

**MINIMIZING THE (CONDITIONAL) VALUE-AT-RISK  
FOR A COUPON-BEARING BOND USING A BOND PUT OPTION.**

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**Abstract**

In this paper, we elaborate a formula for determining the optimal strike price for a bond put option, used to hedge a position in a bond. This strike price is optimal in the sense that it minimizes, for a given budget, either Value-at-Risk or Conditional Value-at-Risk. Formulas are derived for both zero coupon and coupon bonds, which can also be understood as a portfolio of bonds. These formulas are valid for any short rate model with a given distribution of future bond prices.

## **1. INTRODUCTION**

The importance of a sound risk management system can hardly be underestimated. The advent of new capital requirements for both the banking (Basel II) and insurance (Solvency II) industry, are two recent examples of the growing concern of regulators for the financial health of firms in the economy. This paper adds to this goal. In particular, we consider the problem of determining the optimal strike price for a bond put option, which is used to hedge the interest rate risk of an investment in a bond, zero coupon or coupon-bearing. In order to measure risk, we focus on both Value-at-Risk and Conditional Value-at-Risk. Our optimization is constrained by a maximum hedging budget. Alternatively, our approach can also be used to determine the minimal budget a firm needs to spend in order to achieve a predetermined absolute risk level. This paper can be seen as an extension of Ahn et al. (1999), who consider the same problem for an investment in a share.

## 2. LOSS FUNCTION AND RISK MEASURES

Consider a portfolio with value  $W_t$  at time  $t$ .  $W_0$  is then the value or price at which we buy the portfolio at time zero.  $W_T$  is the value of the portfolio at time  $T$ . The loss  $L$  we make by buying at time zero and selling at time  $T$  is then given by  $L = W_0 - W_T$ . The Value-at-Risk of this portfolio is defined as the  $(1 - \alpha)$ -quantile of the loss distribution depending on a time interval with length  $T$ . A formal definition for the  $\text{VaR}_{\alpha,T}$  is

$$\Pr[L \geq \text{VaR}_{\alpha,T}] = \alpha. \quad (1)$$

In other words  $\text{VaR}_{\alpha,T}$  is the loss of the worst case scenario on the investment at a  $(1 - \alpha)$  confidence level at time  $T$ . It is also possible to define the  $\text{VaR}_{\alpha,T}$  in a more general way

$$\text{VaR}_{\alpha,T}(L) = \inf \{Y \mid \Pr(L \geq Y) < \alpha\}. \quad (2)$$

Although frequently used, VaR has attracted some criticisms. First of all, a drawback of the traditional Value-at-Risk measure is that it does not care about the tail behaviour of the losses. In other words, by focusing on the VaR at, let's say a 5% level, we ignore the potential severity of the losses below that 5% threshold. This means that we have no information on how bad things can become in a real stress situation. Therefore, the important question of 'how bad is bad' is left unanswered. Secondly, it is not a coherent risk measure, as suggested by Artzner et al. (1999). More specifically, it fails to fulfil the subadditivity requirement which states that a risk measure should always reflect the advantages of diversifying, that is, a portfolio will risk an amount no more than, and in some cases less than, the sum of the risks of the constituent positions. It is possible to provide examples that show that VaR is sometimes in contradiction with this subadditivity requirement.

Artzner et al. (1999) suggested the use of Conditional VaR (CVaR) as risk measure, which they describe as a coherent risk measure. CVaR is also known as TVaR, or Tail Value-at-Risk and is defined as follows:

$$\text{CVaR}_{\alpha,T} = \frac{1}{\alpha} \int_0^\alpha \text{VaR}_{\beta,T} d\beta. \quad (3)$$

This formula boils down to taking the arithmetic average of the quantiles of our loss, from 0 to  $\alpha$  on, where we recall that  $\text{VaR}_{\beta,T}$  stands for the quantile at the level  $1 - \beta$ , see (1). This formula already makes clear that  $\text{CVaR}_{\alpha,T}(L)$  will always be larger than  $\text{VaR}_{\alpha,T}(L)$ .

If the cumulative distribution function of the loss is continuous, CVaR is also equal to the Conditional Tail Expectation (CTE) which for the loss  $L$  is calculated as:

$$\text{CTE}_{\alpha,T}(L) = E[L \mid L > \text{VaR}_{\alpha,T}(L)].$$

## 3. THE BOND HEDGING PROBLEM

Analogously to Ahn et al. (1999), we assume that we have, at time zero, one bond with maturity  $S$  and we will sell this bond at time  $T$ , which is prior to  $S$ . In case of an increase in interest rates, not hedging can lead to severe losses. Therefore, the company decides to spend an amount  $C$  on

hedging. This amount will be used to buy one or part of a bond put option, so that, in case of a substantial decrease in the bond price, the put option can be exercised in order to prevent large losses. The remaining question now is how to choose the strike price. We will find the optimal strike prices which minimize VaR and CVaR respectively for a given hedging cost. An alternative interpretation of our setup is that it can be used to calculate the minimal hedging budget the firm has to spend in order to achieve a specified VaR or CVaR level. The latter setup was followed in the paper by Miyazaki (2001).

### 3.1. Zero-coupon bond

Let us assume that the institution has an exposure to a bond,  $Y(0, S)$ , with principal  $K = 1$ , which matures at time  $S$ , and that the company has decided to hedge the bond value by using a percentage  $h$  ( $0 < h < 1$ ) of one put option  $P(0, T, S, X)$  with strike price  $X$  and exercise date  $T$  (with  $T \leq S$ ).

Further, we assume that the distribution of  $Y(T, S)$  is known and is continuous and strictly increasing. We will denote its cumulative distribution function (cdf) under the measure in which we measure the VaR or the CVaR by  $F_{Y(T,S)}(\cdot)$ . For example when the short-rate model is one of the following commonly used interest rate models such as Vasicek, one- and two-factor Hull-White, two-factor additive Gaussian model G2++, two-factor Heath-Jarrow-Morton with deterministic volatilities, see e.g. Brigo and Mercurio (2001), then  $Y(T, S)$  has a lognormal distribution.

Analogously to Ahn et al. (1999), we can look at the future value of the hedged portfolio that is composed of the bond  $Y$  and the put option  $P(0, T, S, X)$  at time  $T$  as a function of the form

$$H_T = \max(hX + (1 - h)Y(T, S), Y(T, S)).$$

In a worst case scenario — a case which is of interest to us — the put option finishes in-the-money. Then the future value of the portfolio equals

$$H_T = (1 - h)Y(T, S) + hX.$$

Taking into account the cost of setting up our hedged portfolio, which is given by the sum of the bond price  $Y(0, S)$  and the cost  $C$  of the position in the put option, we get for the value of the loss:

$$L = Y(0, S) + C - ((1 - h)Y(T, S) + hX), \quad (4)$$

and this under the assumption that the put option finishes in-the-money.

Note that this loss function can be seen as a strictly decreasing function  $f$  in  $Y(T, S)$ :

$$f(Y(T, S)) := Y(0, S) + C - ((1 - h)Y(T, S) + hX). \quad (5)$$

#### VaR minimization

We first look at the case of determining the optimal strike  $X$  when minimizing the VaR under a constraint on the hedging cost.

Recalling (1) and (4), the Value-at-Risk at an  $\alpha$  percent level of a position  $H = \{Y, h, P\}$  consisting of a bond  $Y$  and  $h$  put options  $P$  (which are assumed to be in-the-money at expiration)

with a strike price  $X$  and an expiry date  $T$  is equal to<sup>1</sup>

$$\text{VaR}_{\alpha,T}(L) = Y(0, S) + C - ((1 - h)F_{Y(T,S)}^{-1}(\alpha) + hX), \quad (6)$$

where  $F_{Y(T,S)}^{-1}(\alpha)$  is the percentile of the cdf  $F_{Y(T,S)}$ , i.e.  $\Pr[Y(T, S) \leq F_{Y(T,S)}^{-1}(\alpha)] = \alpha$ .

Similar to the Ahn et al. problem, we would like to minimize the risk of the future value of the hedged bond  $H_T$ , given a maximum hedging expenditure  $C$ . More precisely, we consider the minimization problem

$$\min_{X,h} Y(0, S) + C - ((1 - h)F_{Y(T,S)}^{-1}(\alpha) + hX)$$

subject to the restrictions  $C = hP(0, T, S, X)$  and  $h \in (0, 1)$ .

This is a constrained optimization problem with Lagrange function

$$\mathcal{L}(X, h, \lambda) = \text{VaR}_{\alpha,T}(L) - \lambda(C - hP(0, T, S, X)),$$

containing one multiplier  $\lambda$ . Note that the multipliers to include the inequalities  $0 < h$  and  $h < 1$  are zero since these constraints are not binding. Taking into account that the optimal strike  $X^*$  will differ from zero, we find from the Kuhn-Tucker conditions

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial X} = -h + h\lambda \frac{\partial P}{\partial X}(0, T, S, X) = 0 \\ \frac{\partial \mathcal{L}}{\partial h} = -(X - F_{Y(T,S)}^{-1}(\alpha)) + \lambda P(0, T, S, X) = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} = C - hP(0, T, S, X) = 0 \\ 0 < h < 1 \quad \text{and} \quad \lambda > 0 \end{cases}$$

that this optimal strike  $X^*$  should satisfy the following equation

$$P(0, T, S, X) - (X - F_{Y(T,S)}^{-1}(\alpha)) \frac{\partial P}{\partial X}(0, T, S, X) = 0. \quad (7)$$

By a change of numeraire, it is well known that the put option price equals the discounted expectation under the  $T$ -forward measure of the the pay-off:

$$P(0, T, S, X) = Y(0, T) \mathbb{E}^T[(X - Y(T, S))_+].$$

Its first order derivative with respect to the strike  $X$  gives the cumulative distribution function  $F_{Y(T,S)}^T$  of  $Y(T, S)$  under this  $T$ -forward measure, see Breeden and Litzenberger (1978):

$$\frac{\partial P}{\partial X}(0, T, S, X) = Y(0, T) F_{Y(T,S)}^T(X). \quad (8)$$

Hence, (7) is equivalent to

$$P(0, T, S, X) - (X - F_{Y(T,S)}^{-1}(\alpha)) Y(0, T) F_{Y(T,S)}^T(X) = 0.$$

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<sup>1</sup>In case of an unhedged portfolio, take  $C = h = 0$  in (4) and in (6) to obtain the loss function  $L$  with corresponding  $\text{VaR}_{\alpha,T}(L)$ .

**Important remarks**

1. We note that the optimal strike price is independent of the hedging cost  $C$ . This independence implies that for the optimal strike  $X^*$ , VaR in (6) is a linear function of  $h$  (or  $C$ ):

$$\text{VaR}_{\alpha,T}(L) = Y(0, S) - F_{Y(T,S)}^{-1}(\alpha) + h(P(0, T, S, X^*) + F_{Y(T,S)}^{-1}(\alpha) - X^*).$$

So, there is a linear trade-off between the hedging expenditure and the VaR level. It is a decreasing function since in view of (8)  $\frac{\partial P}{\partial X}(0, T, S, X^*) < 1$  and thus according to (7)  $X^* - F_{Y(T,S)}^{-1}(\alpha) > P(0, T, S, X^*)$ .

Although the setup of the paper is determining the strike price which minimizes a certain risk criterion, given a predetermined hedging budget, this trade-off shows that the analysis and the resulting optimal strike price can evidently also be used in the case where a firm is fixing a nominal value for the risk criterion and seeks the minimal hedging expenditure needed to achieve this risk level. It is clear that, once the optimal strike price is known, we can determine, in both approaches, the remaining unknown variable (either VaR, either  $C$ ).

2. We also note that the optimal strike price is higher than the bond VaR level  $F_{Y(T,S)}^{-1}(\alpha)$ . This has to be the case since  $P(0, T, S, X)$  is always positive and the change in the price of a put option due to an increase in the strike is also positive. This result is also quite intuitive since there is no point in taking a strike price which is situated below the bond price you expect in a worst case scenario.

When moreover the optimal strike is smaller than the forward price of the bond, i.e.

$$X^* < \frac{Y(0, S)}{Y(0, T)},$$

then the price of put option to buy will be small.

3. The assumption of continuity and strictly monotonicity of the distribution of  $Y(T, S)$  can be weakened. In that case we should work with the general definition (2) of VaR.

**CVaR minimization**

In this section, we demonstrate the ease of extending our analysis to the alternative risk measure CVaR (3) by integration of (6):

$$\text{CVaR}_{\alpha,T}(L) = Y(0, S) + C - hX - \frac{1}{\alpha}(1 - h) \int_0^\alpha F_{Y(T,S)}^{-1}(\beta) d\beta. \quad (9)$$

We again seek to minimize this risk measure, in order to minimize potential losses. The procedure for minimizing this CVaR is analogue to the VaR minimization procedure. The resulting optimal strike price  $X^*$  can thus be determined from the implicit equation below:

$$P(0, T, S, X) - \left(X - \frac{1}{\alpha} \int_0^\alpha F_{Y(T,S)}^{-1}(\beta) d\beta\right) \frac{\partial P}{\partial X}(0, T, S, X) = 0, \quad (10)$$

or, equivalently by (8), from

$$P(0, T, S, X) - \left(X - \frac{1}{\alpha} \int_0^\alpha F_{Y(T,S)}^{-1}(\beta) d\beta\right) Y(0, T) F_{Y(T,S)}^T(X) = 0.$$

As for the VaR-case the optimal strike  $X^*$  is independent of the hedging cost  $C$  and CVaR can be plotted as a linear function of  $C$  (or  $h$ ) representing a trade-off between the cost and the level of protection.

For the same reason as in the VaR-case, the optimal strike  $X^*$  has to be higher than the bond CVaR level  $\frac{1}{\alpha} \int_0^\alpha F_{Y(T,S)}^{-1}(\beta) d\beta$ .

#### 4. COUPON-BEARING BOND

We consider now the case of a coupon-bearing bond paying cash flows  $\mathcal{C} = [c_1, \dots, c_n]$  at maturities  $\mathcal{S} = [S_1, \dots, S_n]$ . Let  $T \leq S_1$ . The price of this coupon-bearing bond in  $T$  is expressed as a linear combination (or a portfolio) of zero-coupon bonds:

$$\text{CB}(T, \mathcal{S}, \mathcal{C}) = \sum_{i=1}^n c_i Y(T, S_i). \quad (11)$$

As in the previous section, the company wants to hedge its position in this bond by buying a percentage of a put option on this bond with strike  $X$  and maturity  $T$ . In order to determine the strike  $X$ , the VaR or the CVaR of the hedged portfolio at time  $T$  is minimized under a budget constraint. Comparing the results in the previous section for VaR and CVaR minimization for a hedged position in zero-coupon bond we note that both cases can in fact be treated together.

We first have a look at the value of a put option on a coupon-bearing bond as well as at the structure of the loss function.

Since the zero-coupon bonds  $Y(T, S_i)$  all depend on the same short rate at  $T$ , the vector  $(Y(T, S_1), \dots, Y(T, S_n))$  is comonotonic, see Kaas et al. (2000). By the properties of comonotonic vectors, the coupon-bearing bond  $\text{CB}(T, \mathcal{S}, \mathcal{C})$  (11) is a comonotonic sum with cumulative distribution function  $F_{\text{CB}}^T(\cdot)$  under the  $T$ -forward measure. This implies that a European option on a coupon-bearing bond decomposes into a portfolio of options on the individual zero-coupon bonds in the portfolio, which gives in case of a put with maturity  $T$  and strike  $X$ :

$$\text{CBP}(0, T, \mathcal{S}, \mathcal{C}, X) = \sum_{i=1}^n c_i P(0, T, S_i, X_i), \quad \text{with} \quad \sum_{i=1}^n c_i X_i = X. \quad (12)$$

This result, now well-known as the Jamshidian decomposition, was found in Jamshidian (1989) in case of a Vasicek interest rate model. Kaas et al. (2000) obtained this result in a more general framework of stop-loss premiums and gave an explicit expression for the  $X_i$ :

$$X_i = (F_{Y(T,S_i)}^T)^{-1}(F_{\text{CB}}^T(X)). \quad (13)$$

Repeating the reasoning of Section 3.1 we may conclude that in a worst case scenario the loss of the hedged portfolio at time  $T$  composed of the coupon-bearing bond (11) and the put option (12) equals a strictly decreasing function  $f$  of the random variable  $\text{CB}(T, \mathcal{S}, \mathcal{C})$ :

$$L = \text{CB}(0, \mathcal{S}, \mathcal{C}) + C - ((1 - h)\text{CB}(T, \mathcal{S}, \mathcal{C}) + hX) := f(\text{CB}(T, \mathcal{S}, \mathcal{C})). \quad (14)$$

### VaR and CVaR minimization

The VaR of this loss that we want to minimize under the constraints  $0 < h < 1$  and  $C = hCBP(0, T, \mathcal{S}, \mathcal{C}, X)$ , is given by

$$\text{VaR}_{\alpha, T}(L) = f(F_{\text{CB}}^{-1}(\alpha)) = \text{CB}(0, \mathcal{S}, \mathcal{C}) + C - ((1 - h)F_{\text{CB}}^{-1}(\alpha) + hX), \quad (15)$$

where  $F_{\text{CB}}^{-1}$  stands for the inverse cdf of the coupon-bearing bond under the measure in which VaR (and CVaR) is measured.

By integrating this relation (15), after replacing  $\alpha$  by  $\beta$ , with respect to  $\beta$  between the integration bounds 0 and  $\alpha$ , we find for the CVaR of the loss:

$$\text{CVaR}_{\alpha, T}(L) = \text{CB}(0, \mathcal{S}, \mathcal{C}) + C - hX - \frac{1}{\alpha}(1 - h) \int_0^\alpha F_{\text{CB}}^{-1}(\beta) d\beta. \quad (16)$$

Also here we note the similarity in the expressions for the risk measures (RM) VaR and CVaR which could be collected in one expression:

$$\text{RM}_{\alpha, T}(L) = \text{CB}(0, \mathcal{S}, \mathcal{C}) + C - hX - (1 - h)g(F_{\text{CB}}^{-1}(\alpha)) \quad (17)$$

$$\text{with } g(F_{\text{CB}}^{-1}(\alpha)) = \begin{cases} F_{\text{CB}}^{-1}(\alpha) & \text{if RM = VaR} \\ \frac{1}{\alpha} \int_0^\alpha F_{\text{CB}}^{-1}(\beta) d\beta & \text{if RM = CVaR.} \end{cases} \quad (18)$$

Although the marginal distributions  $F_{Y(T, S_i)}$  are known, the distribution  $F_{\text{CB}}$  of the sum can in general not be obtained. However, in the case of a comonotonic sum we have, see again Kaas et al. (2000),

$$F_{\text{CB}}^{-1}(p) = \sum_{i=1}^n c_i F_{Y(T, S_i)}^{-1}(p) \quad \text{for all } p \in [0, 1], \quad (19)$$

and similarly for the inverse cdfs under the  $T$ -forward measure.

We now want to solve the constrained optimization problem

$$\min_{X, h} \text{RM}_{\alpha, T}(L) \quad \text{subjected to } C = hCBP(0, T, \mathcal{S}, \mathcal{C}, X), \quad 0 < h < 1.$$

From the Kuhn-Tucker conditions we find that the optimal strike price  $X^*$  satisfies the following equation

$$\text{CBP}(0, T, \mathcal{S}, \mathcal{C}, X) - (X - g(F_{\text{CB}}^{-1}(\alpha))) \frac{\partial \text{CBP}}{\partial X}(0, T, \mathcal{S}, \mathcal{C}, X) = 0. \quad (20)$$

Rewriting this equation in terms of the put options on the individual zero-coupon bonds cfr. (12), invoking (19) and using the linearity of the function  $g$  (18), leads to the following equivalent set of equations:

$$\sum_{i=1}^n c_i P(0, T, S_i, X_i) - (X - \sum_{i=1}^n c_i g(F_{Y(T, S_i)}^{-1}(\alpha))) \sum_{i=1}^n c_i \frac{\partial P}{\partial X_i}(0, T, S_i, X_i) \frac{\partial X_i}{\partial X} = 0 \quad (21)$$

$$\sum_{i=1}^n c_i X_i = X \quad (22)$$

$$\sum_{i=1}^n c_i \frac{\partial X_i}{\partial X} = 1, \quad (23)$$

where  $X_i$  is defined by (13).

We can further simplify relation (21) by applying relation (8) to the strike  $X_i$  given by (13), i.e.

$$\frac{\partial P}{\partial X_i}(0, T, S_i, X_i) = Y(0, T) F_{Y(T, S_i)}^T ((F_{Y(T, S_i)}^T)^{-1}(F_{CB}^T(X))) = Y(0, T) F_{CB}^T(X).$$

Hence, this derivative is independent of  $i$  which implies in view of (23) that

$$\sum_{i=1}^n c_i \frac{\partial P}{\partial X_i}(0, T, S_i, X_i) \frac{\partial X_i}{\partial X} = Y(0, T) F_{CB}^T(X) \sum_{i=1}^n c_i \frac{\partial X_i}{\partial X} = Y(0, T) F_{CB}^T(X). \quad (24)$$

We introduce the short hand notation

$$A_X := F_{CB}^T(X).$$

Substitution of (13), (22) and (24) in (21) leads to the following equation that we have to solve for  $A_X$ :

$$\sum_{i=1}^n c_i P(0, T, S_i, (F_{Y(T, S_i)}^T)^{-1}(A_X)) - Y(0, T) A_X \sum_{i=1}^n c_i [(F_{Y(T, S_i)}^T)^{-1}(A_X) - g(F_{Y(T, S_i)}^{-1}(\alpha))] = 0. \quad (25)$$

Once, we know  $A_X$  we immediately have the optimal strike  $X^*$  from (22):

$$X^* = \sum_{i=1}^n c_i (F_{Y(T, S_i)}^T)^{-1}(A_X). \quad (26)$$

## Remarks

1. We note that also in the case of a coupon-bearing bond the optimal strike price is independent of the hedging cost and that one can look at the trade-off between the hedging expenditure and the RM level, cfr. Section 3.1.
2. Also here we may weaken the assumption of continuity and strictly monotonicity of the distribution functions  $F_{Y(T, S_i)}$ . In that case we have to invoke Kaas et al. (2000) with a so-called  $\eta$ -inverse distribution of a random variable  $Y$  which is defined as the following convex combination:

$$\begin{aligned} F_Y^{-1(\eta)}(p) &= \eta F_Y^{-1}(p) + (1 - \eta) F_Y^{-1+}(p), \quad p \in (0, 1), \quad \eta \in [0, 1], \\ F_Y^{-1}(p) &= \inf \{y \in \mathbb{R} \mid F_Y(y) \geq p\}, \quad p \in [0, 1], \\ F_Y^{-1+}(p) &= \sup \{y \in \mathbb{R} \mid F_Y(y) \leq p\}, \quad p \in [0, 1]. \end{aligned}$$

Thus relation (12) holds with

$$X_i = (F_{Y(T, S_i)}^T)^{-1(\eta)}(F_{CB}^T(X)),$$

where  $\eta \in [0, 1]$  is determined from

$$\sum_{i=1}^n c_i (F_{Y(T, S_i)}^T)^{-1(\eta)}(F_{CB}^T(X)) = X.$$



## 5. APPLICATION: HULL-WHITE MODEL

As an application, we focus on the Hull-White one-factor model, first discussed by Hull and White in 1990 (see Hull and White (1990)). We choose this model because it is still an often used model in financial institutions for risk management purposes, (see Brigo and Mercurio (2001)).

Hull and White (1990) assume under the risk-neutral measure  $Q$  that the instantaneous interest rate follows a mean reverting process also known as an Ornstein-Uhlenbeck process:

$$dr(t) = (\theta(t) - \gamma(t)r(t))dt + \sigma(t)dZ(t) \quad (27)$$

with  $Z(t)$  a standard Brownian motion under  $Q$ , and with time dependent parameters  $\theta(t)$ ,  $\gamma(t)$ , and  $\sigma(t)$ . The parameter  $\theta(t)$  is the time dependent long-term average level of the spot interest rate around which  $r(t)$  moves,  $\gamma(t)$  controls the mean-reversion speed and  $\sigma(t)$  is the volatility function. By making the mean reversion level  $\theta$  time dependent, a perfect fit with a given term structure can be achieved, and in this way arbitrage can be avoided. In our analysis, we will keep  $\gamma$  and  $\sigma$  constant, and thus time-independent. According to Brigo and Mercurio (2001), this is desirable when an exact calibration to an initial term structure is wanted. This perfect fit then occurs when  $\theta(t)$  satisfies the following condition:

$$\theta(t) = F_t^M(0, t) + \gamma F^M(0, t) + \frac{\sigma^2}{2\gamma}(1 - e^{-2\gamma t}),$$

where,  $F^M(0, t)$  denotes the instantaneous forward rate observed in the market on time zero with maturity  $t$ .

It can be shown (see Hull and White (1990)) that the expectation and variance of the stochastic variable  $r(t)$  are:

$$E[r(t)] = m(t) = r(0)e^{-\gamma t} + a(t) - a(0)e^{-\gamma t}, \quad \text{Var}[r(t)] = s^2(t) = \frac{\sigma^2}{2\gamma}(1 - e^{-2\gamma t}) \quad (28)$$

with the expression  $a(t)$  calculated as follows:

$$a(t) = F^M(0, t) + \frac{\sigma^2}{2} \left( \frac{1 - e^{-\gamma t}}{\gamma} \right)^2.$$

Based on these results, Hull and White developed an analytical expression for the price of a zero-coupon bond with maturity date  $S$

$$Y(t, S) = A(t, S)e^{-B(t, S)r(t)},$$

where

$$B(t, S) = \frac{1 - e^{-\gamma(S-t)}}{\gamma}, \quad A(t, S) = \frac{Y^M(0, S)}{Y^M(0, t)} e^{B(t, S)F^M(0, t) - \frac{\sigma^2}{4\gamma}(1 - e^{-2\gamma t})B^2(t, S)}$$

with  $Y^M$  the bond price observed in the market. Since  $A(t, S)$  and  $B(t, S)$  are independent of  $r(t)$ , the distribution of a bond price at any given time must be lognormal with parameters  $\Pi$  and  $\Sigma^2$ :

$$\Pi(t, S) = \ln A(t, S) - B(t, S)m(t), \quad \Sigma(t, S)^2 = B(t, S)^2 s^2(t),$$

with  $m(t)$  and  $s^2(t)$  given by (28). Thus under the risk neutral measure the inverse cdf of  $Y(T, S)$  is given by

$$F_{Y(T,S)}^{-1}(p) = e^{\Pi(T,S) + \Sigma(T,S)\Phi^{-1}(p)}, \quad p \in [0, 1], \quad (29)$$

and we can compute the (standard) integral

$$\int_0^\alpha F_{Y(T,S)}^{-1}(\beta) d\beta = e^{\Pi(T,S)} \int_0^\alpha e^{\Sigma(T,S)\Phi^{-1}(\beta)} d\beta = e^{\Pi(T,S) + \frac{1}{2}\Sigma^2(T,S)} \Phi(\Phi^{-1}(\alpha) - \Sigma(T,S)). \quad (30)$$

By a change of numeraire it can be shown that  $Y(T, S)$  remains lognormally distributed under the  $T$ -forward measure but now with parameters  $\Pi^T$  and  $(\Sigma^T)^2$  given by:

$$\Pi^T(T, S) = \ln\left(\frac{Y(0, S)}{Y(0, T)}\right) - \frac{1}{2}(\Sigma^T(T, S))^2, \quad \Sigma^T(T, S) = \Sigma(T, S). \quad (31)$$

Hence, the inverse cdf of  $Y(T, S)$  under the  $T$ -forward measure is known explicitly:

$$(F_{Y(T,S)}^T)^{-1}(p) = e^{\Pi^T(T,S) + \Sigma(T,S)\Phi^{-1}(p)}, \quad p \in [0, 1], \quad (32)$$

as well as the put option price and its derivative with respect to the strike:

$$\begin{aligned} P(0, T, S, X) &= -Y(0, S)\Phi(-d_1(X)) + XY(0, T)\Phi(-d_2(X)), \\ \frac{\partial P}{\partial X}(0, T, S, X) &= Y(0, T)\Phi(-d_2(X)), \end{aligned}$$

with, when taking (31) into account,

$$d_1(X) = \frac{1}{\Sigma(T, S)} \left[ \ln\left(\frac{Y(0, S)}{Y(0, T)}\right) - \ln(X) \right] + \frac{1}{2}\Sigma(T, S) = \frac{\Pi^T(T, S) - \ln(X)}{\Sigma(T, S)} + \Sigma(T, S) \quad (33)$$

$$d_2(X) = d_1(X) - \Sigma(T, S) = \frac{\Pi^T(T, S) - \ln(X)}{\Sigma(T, S)}. \quad (34)$$

For the **zero-coupon case**, substitution of the relations above in (7) and in (10) gives the following implicit relation for the optimal strike  $X^*$ :

$$G(\Phi^{-1}(\alpha)) = \frac{Y(0, S)\Phi(-d_1(X))}{Y(0, T)\Phi(-d_2(X))},$$

with

$$G(\Phi^{-1}(\alpha)) = \begin{cases} e^{\Pi(T,S) + \Sigma(T,S)\Phi^{-1}(\alpha)} & \text{if VaR} \\ \frac{1}{\alpha} e^{\Pi(T,S) + \frac{1}{2}\Sigma^2(T,S)} \Phi(\Phi^{-1}(\alpha) - \Sigma(T,S)) & \text{if CVaR.} \end{cases} \quad (35)$$

For the **coupon-bearing bond case**, the above relations for the distribution and the put option price hold but with  $S$  and  $X$  replaced by  $S_i$  and  $X_i$ . The expressions (33) and (34) for  $d_1(X_i)$  and  $d_2(X_i)$  can further be simplified in view of (13) and (30):

$$d_1(X_i) = \Sigma(T, S_i) - \Phi^{-1}(A_X), \quad d_2(X_i) = -\Phi^{-1}(A_X).$$

In this way, the set of equations (25)-(26) to find the optimal strike  $X^*$  are equivalent with:

$$\begin{aligned} & \sum_{i=1}^n c_i \left[ -Y(0, S_i) \Phi(\Phi^{-1}(A_X) - \Sigma(T, S_i)) + Y(0, T) A_X e^{\Pi^T(T, S_i) + \Sigma(T, S_i) \Phi^{-1}(A_X)} \right] \\ & = Y(0, T) A_X \sum_{i=1}^n c_i \left[ e^{\Pi^T(T, S_i) + \Sigma(T, S_i) \Phi^{-1}(A_X)} - G_i(\Phi^{-1}(\alpha)) \right] \\ X^* & = \sum_{i=1}^n c_i e^{\Pi^T(T, S_i) + \Sigma(T, S_i) \Phi^{-1}(A_X)}, \end{aligned}$$

where  $G_i(\Phi^{-1}(\alpha))$  is defined by (35) when replacing  $S$  by  $S_i$ .

For a complete numerical example we refer to Deelstra et al. (2005) and Heyman et al. (2006).

## 6. CONCLUSIONS

We provided a method for minimizing the risk of a position in a bond (zero coupon or coupon-bearing) by buying (a percentage of) a bond put option. Taking into account a budget constraint, we determine the optimal strike price, which minimizes a Value-at-Risk or Tail-Value-at-Risk criterion. Alternatively, our approach can be used when a nominal risk level is fixed, and the minimal hedging budget to fulfil this criterion is desired. From the class of short rate models which result in lognormally distributed future bond prices, we have selected the Hull-White one-factor model for an illustration of our optimization.

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