f_j K$,
2. Numerical search of β_2^- , β_1^- , β_1^+ ,β_2^+, λ_- and λ_+ defining the Wiener Hopf factors,$\kappa_{q_j}^-(z)$, $\kappa_{q_j}^+(z)$
3. Valuation of $\mathcal{E}_{q_j}^- g_j(x)$ or of $\mathcal{E}_{q_j}^+ g_j(x)$,
4. Numerical search of $h_j^* = \ln \frac{b_j}{W_0}$, root of $\mathcal{E}_{q_j}^- g_j(x) = 0$ or $\mathcal{E}_{q_j}^+ g_j(x) = 0$,
5. Valuation of the option to annuitize $\mathcal{E}_{q_j}^+ \mathbf{1}_{(-\infty, \ln \frac{b_j}{W_0}]} \mathcal{E}_{q_j}^- g_j(x)$ or $\mathcal{E}_{q_j}^- \mathbf{1}_{([\ln \frac{b_j}{W_0}, \infty,)} \mathcal{E}_{q_j}^+ g_j(x)$,
6. Update of the value function: $v_j(x) = (W_0 e^x - K)\, f_j + \textit{option to annuitize}$

next j

7 Probability of annuitization.

As the investor has the right to withdraw his money from the mutual fund at any moment to purchase a life annuity, the fund manager faces in certain circumstances a surrender risk. For example, in France, due to tax incentives, insurers and bankers are encouraged to invest their savings in funds with private equity. These funds with non listed stocks issued by SME's (small and medium enterprises) provide a higher return in exchange for their liquidity risk. However if the motivation to withdraw money becomes strong, large outflows of money can cause liquidity shortages. Understanding the probabilities of annuitization are thus helpful to manage this risk. They can either be calculated by Monte Carlo simulations or by inverting the Laplace transform of the hitting time τ. The second approach is considered here. By definition, for a given constant γ, the Laplace transform of τ is given by

$$\begin{aligned} \mathbb{E}\left(e^{-\gamma\tau} \,|\, \mathcal{F}_t\right) &= \gamma \int_t^{+\infty} e^{-\gamma s} P(\tau \leq s \,|\, \mathcal{F}_t) ds \\ &= \gamma \mathcal{L}_\gamma (P(\tau \leq s \,|\, \mathcal{F}_t)) \end{aligned} \tag{7.1}$$

where $\mathcal{L}_\gamma$ is the Laplace operator. The probability that the individual leaves the mutual fund to purchase to a life annuity, is then obtained by inverting this operator:

$$\begin{aligned} P(\tau \leq s \,|\, \mathcal{F}_t) &= \mathcal{L}_\gamma^{-1} \left(\frac{1}{\gamma} \mathbb{E}\left(e^{-\gamma\tau} \,|\, \mathcal{F}_t\right) \right) \\ &= \frac{1}{2\pi i} \lim_{T \to \infty} \int_{\gamma_0 - iT}^{\gamma_0 + iT} e^{\gamma s} \frac{1}{\gamma} \mathbb{E}\left(e^{-\gamma\tau} \,|\, \mathcal{F}_t\right) d\gamma \end{aligned}$$

where γ_0 is larger than the real part of all singularities of $\mathbb{E}\left(e^{-\gamma\tau} \,|\, \mathcal{F}_t\right)$. It is known that the Laplace transform is a function of the fund return, X_t:

$$\mathbb{E}\left(e^{-\gamma\tau} \,|\, \mathcal{F}_t\right) := u(t, X_t)$$

and it is solution of the following system

$$\begin{cases} \frac{\partial u(s,x)}{\partial s} + (\mathcal{L} - \gamma)\, u(s,x) = 0 & \text{if } \; x \in \mathcal{C} \\ u(t,x) = 1 & \text{if } \; x \in \bar{\mathcal{C}}. \end{cases} \tag{7.2}$$

where $\mathcal{L}$ is the infinitesimal generator of X_t. The authors are unaware of any analytical solutions for this system, but it is possible to compute numerical estimates by time stepping. Once again, the time interval $[t, \tilde{T}_m]$ is split into n subperiods of time: $t = t_0 < t_1 < ... < t_n = \tilde{T}_m$. The term Δ_j is the length of the time interval between t_j and t_{j+1}. On these intervals, b_s is assumed constant and $b_s = b_j$ if $s \in [t_j, t_{j+1})$. Discretizing the derivative with respect to time in equation (7.2) and denoting $u(t_j, x)$ by $u_j(x)$, lead to

$$\begin{cases} u_{j+1}(x) - (1 + \Delta_j \gamma - \Delta_j \mathcal{L})\, u_j(x) = 0 & \text{for } \; x \in \mathcal{C} \\ u_j(x) = 1 & \text{for } \; x \in \bar{\mathcal{C}} \end{cases} \tag{7.3}$$

where $u_n(x) = 0$. The Laplace transform can be obtained in terms of EPV operators, as shown previously in Section 6. To achieve this, the following function is introduced

$$\tilde{u_j}(x) = u_j(x) - 1 \tag{7.4}$$

and equations (7.3) are rewritten as follows:

$$\begin{cases} \left(\frac{1}{\Delta_j} + \gamma - \mathcal{L}\right) \tilde{u}_j(x) = \frac{1}{\Delta_j} u_{j+1}(x) - \left(\frac{1}{\Delta_j} + \gamma\right) & \text{for } \; x \in \mathcal{C} \\ \tilde{u}_j(x) = 0 & \text{for } \; x \in \bar{\mathcal{C}}. \end{cases} \tag{7.5}$$

Since this last system is similar to (6.2), the following results are inferred:

Corollary 7.1. *Defining the function $g_j^u(.)$ as follows*

$$g_j^u(x) \;=\; \frac{1}{\Delta_j} u_{j+1}(x) - q_j^u \tag{7.6}$$

where $q_j^u = \frac{1}{\Delta_j} + \gamma$. If $\mathcal{C} = \left\{(t,x) \,|\, 0 \le t \le \tilde{T}_m \,,\; x \le \ln\left(\frac{b_j}{W_0}\right)\right\}$, the Laplace transform at time t_j for $j = n-1, n-2, \ldots, 0$ is equal to

$$u_j(x) \;=\; 1 + \left(q_j^u\right)^{-1} \left(\mathcal{E}_{q_j^u}^{+} \mathbf{1}_{(-\infty, \ln \frac{b_j}{W_0}]} \mathcal{E}_{q_j^u}^{-} g_j^u\right)(x). \tag{7.7}$$

If $\mathcal{C} = \left\{(t,x) \,|\, 0 \le t \le \tilde{T}_m \,,\; x \ge \ln\left(\frac{b_j}{W_0}\right)\right\}$, then

$$u_j(x) \;=\; 1 + \left(q_j^u\right)^{-1} \left(\mathcal{E}_{q_j^u}^{-} \mathbf{1}_{[\ln \frac{b_j}{W_0}, \infty)} \mathcal{E}_{q_j^u}^{+} g_j^u\right)(x). \tag{7.8}$$

Proof. According to equations (7.5), the function $\tilde{u}_j(x)$ is solution of the following system

$$\begin{cases} \left(q_j^u - \mathcal{L}\right) \tilde{u}_j(x) = g_j^u(x) & \text{if } \; x \in \mathcal{C} \\ \tilde{u}_j(x) = 0 & \text{if } \; x \in \bar{\mathcal{C}}. \end{cases} \tag{7.9}$$

Given that $\mathcal{E}_{q_j^u}^{-1} = q_j^{u-1}\left(q_j^u - \mathcal{L}\right)$, the system (7.9) implies that

$$\mathcal{E}_{q_j^u}^{-1} \quad \tilde{u}_j(x) = \quad q_j^{u-1} g_j^u(x) + g_j^{u+}(x)$$

where $g_j^{u+}(x) := \mathcal{E}_{q_j^u}^{-1}\tilde{u}_j(x) - q_j^{u-1} g_j^u(x)$ is a function vanishing on $\mathcal{C}$. As $\mathcal{E}_{q_j^u}^{-1} = \left(\mathcal{E}_{q_j^u}^{-}\right)^{-1}\left(\mathcal{E}_{q_j^u}^{+}\right)^{-1}$ and $\mathcal{E}_{q_j^u}^{-1} = \left(\mathcal{E}_{q_j^u}^{+}\right)^{-1}\left(\mathcal{E}_{q_j^u}^{-}\right)^{-1}$, the last equation leads to the following observations:

$$\begin{aligned}
\left(\mathcal{E}_{q_j^u}^{+}\right)^{-1} \tilde{u}_j(x) &= q_j^{u-1}\mathcal{E}_{q_j^u}^{-} g_j^u(x) + \mathcal{E}_{q_j^u}^{-} g_j^{u+}(x) \\
\left(\mathcal{E}_{q_j^u}^{-}\right)^{-1} \tilde{u}_j(x) &= q_j^{u-1}\mathcal{E}_{q_j^u}^{+} g_j^u(x) + \mathcal{E}_{q_j^u}^{+} g_j^{u+}(x).
\end{aligned}$$

If

$$\mathcal{C} = \left\{(t_j, x)\,|\, 0 \le t_j \le \bar{T}_m\,,\ x \le \ln\left(\frac{b_j}{W_0}\right)\right\},$$

then $g_j^{u+}(x)$ is null for $x \le \ln\left(\frac{b_j}{W_0}\right)$. By construction, $\mathcal{E}_{q_j}^{-} g_j^{u+}(x)$ and $\tilde{u}_j(x)$ are respectively null below and above $\ln\left(\frac{b_j}{W_0}\right)$. Then,

$$\tilde{u}_j(x) = q_j^{u-1}\left(\mathcal{E}_{q_j^u}^{+}\mathbf{1}_{(-\infty, \ln\frac{b_j}{W_0}]}\mathcal{E}_{q_j^u}^{-} g_j^u\right)(x).$$

In the same way, if

$$\mathcal{C} = \left\{(t_j, x)\,|\, 0 \le t_j \le \bar{T}_m\,,\ x \ge \ln\left(\frac{b_j}{W_0}\right)\right\},$$

then $g_j^{u+}(x)$ is null for $x \ge \ln\left(\frac{b_j}{W_0}\right)$. By construction, $\mathcal{E}_{q_j^u}^{+} g_j^{u+}(x)$ and $\tilde{u}_j(x)$ are respectively null above and below $\ln\left(\frac{b_j}{W_0}\right)$. This leads to the result which remains to be proven:

$$\tilde{u}_j(x) = q_j^{u-1}\left(\mathcal{E}_{q_j^u}^{-}\mathbf{1}_{[\ln\frac{b_j}{W_0}, \infty)}\mathcal{E}_{q_j^u}^{+} g_j^u\right)(x).$$

□

If $\beta_k^{\pm}$ denotes the roots of the numerator of $q_j^u - \psi(z)$ and $a_k^{\pm}$ are the related coefficients such as defined by equations (5.12), then the following corollary provides an analytical expression for EPV operators:

Corollary 7.2. *The EPV operators* $\left(\mathcal{E}_{q_j^u}^{-} g_j^u\right)$ *and* $\left(\mathcal{E}_{q_j^u}^{+} g_j^u\right)$ *are equal to*

$$\left(\mathcal{E}_{q_j^u}^{-} g_j^u\right)(x) = -\frac{1}{\Delta_j}\sum_{k=1}^{2} a_k^{-}\beta_k^{-} w_{k,j+1}^{u-}(x) - q_j^u \tag{7.10}$$

$$\left(\mathcal{E}_{q_j^u}^{+} g_j^u\right)(x) = \frac{1}{\Delta_j}\sum_{k=1}^{2} a_k^{+}\beta_k^{+} w_{k,j+1}^{u+}(x) - q_j^u \tag{7.11}$$

where the functions $w^{u-}_{k,j+1}$ *and* $w^{u+}_{k,j+1}$ *are defined by*

$$w^{u-}_{k,j+1}(.) \quad = \quad e^{\beta^-_k x} \int_{-\infty}^{x} e^{-\beta^-_k y} u_{j+1}(y) dy \tag{7.12}$$

$$w^{u+}_{k,j+1}(.) \quad = \quad e^{\beta^+_k x} \int_{x}^{\infty} e^{-\beta^+_k y} u_{j+1}(y) dy. \tag{7.13}$$

Corollary 7.3. *In the second terms of (7.7) and (7.8), the EPV operators are equal to*

$$\begin{aligned}\left(\mathcal{E}^+_{q_j} \mathbf{1}_{(-\infty, \ln \frac{b_j}{W_0}]} \mathcal{E}^-_{q_j} g_j\right)(x) = & -\frac{1}{\Delta_j} \sum_{k=1}^{2} \sum_{l=1}^{2} a^-_k \beta^-_k a^+_l \beta^+_l z^{u+}_{k,l,j+1}(x) \\ & + q^u_j \sum_{l=1}^{2} a^+_l \left(e^{\beta^+_l (x - \ln \frac{b_j}{W_0})} - 1 \right)\end{aligned} \tag{7.14}$$

$$\begin{aligned}\left(\mathcal{E}^-_{q^u_j} \mathbf{1}_{[\ln \frac{b_j}{W_0}, \infty)} \mathcal{E}^+_{q^u_j} g^u_j\right)(x) = & -\frac{1}{\Delta_j} \sum_{k=1}^{2} \sum_{l=1}^{2} a^-_k \beta^-_k a^+_l \beta^+_l z^{u-}_{k,l,j+1}(x) \\ & + q^u_j \sum_{l=1}^{2} a^-_l \left(e^{\beta^-_l (x - \ln \frac{b_j}{W_0})} - 1 \right)\end{aligned} \tag{7.15}$$

where the functions $z^{u+}_{k,l,j+1}(.)$ *and* $z^{u-}_{k,l,j+1}(.)$ *are given by*

$$z^{u+}_{k,l,j+1}(.) \quad = \quad e^{\beta^+_l x} \int_{x}^{\ln \frac{b_j}{W_0}} e^{-\beta^+_l y} w^{u-}_{k,j+1}(y) dy \quad x \leq \ln \frac{b_j}{W_0} \tag{7.16}$$

$$z^{u-}_{k,l,j+1}(.) \quad = \quad e^{\beta^-_l x} \int_{\ln \frac{b_j}{W_0}}^{x} e^{-\beta^-_l y} w^{u+}_{k,j+1}(y) dy \quad x \geq \ln \frac{b_j}{W_0} \tag{7.17}$$

and are null everywhere else.

Proofs of these Corollaries 7.2 and 7.3 are identical to those for Propositions 6.2 and 6.5. Because, the Laplace transform of the default time is known, the Gaver-Stehfest algorithm can be used to numerically invert it. This approach is detailed by Davies (2002, chapter 19). Denoting $F(\gamma) = \frac{1}{\gamma} \mathbb{E}\left(e^{-\gamma\tau} \,|\, \mathcal{F}_t\right)$. Let N be an integer. Then, an approximation of the inverse is provided by the following sum:

$$P(\tau \leq s \,|\, \mathcal{F}_t) \quad \approx \quad \frac{\ln 2}{(s-t)} \sum_{j=1}^{N} \gamma_j F\left(\frac{\ln 2}{(s-t)} j\right) \tag{7.18}$$

where

$$\gamma_j \quad = \quad (-1)^{N/2+j} \sum_{k=\left[\frac{j+1}{2}\right]}^{min(j,N/2)} \frac{k^{N/2}(2k)!}{(N/2-k)!k!(k-1)!(j-k)!(2k-j)!}.$$

In numerical applications, it is recommended to work with N set as 12. Note that the Gaver-Stehfest algorithm is sometimes numerically unstable. In this case, probabilities of annuitization can be obtained from Monte Carlo simulation of X_t.

8 Numerical application.

This section presents annuitization regions for a male individual investing his savings in a mutual fund tracking the S&P 500 index. The related mortality rates $\mu(\eta + t)$ are represented by a Gompertz Makeham distribution such as detailed in Appendix A. The annuitization must occur before the age of 80 ($\eta + \tilde{T}_m = 80$). This choice is motivated by the fact that insurers refuse to sell annuities to the elderly in order to limit the risk of anti-selection. The time step used in the time stepping procedure is chosen to be equal to a half year ($\Delta_i = 0.5$).

The jump diffusion process that models the mutual fund return is fitted (by loglikelihood maximization), to daily figures of the S&P500, from June 2003 to June 2013. The parameters are presented in Table 8.1. The drift θ of X_t is high (16.15%) but the average yearly return, without dividend, is equal[1] to 2.38%. The difference between this drift and this average return, 13.77%, corresponds to the yearly expected growth of the jump component. The volatility of the Brownian motion is 3.92% but the standard deviation of the yearly return is greater at 8.61%. In order to assess the impact of jumps on the optimal boundaries, the jump diffusion model will be compared later with a pure Brownian model set up, with the same mean and volatility.

The discount rate and initial wealth are set as $\rho = 3\%$ and $W_0 = 100$. In a first scenario, the function $f(t)$ as defined by equation (3.6) is constant ($f(t) = 100\%$), K is a positive fee ($K = 2$, 2% of W_0) and the dividend rate is $\alpha = 0.5\%$. The average return of the mutual fund after dividends, is in this case $\mathbb{E}(X_1) = 1.88\%$. As $f(t)$ is equal to 100%, the individual has the same anticipation regarding his own survival as that viewed by the insurer, (or at least he is not suspicious about the purchase of an annuity given its irreversibility). With these assumptions, the necessary conditions (6.22) are satisfied. Moreover, numerical tests reveal that all functions $g_j(x)$ are all monotone decreasing, with a single root (this has to be checked because conditions (6.21) and (6.22) are necessary but not sufficient). As demonstrated in Proposition 6.1, the continuation region is delimited by an upper boundary.

In a second scenario, K is a tax incentive ($K = -2$, -2% of W_0), the dividend rate is set to $\alpha = 1\%$ and the drift θ is slightly increased to 16.65%. Under these assumptions, the average mutual fund return remains unchanged when compared with the first scenario ($\mathbb{E}(X_1) = 1.88\%$), but higher dividends are expected. These assumptions ensure that conditions (6.21) are satisfied. Furthermore, numerical tests reveal that all functions $g_j(x)$ are monotone increasing with one single root. The continuation region in this scenario is delimited by a lower boundary. Therefore annuitization should occur only if the accrued return falls off too sharply.

In both considered scenarios, the money's worth is constant $f(t) = f$, and $\rho + \mu_j > 0\,\forall j$.

[1] $\mathbb{E}(X_1) = \theta - \alpha + \lambda(p\frac{1}{\lambda^+} - (1-p)\frac{1}{\lambda^-})$ and $\sigma(X_1) = \mathbb{V}(X_1)^{1/2}$ with $\mathbb{V}(X_1) = \sigma^2 + 2\lambda(p\frac{1}{(\lambda^+)^2} + (1-p)\frac{1}{(\lambda^-)^2})$

It follows that necessary conditions (6.21) and (6.22) can respectively be restated as:

$$\frac{(1-f)}{f}(\mu_j+\alpha)+\ln\mathbb{E}(e^{X_1+\alpha})>\rho \qquad K<0, \tag{8.1}$$

$$\frac{(1-f)}{f}(\mu_j+\alpha)+\ln\mathbb{E}(e^{X_1+\alpha})<\rho \qquad K>0. \tag{8.2}$$

In these equations,

$$\begin{aligned}\ln\mathbb{E}(e^{X_1+\alpha}) &= \ln\mathbb{E}(\frac{W_0e^{X_1+\alpha}}{W_0}) \\ &= \theta+\frac{1}{2}\sigma^2+\lambda\left(\phi_Y(1)-1\right)\end{aligned} \tag{8.3}$$

is a kind of measure of financial performance, and is called log-average return in the remainder of this paragraph. This estimates the global performance of the fund prior dividends, and is independent from the dividends rate. In practice, the spread between mortality rates of the individual and of the reference population for the insurer, is never huge and f is close to one. Therefore, the first terms of equations (8.1) or (8.2) are nearly insignificant. Unless a high withdrawal or contribution rate, α has a marginal effect on the necessary conditions. When $f=1$, the existence of a lower boundary is only conditioned to the fact that the log-average return dominates the risk free rate $\ln\mathbb{E}(e^{X_1+\alpha})>\rho$, and that a tax incentive exists, $K<0$. In absence of a such incentive, the lower boundary does not exist if the log-average return is greater than the risk free rate. Indeed, as discussed in the paragraph following Proposition 6.4, it is then never recommended to annuitize before $\tilde{T}_m$ because the option to delay the annuitization is positive, whatever the accrued return.

On another hand, when $f=1$, an upper boundary exists under the conditions that the log-average return $\ln\mathbb{E}(e^{X_1+\alpha})$ is smaller than ρ and that there is a positive acquisition fee, $K>0$. In absence of a such fee, or in presence of a tax incentive, the upper boundary cannot exist when $\ln\mathbb{E}(e^{X_1+\alpha})<\rho$. In this case, whatever the accrued return, the option to delay the annuitization is negative as mentioned in the discussion following Proposition 6.4. The right decision consists thus in converting the fund immediately in an annuity.

	Jump Diffusion		Brownian
θ	16.15%/16.65%	$\tilde{\theta}$	2.38%
σ	3.92%	$\tilde{\sigma}$	8.61%
p	0.3825		
λ	148.2928		
λ^+	217.1081		
λ^-	-229.5335		
Log. Lik.	10200	Log. Lik.	9720

Table 8.1: Parameters fitting the S&P 500 index

α	0.5% /1%	$\eta+\tilde{T}_m$	80
W_0	100	K	$+2/-2$
ρ	3%	$f(t)$	1.00

Table 8.2: Other parameters.

Figure 8.1 presents optimal boundaries in the domain time-accrued return and probabilities of annuitization, for different initial ages, η set as 40, 50 and 60 years. Left and right upper graphs show these boundaries in respectively the first and the second scenario. The annuitization occurs before 80 years old, if the path followed by the accrued return starting from $X_0 = 0$, crosses one of these boundaries, either from below (left graph) or from above (right graph).

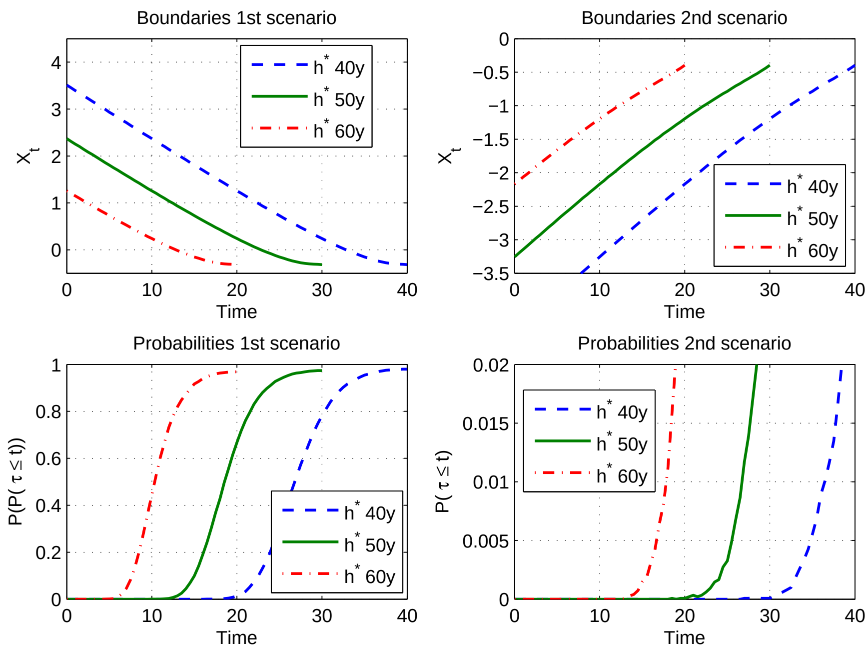

Figure 8.1: Optimal boundaries triggering the annuitization and the probabilities of annuitization, for different initial ages.

In the first scenario, the purchase of a life annuity is postponed till the financial return achieved is high enough. The individual waits until the rise in capital can ensure a comfortable annuity. If the fund performs poorly ($X_t \leq -0.30$, or $W_t \leq 74$), the annuitization should be delayed to the limiting age of 80 years. However, the probability of such a late annuitization is less than 2%. Furthermore, an analysis of probabilities graphs reveals that annuitization occurs in 95% of cases before the investor is 75 years old. On average, (as shown in Table 8.3), the annuity is purchased between the ages of 67 and 71 years.

In the second scenario, the purchase of the annuity is postponed unless the accrued return falls off too rapidly. The probability to annuitize before the age of 80 years, is lower than 2%. Moreover, on average, (as shown in Table 8.3), the annuity is purchased between 79 and 80 years old. Despite a tax incentive, the individual has no interest in investing too early in a fixed payout annuity, except if the mutual fund slumps. This is mainly explained by the higher dividend rate paid in this second scenario (1% instead of 0.5%).

Age	$\mathbb{E}(\eta + \tau \,|\, \eta)$, 1st scenario	Age	$\mathbb{E}(\eta + \tau \,|\, \eta)$, 2nd scenario
$\eta = 40\, years$	67.77	$\eta = 40\, years$	79.95
$\eta = 50\, years$	69.41	$\eta = 50\, years$	79.96
$\eta = 60\, years$	71.14	$\eta = 60\, years$	79.97

Table 8.3: Average age for annuitization, as a function of the initial age of the individual. These expected ages are computed with probabilities of annuitization, presented in Figure 8.1.

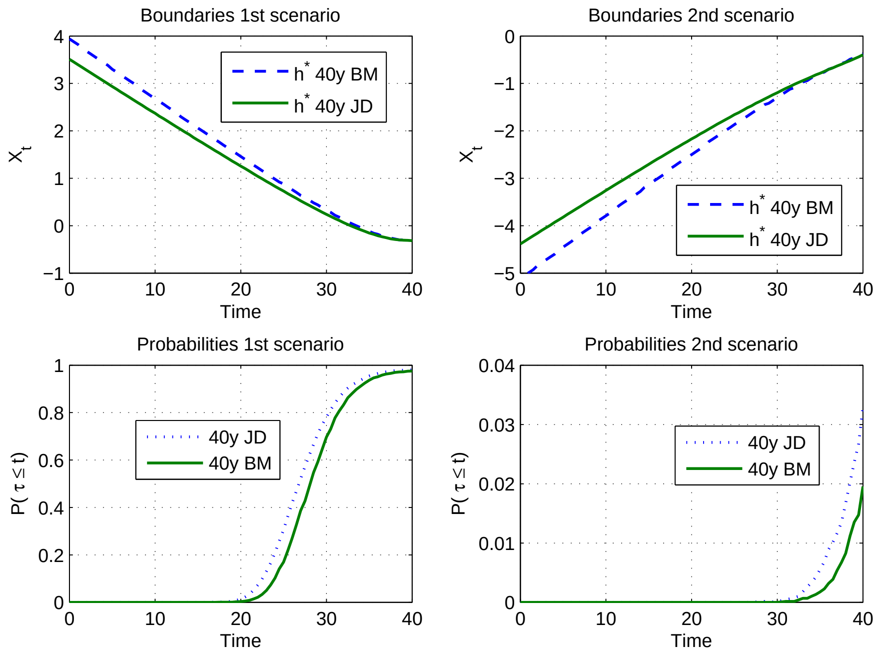

Figure 8.2: Comparison of boundaries and probabilities of annuitization for Brownian Motion and Jump Diffusion processes. Initial age : 40 years.

Figure 8.2 compares optimal boundaries and probabilities of annuitization, when the S&P 500 return is modeled by pure Brownian motion (blue dotted line) compared to a jump diffusion process (green continuous line). The optimal boundaries in the Brownian model are set as described in Appendix B. The presence of jumps in the fund dynamics influences the shape of optimal boundaries. In the first scenario (left graph), the Brownian boundary is higher than the one for the jump diffusion model. On the other hand, for the second scenario (right graph), the Brownian boundary is dominated by the one of jump diffusion. This leads to different probabilities for annuitization. For a given maturity, the probability to annuitize is predicted in a Brownian framework to be lower than in a jump diffusion model. For the first scenario described above, a comparison of Tables 8.3 and 8.4 reveals that for the Brownian model, the annuity is purchased on average 1 year later than in case of the jump diffusion model. For the second scenario presented, annuitization is delayed until reaching 80 years old, whatever the chosen model.

Age	$\mathbb{E}(\eta+\tau \mid \eta)$, 1st scenario	Age	$\mathbb{E}(\eta+\tau \mid \eta)$, 2nd scenario
$\eta = 40\, years$	68.77	$\eta = 40\, years$	79.96
$\eta = 50\, years$	70.30	$\eta = 50\, years$	79.96
$\eta = 60\, years$	71.88	$\eta = 60\, years$	79.97

Table 8.4: The average ages for annuitization when X_t is modeled by a Brownian motion. Different initial ages of the investor are used.

Intuitively, these results can be explained as follows. Despite that both processes have the same averages and volatilities, the jump diffusion has heavier tails than the Brownian motion. The tails of the distribution decays slowly at infinity and very large moves have a significant probability of occurring. Due to these large moves, the process X_t may reach the boundary at an earlier point than a pure Brownian motion. This triggers an anticipate annuitization and raises probabilities of conversion, for a given maturity. On another hand, a jump diffusion can generate sudden, discontinuous moves in prices, contrary to a Brownian motion. Therefore, sometimes it may incur an 'overshoot' over the boundary. Optimal boundaries are then adjusted to mitigate the risk to annuitize when X_t is already deeply in the stopping region.

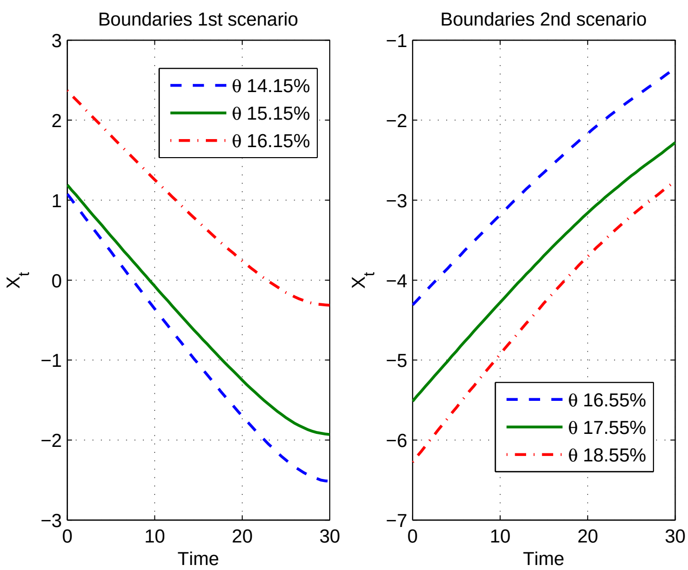

Figure 8.3: Influence of the drift factor θ on the location of the optimal boundaries.

	$\mathbb{E}(\eta+\tau \mid \eta)$, 1st scenario		$\mathbb{E}(\eta+\tau \mid \eta)$, 2nd scenario
$\theta = 14.15\%$	56.95	$\theta = 16.15\%$	79.96
$\theta = 15.15\%$	58.55	$\theta = 17.15\%$	80.00
$\theta = 16.15\%$	69.41	$\theta = 18.15\%$	80.00

Table 8.5: Average age for annuitization, for various drift factors.

Figure 8.3 shows the boundaries in case of a 50 year old man and for different drift factors θ. In the first scenario (left graph) with a drift of 14.15%, the average fund return (after dividends) is close to zero (-0.12%). The lack of expected capital gains does not encourage an investment in the mutual fund. This absence of incentive pushes down the upper boundary in comparison with higher drift rates. Moreover, (as illustrated in Table 8.5), the annuitization occurs on average at younger ages.

In the second scenario (right graph), if $\theta = 18.55\%$, the yearly fund return (after dividends) is 3.88%. These high expected capital gains represent an important incentive for investing in the mutual fund and the high dividends ensure a comfortable income before annuitization. Therefore, there is no reason in this case to purchase a fixed payout annuity, except if the financial markets slump. When the drift increases in the second scenario, the delimiting boundary is pushed down and annuitization is postponed.

Since the recent financial crisis, people fear to invest in mutual funds because of their volatility. As illustrated in Figure 8.4, the volatility is also involved in the decision to annuitize. The right and left graphs analyze for the first and second scenarios, the sensitivity of the boundaries to the Brownian motion volatility (σ) in the jump diffusion setting. In both cases, when σ rises, the steepness of the boundaries decreases. Table 8.6 shows that on average for the first scenario, a higher volatility delays the annuitization.

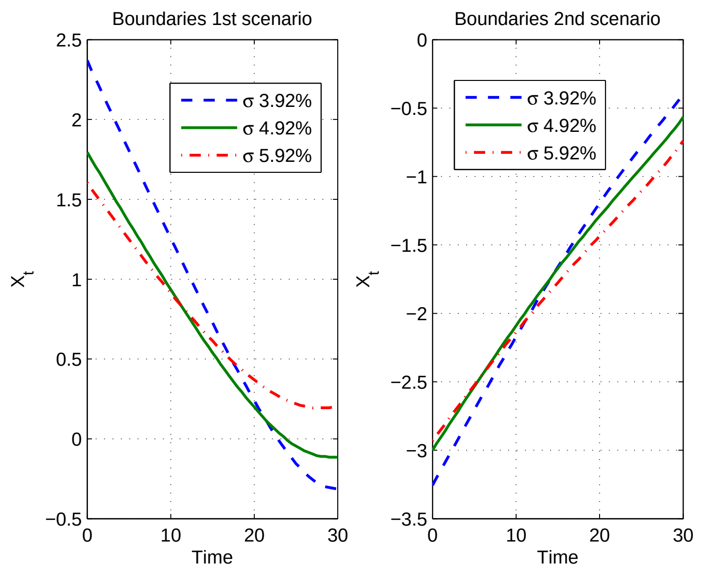

Figure 8.4: Influence of volatility on optimal boundaries.

	$\mathbb{E}(\eta + \tau \mid \eta)$, 1st scenario		$\mathbb{E}(\eta + \tau \mid \eta)$, 2nd scenario
$\sigma = 3.92\%$	69.41	$\sigma = 3.92\%$	79.95
$\sigma = 4.92\%$	68.84	$\sigma = 4.92\%$	79.98
$\sigma = 5.92\%$	71.23	$\sigma = 5.92\%$	79.99

Table 8.6: Average age for annuitization, for various volatilities.

The money's worth $f(t)$ measures the spread between individual's mortality rates $\mu(x+t)$, and these of the insurer's reference population, $\mu^{tf}(x+t)$. If $f(t)$ is above or below 100%, the expected present value of annuity payments is respectively greater or lower than the price paid for the annuity. It plays an important role in the decision to annuitize, as illustrated by Figure 8.5. In the first scenario (left graph), increasing $f(t)$ pushes down the upper boundary. Because $f(t)$ is not involved in the dynamics of X_t, this process will on average reach the boundary at an earlier point when $f(t)$ is high. Therefore, the annuity is purchased earlier on average. This conclusion is supported by results of Table 8.7: the annuitization occurs on average at younger age when $f(t)$ is significantly higher than 100%. In this case, the annuity is indeed underpriced and the annuitant benefits from the asymmetry of information between the insurance company and himself. This represents a strong incentive to annuitize. In the second scenario (right graph), increasing $f(t)$ pushes up the boundary. As shown in Table 8.7, the consequence of such movement is similar to the one observed in the first scenario: on average the annuitization happens earlier, but the impact is less important. This leads to the conclusion that whatever the type of boundary, an individual who has a better longevity than an average person of the insurer's reference population, will be interested in purchasing a life annuity at an earlier point.

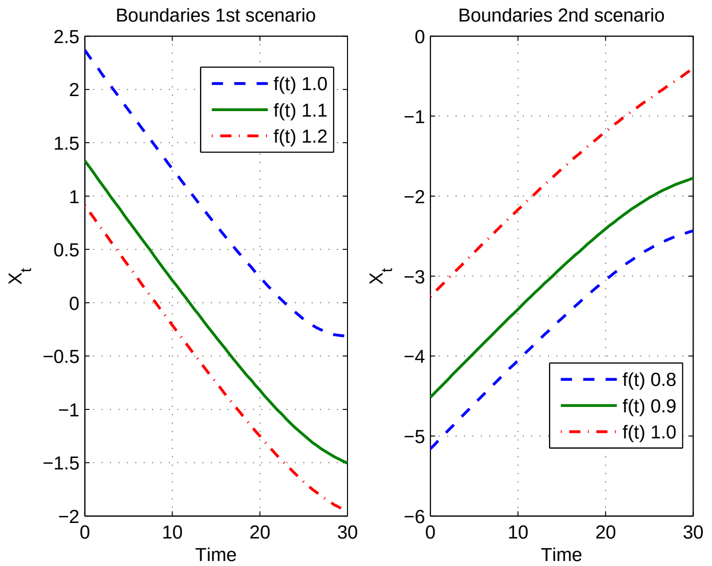

Figure 8.5: Influence of $f(t)$ on optimal boundaries.

	$\mathbb{E}(\eta + \tau \,\vert\, \eta)$, 1st scenario		$\mathbb{E}(\eta + \tau \,\vert\, \eta)$, 2nd scenario
$f(t) = 1.0$	69.41	$f(t) = 1.0$	79.96
$f(t) = 1.1$	60.62	$f(t) = 0.9$	79.98
$f(t) = 1.2$	57.33	$f(t) = 0.8$	79.99

Table 8.7: Average age for annuitization, for various values of $f(t)$.

9 Conclusions.

The literature provides a great deal of evidence that an investor who intends to purchase a life annuity (in an 'all or nothing' format) will be induced to delay if alternative financial investments are available. This paper presents some new aspects of this optimal timing problem, for an individual looking to optimize the market value of his investment strategy.

The expected financial return from assets purchased before annuitization is driven by a jump diffusion process, whereas most of existing studies use a Brownian motion framework. A case study is presented that reveals that the presence of jumps in asset dynamics substantially modifies the shape of the boundaries delimiting the annuitization region.

The solution is presented in terms of Expected Present Value (EPV) operators. These were initially developed to price American options by Boyarchenko and Levendorskii (2007) but such operators are not widely used in the actuarial literature, despite their efficiency. A procedure to estimate the probability of conversion has been developed.

However, the main contribution from the current study has been to show the existence of upper or lower mutually exclusive boundaries, which define the continuation region in the space time versus realized returns. Contrary to working with American options, it is not known beforehand if the boundary delimiting the exercise region is an upper or a lower barrier. Propositions are set out that bind the type of limits to assumptions on (or relations between) the actuarial and financial parameters. When the financial fund tracks the S&P 500 and under realistic mortality assumptions, two different scenarios are numerically considered. In the first, the annuitization only occurs if the achieved return reaches an upper boundary, whereas in the second (with only slightly higher dividends), the annuitization only occurs in the case of poor financial performances.

There are several relevant topics for future research. One would be to consider a partial annuitization of the individual's wealth. Another improvement could be to model the fact that before the age of retirement, an investor should buy deferred annuities, which (by definition) only start paying out from the age of retirement (since annuitizing before the age of retirement has indeed only little practical sense). Finally, the utility optimization of consumption deserves a deeper investigation since this problem leads to a Bellman equation that appears unsolvable by EPV operators.

Appendix A, mortality assumptions.

In the examples presented in this paper, the real mortality rates $\mu(x+t)$ are assumed to follow a Gompertz Makeham distribution. The chosen parameters are those defined by the Belgian regulator ("Arrêté Vie 2003") for the pricing of life annuities purchased by males. For an individual of age x, the mortality rate is given by:

$$\mu(x) = a_\mu + b_\mu . c_\mu^x \qquad a_\mu = -\ln(s_\mu) \qquad b_\mu = \ln(g_\mu). \ln(c_\mu)$$

where the parameters s_μ, g_μ, c_μ take the values given in Table 9.1. As an example Table 9.2 presents the progression of mortality rates with age for the male individual.

Table 9.1: Belgian legal parameters for modeling mortality rates, for life insurance products, targetting a male population.

s_μ:	0.999441703848
g_μ:	0.999733441115
c_μ :	1.101077536030

Table 9.2: Mortality rates, predicted by the Gompertz Makeham model based on parameters of table 9.1.

Age x	$\mu(x)$
30	0.10%
40	0.18%
50	0.37%
60	0.88%
70	2.23%
80	5.74%

Appendix B, Pure Brownian motion.

This appendix presents results when the financial return on assets is driven by a pure Brownian motion without any jumps. These are used in the preceding numerical applications section to estimate the impacts of jumps on the boundaries delimiting the annuitization area. For the remainder of this section, the dynamics of X_t are reduced to:

$$dX_t \;=\; (\tilde{\theta}-\alpha)dt + \tilde{\sigma}d\tilde{W}_t \quad \text{with} \quad X_0 = 0, \tag{9.1}$$

and its characteristic exponent $\psi(z)$ is a second order polynomial:

$$\psi(z) \;=\; (\tilde{\theta}-\alpha)z + \frac{1}{2}z^2\tilde{\sigma}^2.$$

If β^+ and β^- are respectively positive and negative roots of $q-\psi(z)=0$,

$$\beta^+ \;=\; \frac{-(\tilde{\theta}-\alpha)+\sqrt{(\tilde{\theta}-\alpha)^2+2\tilde{\sigma}^2 q}}{\sigma^2}$$

$$\beta^- \;=\; \frac{-(\tilde{\theta}-\alpha)-\sqrt{(\tilde{\theta}-\alpha)^2+2\tilde{\sigma}^2 q}}{\sigma^2}$$

the Wiener-Hopf factors are provided by:

$$\kappa_q^+(z) \;=\; \frac{\beta^+}{\beta^+-z} \tag{9.2}$$

$$\kappa_q^-(z) \;=\; \frac{\beta^-}{\beta^--z}. \tag{9.3}$$

In this case, the EPV operators $\mathcal{E}_q^+$ and $\mathcal{E}_q^-$ act on bounded measurable functions $g(.)$ as follows:

$$(\mathcal{E}_q^+ g)(x) \;=\; \int_x^{+\infty} \beta^+ e^{\beta^+(x-y)} g(y)dy,$$

$$(\mathcal{E}_q^- g)(x) \;=\; \int_{-\infty}^{x} (-\beta^-) e^{\beta^-(x-y)} g(y)dy.$$

The value function, rewritten in terms of EPV operators, is still provided by Proposition 6.1 (given earlier) if we define δ_j as the following constant on the interval of time $[t_j, t_{j+1})$:

$$\delta_j \;:=-\,(\alpha+\mu_j) + \;f_j\left(\frac{1}{\Delta_j}+\rho+\mu_j-(\tilde{\theta}-\alpha)-\frac{1}{2}\tilde{\sigma}^2\right). \tag{9.4}$$

Proposition 6.2 has the following analogue in the Brownian motion model:

Corollary 9.1. *The value of* $\left(\mathcal{E}^{-}_{q_j} g_j\right)(x)$ *and* $\left(\mathcal{E}^{+}_{q_j} g_j\right)(x)$ *in the Brownian model, are given by*

$$\left(\mathcal{E}^{-}_{q_j} g_j\right)(x) = -\frac{1}{\Delta_j}\beta^{-} w^{-}_{j+1}(x) - \delta_j W_0 \kappa^{-}_{q_j}(1)\, e^{x} + q_j f_j K \tag{9.5}$$

$$\left(\mathcal{E}^{+}_{q_j} g_j\right)(x) = \frac{1}{\Delta_j}\beta^{+} w^{+}_{j+1}(x) - \delta_j W_0 \kappa^{+}_{q_j}(1)\, e^{x} + q_j f_j K \tag{9.6}$$

where the functions $w^{-}_{j+1}(.)$ *and* $w^{+}_{j+1}(.)$ *are defined as follows*

$$w^{-}_{j+1}(.) = e^{\beta^{-} x}\int_{-\infty}^{x} e^{-\beta^{-} y} v_{j+1}(y)dy, \tag{9.7}$$

$$w^{+}_{j+1}(.) = e^{\beta^{+} x}\int_{x}^{+\infty} e^{-\beta^{+} y} v_{j+1}(y)dy. \tag{9.8}$$

Furthermore, the optimal boundary is given by a simplified version of Corollary 6.3 (given earlier).

Corollary 9.2. *When* $g_j(.)$ *are respectively monotone decreasing or monotone increasing functions with one root, the optimal boundaries* $h^{*}_{j} = \ln\left(\frac{b_j}{W_0}\right)$ *are respectively the solutions of*

$$\left(\mathcal{E}^{-}_{q_j} g_j\right)(h^{*}_{j}) = -\frac{1}{\Delta_j}\beta^{-} w^{-}_{j+1}(h^{*}_{j}) - \delta_j W_0 \kappa^{-}_{q_j}(1)\, e^{h^{*}_{j}} + q_j f_j K = 0. \tag{9.9}$$

$$\left(\mathcal{E}^{+}_{q_j} g_j\right)(h^{*}_{j}) = \frac{1}{\Delta_j}\beta^{+}_{k} w^{+}_{j+1}(h^{*}_{j}) - \delta_j W_0 \kappa^{+}_{q_j}(1)\, e^{h^{*}_{j}} + q_j f_j K = 0. \tag{9.10}$$

Proposition 6.4 (given earlier) remains valid if the return is modeled by a pure Brownian motion and therefore, some necessary conditions satisfied when $g_j(x)$ are monotone increasing or decreasing are given by (6.21) and (6.22) with the appropriate parameters such as δ_j in (9.4).

Appendix C, Numerical calculation of the density.

The jump diffusion process is adjusted by a loglikelihood maximization to daily figures of the S&P 500 from June 2003 to June 2013. Since the density of the returns has no closed form expression, it is retrieved numerically by a discrete Fourier Transform. Indeed, the density, denoted by $f_{X_t}(.)$, is approached on the interval $[-x_{max}, x_{max}]$ by a sum as stated in Proposition 9.3 (below).

Proposition 9.3. *Let* N *be the number of steps used in the Discrete Fourier Transform (DFT) and* $\Delta_x = \frac{2x_{max}}{N-1}$ *be the step of stepping. Let us denote* $\delta_j = \frac{1}{2}1_{\{j=1\}} + 1_{\{j\neq 1\}}$, $\Delta_z = \frac{2\pi}{N\,\Delta_x}$ *and* $z_j = (j-1)\,\Delta_z$. *The values of* $f_{X_t}(.)$ *at points* $x_k = -\frac{N}{2}\Delta_x + (k-1)\Delta_x$ *are approached by*

$$f_{X_t}(x_k) = \frac{2}{N\,\Delta_x}\sum_{j=1}^{N}\delta_j\left(e^{t\psi(i z_j)}(-1)^{j-1}\right)e^{-i\frac{2\pi}{N}(j-1)(k-1)} \tag{9.11}$$

where $\psi(z)$ *is defined by equation (2.8).*

Proof. By definition, the characteristic function of X_t, denoted $M_{X_t}(z)$, is the inverse Fourier transform of the density multiplied by 2π:

$$\begin{aligned} M_{X_t}(z) &= \int_{-\infty}^{+\infty} f_{X_t}(x) e^{izx} dx \\ &:= 2\pi \mathcal{F}^{-1}[f_{X_t}(x)](z). \end{aligned}$$

The density is retrieved by calculating the Fourier transform of $M_{X_t}(z) = e^{t\psi(iz)}$ as follows

$$\begin{aligned} f_{X_t}(x) &= \frac{1}{2\pi} \mathcal{F}[e^{t\psi(iz)}](x) \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{t\psi(iz)} e^{-ixz} dz \\ &= \frac{1}{\pi} \int_{0}^{+\infty} e^{t\psi(iz)} e^{-ixz} dz. \end{aligned}$$

The last equality arises from the fact that $\psi(iz)$ and $\psi(-iz)$ are complex conjugate. Approaching this last integral with the trapezoid rule $\int_a^b h(x)dx = \left[\frac{h(a)+h(b)}{2} + \sum_{k=1}^{N-1} h(a + k\,\Delta_x)\right] \Delta_x$, leads to the result.

$\square$

Acknowledgments

The authors acknowledges support of the ARC grant IAPAS "Interaction between Analysis, Probability and Actuarial Sciences" 2012-2017. The authors also wish to thank Enrico Biffis, Elena Vigna and Lane Hughston for their advices. They are also grateful to Serguei Levendorskii for his encouragement and suggestions. The authors also thank the referees sincerely for their constructive remarks.

References

[1] Boyarchenko S., Levendorskii S. 2007. Irreversible Decisions under Uncertainty. Springer Editions.

[2] Brody D.C., Hughston, L.P. 2013. Social Discounting and the Long Rate of Interest. ArXiv: 1306.5145.

[3] Carr P. 1998. Randomization and the American Put. Review of Financial Studies, 11, 597-626.

[4] Davidoff T., Brown J., Diamond P. 2003., Annuities and individual welfare. NBER Working Paper 9714, May 2003.

[5] Davies B. 2002. Integral Transforms and Their Applications. Springer Third Edition.

[6] Di Giacinto M., Vigna E. 2012. On the sub-optimality cost of immediate annuitization in DC pension fund. Central European Journal of Operations Research, 20(3), 497-527.

[7] Emms P., Haberman S. 2008. Income drawdown schemes for a defined-contribution pension plan. The Journal of Risk and Insurance, 75 (3), 739-761.

[8] Fleming W.H., R.W.Rishel. 1975. Deterministic and Stochastic Optimal Control. Springer Editions.

[9] Gerrard R., Højgaard B., Vigna E. 2012. Choosing the optimal annuitization time post-retirement. Quantitative Finance, 12(7), 1143-1159.

[10] Hainaut D., Devolder P. 2006. Life annuitization: why and how much? ASTIN bulletin, 36(2), 629-654.

[11] Horneff W. J., Maurer R. H., Stamos M. Z. 2008. Optimal Gradual Annuitization: Quantifying the Costs of Switching to Annuities. Journal of Risk and Insurance, 75(4), 1019-1038.

[12] Huang H., Milevsky M. A., Salisbury T.S. 2013. Optimal Initiation of a GLWB in a Variable Annuity: No Arbitrage Approach. ArXiv:1304.1821 [q-fin.PM]

[13] Kou S.G., Wang H. 2003. First Passage Times of a Jump Diffusion Process. Adv. Appl. Prob., 35, 504-531.

[14] Kou S.G., Wang H. 2004. Option Pricing under a Double Exponential Jump Diffusion Model. Management Science, 50(9), 1178-1192.

[15] Levendorskii S.Z. 2004. Pricing of the American put under Lévy processes. Int. J. Theor. Appl. Finan., 7(3), 303-335.

[16] Lipton A. 2002. Assets with Jumps. Risk, September, 149–153.

[17] Milevsky M. A. 1998. Optimal Asset Allocation towards the End of the Life Cycle: To Annuitize or Not to Annuitize? The Journal of Risk and Insurance, 65(3), 401-426.

[18] Milevsky, M. A., Moore, K. S., Young, V. R. 2006. Optimal asset allocation and ruin-minimization annuitization strategies, Mathematical Finance, 16, 647-671.

[19] Milevsky, M. A., Young, V. R. 2007. Annuitization and asset allocation, Journal of Economic Dynamics and Control 31: 3138{3177.

[20] Mitchel, O.S., Poterba J.M., Warshawsky M.J. and Brown J.R. 1999. New Evidence on the Money's Worth of Individual Annuities. American Economic Review, 89, 1299-1318.

[21] Purcal S., Piggott J. 2008. Explaining Low Annuity Demand: An Optimal Portfolio Application to Japan. Journal of Risk and Insurance, 75(2), 493-516.

[22] Rothschild, M., Stiglitz, J. 1976. Equilibrium in competitive insurance markets: An essay on the economics of imperfect information . Quarterly Journal of Economics , 90(4), 629 - 649.

[23] Shreve S. 2004. Stochastic Calculus for Finance II: Continuous-Time Models. Springer Finance Editions.

[24] Stabile G. 2006. Optimal timing of the annuity purchase: combined stochastic control and optimal stopping problem. Int. J. Theor. Appl. Finan., 9 (2), 151-170.

[25] Stanton, R. 2000. From cradle to grave: How to loot a 401(k) plan. Journal of Financial Economics, 56, 485-516.

[26] Usabel M. 1999. Calculating Multivariate Ruin Probabilities via Gaver-Stehfest Inversion Technique. Insurance: Mathematics and Economics 25(2), 133-142.

[27] Yaari M.E. 1965. Uncertain lifetime, life insurance and the theory of the consumer, Review of Economic Studies, 32, 137-150.