

Explosion time for some Laplace transforms of the Wishart process

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Abstract

In this paper, we focus upon a family of matrix valued stochastic processes and study the problem of determining the smallest time such that their Laplace transforms become infinite. In particular, we concentrate upon the class of Wishart processes, which have proved to be very useful in different applications by their ability in describing non-trivial dependence. Thanks to this remarkable property we are able to explain the behavior of the explosion times for the Laplace transforms of the Wishart process and its time integral in terms of the relative importance of the involved factors and their correlations.

Key words: Wishart process, Laplace transform, explosion time.

1 Introduction

In this paper, we study the explosion times of different Laplace transforms of a class of Wishart processes: these are matrix valued stochastic processes which are very useful in various applications by their ability in describing non-trivial dependence. The Wishart process has been introduced by Bru [5] and can be viewed as the matrix analogue of the squared Bessel processes (see e.g. Jeanblanc et al. [25]). The Wishart process belongs to the class of affine processes on positive semidefinite matrices characterized by Cuchiero et al. [10], namely time homogeneous and stochastically continuous Markov processes for which the Laplace transform has exponential-affine dependence on the initial state. In the following, we will always consider Wishart processes without jumps. They can be seen as matrix extensions of the Cox–Ingersoll–Ross (CIR) process (Cox et al. [9]) or the Heston [24] volatility process. The Wishart process has been introduced into the domain of finance by Gouriéroux and Sufana [20, 21] and Gouriéroux et al. [22]. It has found applications in various fields like multivariate option pricing (see e.g. Da Fonseca et al. [12]), yield curve modelling (see e.g. Buraschi et al. [6], Gnoatto [18], Chiarella et al. [7]), credit risk (Gouriéroux and Sufana [20]) and commodity derivative pricing (Chiu et al. [8]).

The great advantage of the Wishart specification in comparison with classical affine processes with state space $\mathbb{R}_+^m \times \mathbb{R}^n$ lies in the flexibility to catch non-trivial correlation among positive factors (i.e. the diagonal elements of the Wishart process). For example, it is well-known that multi-dimensional CIR processes should be driven by independent Brownian motions in order to remain in the class of affine processes (see e.g. Grasselli and Tebaldi [23]). In other words, the only dependence between multivariate CIR processes can be obtained by introducing dependence

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in the drift terms, which is well-known to be a rather weak form of dependence, see e.g. Dai and Singleton [14].

Actually, in the literature there are already explicit expressions for a few Laplace transforms of particular Wishart specifications. Indeed, Bru [5] derives under some assumptions the joint Laplace transform of the Wishart process and its integral. Afterwards, e.g. Ahdida and Alfonsi [1] consider the general (continuous) version of the Wishart case and they derive the explicit Laplace transform of the process alone. Gauthier and Possamai [16] study special dynamics of the Wishart specification and provide a closed-form expression for the Laplace transform of the integrated process. Gnoatto and Grasselli [19] extend the result of Bru [5] on the joint Laplace transform under weaker assumptions on the Wishart family, which is finally extended by Alfonsi et al. [2]. As far as we know, no explicit characterization for the corresponding explosion times is available. One remarkable exception is the paper of Da Fonseca [11], who adopted standard techniques from stochastic control (see e.g. Wonham [35]) in order to provide a sufficient condition for the non-explosion of such joint Laplace transforms of the Wishart process and its integral. Da Fonseca [11] employed the same commutativity assumption on the parameters that has been introduced by Gnoatto and Grasselli [19] in order to obtain a closed-form solution for the joint transform. Unfortunately, such assumption is not satisfied ex-post in any of the calibration experiments on real data (see Subsections 4.1 and 4.2 in Da Fonseca [11]), thus meaning that one should go beyond the standard techniques for Riccati equations and investigate in more detail the subject in order to be able to say something more precise on the explosion of the corresponding Riccati ODEs. This is a highly non-trivial problem since it involves all the parameters of the process together with the frequencies of the Laplace transform. In this paper we fix a first step in this new direction. We will concentrate on some parameter restrictions which already go beyond the correlation modelling by the classical affine family (see e.g. Da Fonseca et al. [13]). In this perspective, our paper already represents a relevant contribution to the literature on Wishart processes and gives some intuitions on the role of correlations in the explosion of the transform typically encountered in the previous models. In the numerical illustrations, we focus on the special case where there is no mean-reversion: we notice that these processes received a lot of interest in many fields despite the fact that they have a rather simple structure. For example in insurance, stochastic mortality processes are very well calibrated by non mean-reverting processes, see e.g. Luciano and Vigna [32], or Jevtić et al. [27]. Also in mathematical physics, Katori and Tanemura [28] consider non mean-reverting Wishart processes for studying noncolliding diffusion particle systems.

Let us now introduce the notation in order to state formally the problem we are investigating. Given a filtered probability space and a $d \times d$ matrix Brownian motion W (i.e. a matrix whose entries are independent Brownian motions under \mathbb{P}), the Wishart process X_t is defined as the solution of the $d \times d$ -dimensional stochastic differential equation of the form

$$dX_t = \delta \Sigma^2 dt + M X_t dt + X M^T dt + \sqrt{X_t} dW_t \Sigma + \Sigma dW_t^T \sqrt{X_t}, \quad t \geq 0, \quad (1)$$

$$X_0 = x \in S_d^+, \quad (2)$$

where $\Sigma \in S_d \cap GL_d$ (the set of symmetric and invertible $d \times d$ matrices), $M \in M_d$ (the set of $d \times d$ matrices) and $\delta \geq d - 1$. In the case where $\delta \geq d + 1$, the process takes values in S_d^{++} , i.e. the interior of the cone of positive semidefinite symmetric $d \times d$ matrices denoted by S_d^+ . This family of processes is denoted by $\text{WIS}(\delta, M, \Sigma, d, x)$. In the case where the matrix M is negative semidefinite the mean-reversion ensures that the process is stationary.

In this paper, we start by proving an explicit expression of the joint Laplace transform $\Psi_{u,\mu}$ for $X \in \text{WIS}(\delta, M, \Sigma, d, x)$ which is defined as follows

$$\Psi_{u,\mu}(t) = \mathbb{E}_x \left[e^{-\text{Tr}(uX_t) - \text{Tr}(\int_0^t \mu X_s ds)} \right] \quad u, \mu \in S_d, \quad t \geq 0, \quad (3)$$

where Tr denotes the trace operator. The time of explosion for the Laplace transform is defined as

$$T_{\text{exp}} = \inf\{t \in \mathbb{R}^+ \text{ such that } \Psi_{u,\mu}(t) = +\infty\} \quad (4)$$

and $T_{\text{exp}} = +\infty$ if the previous set is empty. These times are in a natural way related with applications in finance. The explosion time of the Laplace transforms of the log-price is indeed deeply related with the moment explosion time for the corresponding asset price. For example, Andersen and Piterbarg [3] demonstrate that the Heston model has the undesirable property that moments of order higher than one can become infinite in finite time. As arbitrage-free price computation for a number of important fixed income products involves calculating expectations of functions with super-linear growth, such lack of moment stability is of significant practical importance. Other references dealing with moment explosions are Lee [31], Friz and Keller-Ressel [15], Glasserman and Kim [17], Jena et al. [26], Keller-Ressel and Mayerhofer [30]. We notice that in the multivariate case, we are not aware of analytical expressions for the corresponding explosion times.

The rest of this paper is organized as follows. In Section 2, we state our main result of the paper, namely an explicit expression of the time of explosion of the joint Laplace transform. Next, we concentrate upon the explosion time of the Laplace transform of the integrated process. For completeness, we focus also upon the Laplace transform of the integrated process belonging to $\text{WIS}(\delta, \Sigma, d, x)$, corresponding to Wishart processes without drift, i.e. $M = 0$. We derive in this case bounds in terms of eigenvalues of the diffusion matrix Σ^2 and the frequency matrix μ by using a result due to Ostrowski [33]. In Section 3, a numerical example gives some insights into the importance of the Wishart dependence structure in explaining the explosion time. We end this paper with some conclusions in Section 4.

2 The joint Laplace transform of the process

In this section we focus upon the joint Laplace transform of the Wishart process in $\text{WIS}(\delta, M, \Sigma, d, x)$ in (1) and its integral and we study the corresponding explosion time. Expressions for the joint Laplace transform for special cases of Wishart processes are given for instance in Bru [5], Gnoatto and Grasselli [19] and Alfonsi et al. [2].

2.1 Explicit expression for the joint Laplace transform

In the following proposition we recall the explicit expression of the joint Laplace transform $\Psi_{u,\mu}$ of $X \in \text{WIS}(\delta, M, \Sigma, d, x)$ from Alfonsi et al. [2].

Proposition 1. *Let $\Psi_{u,\mu}$ be the joint Laplace transform of $X \in \text{WIS}(\delta, M, \Sigma, d, x)$ in (3), where*

$$dX_t = \delta \Sigma^2 dt + M X_t dt + X M^T dt + \sqrt{X_t} dW_t \Sigma + \Sigma dW_t^T \sqrt{X_t}, \quad t \geq 0, \quad (5)$$

$$X_0 = x \in S_d^+, \quad (6)$$

and where we assume that

$$\begin{cases} \Sigma \in S_d \cap GL_d \\ M \Sigma^2 = \Sigma^2 M^T. \end{cases} \quad (7)$$

Then, for $t \geq 0$, we have:

$$\begin{aligned} \Psi_{u,\mu}(t) &= \mathbb{E}_x \left[e^{-\text{Tr}(uX_t) - \text{Tr}(\int_0^t \mu X_s ds)} \right] \\ &= \frac{\exp\left(-\frac{\delta}{2} \text{Tr}(M)t\right)}{(\det(F_{u,\mu}(t)))^{\frac{\delta}{2}}} \exp\left(-\frac{1}{2} \text{Tr}\left(F'_{u,\mu}(t) F_{u,\mu}(t) + \Sigma^{-1} m \Sigma\right) \Sigma^{-1} x \Sigma^{-1}\right) \end{aligned} \quad (8)$$

with

$$F_{u,\mu}(t) = \left(\sum_{n=0}^{\infty} \frac{\tilde{\mu}^n}{(2n+1)!} t^{2n+1} \right) \tilde{u} + \sum_{n=0}^{\infty} \frac{\tilde{\mu}^n}{(2n)!} t^{2n} \quad (9)$$

and $\tilde{\mu} = 2\Sigma\mu\Sigma + \Sigma^{-1}M^2\Sigma$ and $\tilde{u} = 2\Sigma u\Sigma - \Sigma^{-1}M\Sigma$.

Remark 1. 1. If $\tilde{\mu}$ is positive semidefinite:

$$F_{u,\mu}(t) = \cosh(t\sqrt{\tilde{\mu}}) + \sqrt{\tilde{\mu}}^{-1} \sinh(t\sqrt{\tilde{\mu}})\tilde{u}, \quad (10)$$

$$F'_{u,\mu}(t) = \cosh(t\sqrt{\tilde{\mu}})\tilde{u} + \sinh(t\sqrt{\tilde{\mu}})\sqrt{\tilde{\mu}}. \quad (11)$$

2. If $\tilde{\mu}$ is negative semidefinite:

$$F_{u,\mu}(t) = \cos(t\sqrt{-\tilde{\mu}}) + \sqrt{-\tilde{\mu}}^{-1} \sin(t\sqrt{-\tilde{\mu}})\tilde{u}, \quad (12)$$

$$F'_{u,\mu}(t) = \cos(t\sqrt{-\tilde{\mu}})\tilde{u} - \sin(t\sqrt{-\tilde{\mu}})\sqrt{-\tilde{\mu}}. \quad (13)$$

2.2 Explicit explosion time for the joint Laplace transform

In the following theorem we derive an explicit expression for the explosion time of the joint Laplace transform of the Wishart process $X \in \text{WIS}(\delta, M, \Sigma, d, x)$ and its integral.

Theorem 1. Let $\Psi_{u,\mu}$ be the joint Laplace transform of $X \in \text{WIS}(\delta, M, \Sigma, d, x)$ in (3), where we make the same assumption (7) as in Proposition 1. Then, under the constraints that

$$\begin{cases} u\Sigma^2\mu = \mu\Sigma^2u \\ M^2\Sigma^2u = \Sigma^2uM^2 \\ M^T\mu = \mu M, \end{cases} \quad (14)$$

the explosion time T_{exp} of $\Psi_{u,\mu}$ is given by

$$\begin{aligned} T_{\text{exp}} = & \inf_{i \in \mathcal{I}} \left\{ \mathbf{1}_{\{\lambda_i \geq 0, \beta_i \geq 0\}}(+\infty) + \mathbf{1}_{\{\lambda_i \leq 0\}} \left[\frac{1}{\sqrt{-\lambda_i}} \arctan \left(\frac{-\sqrt{-\lambda_i}}{\beta_i} \right) + \mathbf{1}_{\{\beta_i \geq 0\}} \frac{\pi}{\sqrt{-\lambda_i}} \right] \right. \\ & \left. + \mathbf{1}_{\{\lambda_i \geq 0, \beta_i < 0\}} \left[\mathbf{1}_{\{\sqrt{\lambda_i} \geq |\beta_i|\}}(+\infty) + \mathbf{1}_{\{\sqrt{\lambda_i} < |\beta_i|\}} \frac{1}{\sqrt{\lambda_i}} \operatorname{artanh} \left(\frac{-\sqrt{\lambda_i}}{\beta_i} \right) \right] \right\}, \quad (15) \end{aligned}$$

where $\mathcal{I} = \{1, 2, \dots, d\}$ and λ_i (resp. β_i) denote the eigenvalues of $\tilde{\mu} = 2\Sigma\mu\Sigma + \Sigma^{-1}M^2\Sigma$ (resp. $\tilde{u} = 2\Sigma u\Sigma - \Sigma^{-1}M\Sigma$).

Proof. It is easy to check that the conditions (14) imply that the commutative assumption $\tilde{\mu}\tilde{u} = \tilde{u}\tilde{\mu}$ holds. Indeed,

$$\begin{aligned} \tilde{\mu}\tilde{u} &= 4\Sigma\mu\Sigma^2u\Sigma + 2\Sigma^{-1}M^2\Sigma^2u\Sigma - 2\Sigma\mu M\Sigma - \Sigma^{-1}M^3\Sigma, \\ \tilde{u}\tilde{\mu} &= 4\Sigma u\Sigma^2\mu\Sigma + 2\Sigma uM^2\Sigma - 2\Sigma^{-1}M\Sigma^2\mu\Sigma - \Sigma^{-1}M^3\Sigma. \end{aligned}$$

The fact that the first (resp. second) terms are equal follows immediately from the use of the first (resp. second) condition in (14), whereas the equality of the third terms follows from using the inverse of Σ , next the third condition in (14), and finally the assumption (7):

$$-2\Sigma\mu M\Sigma = -2\Sigma^{-1}\Sigma^2\mu M\Sigma = -2\Sigma^{-1}\Sigma^2M^T\mu\Sigma = -2\Sigma^{-1}M\Sigma^2\mu\Sigma.$$

Since the commutative assumption $\tilde{\mu}\tilde{u} = \tilde{u}\tilde{\mu}$ is satisfied, the matrices $\tilde{\mu}$ and \tilde{u} are simultaneously diagonalizable, in particular with real eigenvalues.

We denote the eigenvalues of $\tilde{\mu}$ by λ_j , the ones of \tilde{u} by β_j , for $j = 1, \dots, d$ and by $\Lambda_1 = \text{diag}(\lambda_j)$ and $\Lambda_2 = \text{diag}(\beta_j)$ the diagonal matrices in eigenbasis.

From equation (8) we notice that the explosion time is given by the first time t that $\det(F_{u,\mu}(t)) = 0$. Using the simultaneous diagonalization, the determinant of $F_{u,\mu}(t)$ can be expressed as the following product:

$$\begin{aligned} \det(F_{u,\mu}(t)) &= \det\left(\sum_{n=0}^{\infty} \frac{\Lambda_1^n}{(2n)!} t^{2n} + \Lambda_2 \sum_{n=0}^{\infty} \frac{\Lambda_1^n}{(2n+1)!} t^{2n+1}\right) \\ &= \prod_{j=1}^d \left(\sum_{k=0}^{\infty} \left(\frac{\lambda_j^k}{(2k)!} t^{2k} + \beta_j \frac{\lambda_j^k}{(2k+1)!} t^{2k+1}\right)\right). \end{aligned}$$

Therefore there is an explosion time if at least one of the factors in this product becomes zero. Without loss of generality we focus upon the i^{th} factor of this product and we assume that all other factors in the product are different from zero. We denote the i^{th} factor of this product by $s_i(t)$:

$$s_i(t) := \sum_{k=0}^{\infty} \frac{\lambda_i^k}{(2k)!} t^{2k} + \beta_i \frac{\lambda_i^k}{(2k+1)!} t^{2k+1}. \quad (16)$$

We then consider several cases in order to determine a general expression for the explosion time.

- If $\lambda_i \geq 0$ then $s_i(t)$ can be expressed as:

$$s_i(t) = \cosh(\sqrt{\lambda_i}t) \left(1 + \beta_i \sqrt{\lambda_i}^{-1} \tanh(\sqrt{\lambda_i}t)\right), \quad (17)$$

therefore if $\lambda_i \geq 0$ and $\beta_i \geq 0$ it follows that $s_i(t)$ is strictly positive and as a consequence there exists no explosion time in this case (which is also evident by looking at (16)). If $\lambda_i \geq 0$ and $\beta_i < 0$ then the function $s_i(t)$ can be equal to zero since $\text{artanh}(x)$ is defined for $x \in]-1, 1[$. Indeed we are looking at the smallest time instant t , so that the explosion time is given by

$$T_{\text{exp}} = \frac{1}{\sqrt{\lambda_i}} \text{artanh}\left(\frac{-\sqrt{\lambda_i}}{\beta_i}\right),$$

provided that $|\beta_i| > \sqrt{\lambda_i}$. If $\lambda_i = 0$ and $\beta_i < 0$, we notice that by taking the limit for $\lambda_i \rightarrow 0$ we get $T_{\text{exp}} = -1/\beta_i$ which also agrees with (16). Finally, in the case where $\lambda_i \geq 0$, $\beta_i < 0$ and $|\beta_i| \leq \sqrt{\lambda_i}$ of course there exists no explosion time.

- If $\lambda_i < 0$ then $s_i(t)$ can be expressed as:

$$s_i(t) = \cos(\sqrt{-\lambda_i}t) + \beta_i \sqrt{-\lambda_i}^{-1} \sin(\sqrt{-\lambda_i}t).$$

Noticing that $\arctan(x) \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ for all $x \in \mathbb{R}$ and that the explosion time should be a positive value, we will solve the equation

$$\frac{\sqrt{-\lambda_i}}{-\beta_i} = \tan\left(\sqrt{-\lambda_i}t - k\pi\right), \quad k \in \mathbb{Z}.$$

If $\beta_i < 0$, then $k = 0$ will already lead to a positive explosion time

$$T_{\text{exp}} = \frac{1}{\sqrt{-\lambda_i}} \arctan\left(\frac{-\sqrt{-\lambda_i}}{\beta_i}\right).$$

If $\beta_i \geq 0$, then $k = 1$ will lead to the smallest positive time of explosion

$$T_{\text{exp}} = \frac{1}{\sqrt{-\lambda_i}} \arctan\left(\frac{-\sqrt{-\lambda_i}}{\beta_i}\right) + \frac{\pi}{\sqrt{-\lambda_i}} \mathbf{1}_{\{\beta_i \geq 0\}}.$$

Finally, the result follows by taking the infimum over the times that annulate one of the factors. \square

In order to give a link with the existing literature, we notice that in the one-dimensional Heston volatility setting, the analogue of formula (15) has already been derived by Andersen and Piterbarg [3], see also Keller-Ressel [29].

Further, the results of Theorem 1 are useful in the context of Section 4.1.1 of Gnoatto and Grasselli [19]. Indeed, following the lines of this paper, the moment explosion problem related to the calculation of moments for the stochastic volatility model proposed in Da Fonseca et al. [13] reduces to the computation of the explosion time of a Laplace transform of a Wishart process as in equation (1). As a consequence, the corresponding explosion time can be determined using Theorem 1.

2.3 The Laplace transform of the integrated process

In this subsection, we first study the explosion time of the Laplace transform of the integrated process $X \in \text{WIS}(\delta, M, \Sigma, d, x)$, denoted by Ψ_μ . We limit ourselves to $M \in S_d^-$, the cone of negative semidefinite symmetric $d \times d$ matrices, which is the most interesting case in finance since then the process is mean-reverting and stationary. Note that $\Psi_\mu = \Psi_{0,\mu}$, so that in this case the first two assumptions of (14) are always satisfied.

Corollary 2. *Let Ψ_μ be the Laplace transform of the integrated process of $X \in \text{WIS}(\delta, M, \Sigma, d, x)$*

$$\Psi_\mu(t) = \mathbb{E}_x \left[e^{-\text{Tr}(\int_0^t \mu X_s ds)} \right] \quad \mu \in S_d, \quad t \geq 0,$$

where we make the same assumption (7) as in Proposition 1. Then, under the constraints that $M \in S_d^-$ and that

$$M^T \mu = \mu M, \tag{18}$$

the explosion time T_{exp} of Ψ_μ is given by

$$T_{\text{exp}} = \inf_{i \in \mathcal{I}} \left\{ \mathbf{1}_{\{\lambda_i \geq 0\}}(+\infty) + \mathbf{1}_{\{\lambda_i \leq 0\}} \left[\frac{1}{\sqrt{-\lambda_i}} \arctan\left(\frac{-\sqrt{-\lambda_i}}{\beta_i}\right) + \frac{\pi}{\sqrt{-\lambda_i}} \right] \right\}, \tag{19}$$

where $\mathcal{I} = \{1, 2, \dots, d\}$ and λ_i (resp. β_i) denote the eigenvalues of $\tilde{\mu} = 2\Sigma\mu\Sigma + \Sigma^{-1}M^2\Sigma$ (resp. $\tilde{u} = -\Sigma^{-1}M\Sigma$).

Proof. Since $u = 0$, we have that $\tilde{u} = -\Sigma^{-1}M\Sigma$. Moreover, the matrices $\Sigma^{-1}M\Sigma$ and M are similar and thus share the same eigenvalues. Since M is negative semidefinite in this mean-reverting case, \tilde{u} has positive eigenvalues β_i and thus (15) reduces to (19). \square

Remark 2. *Let us now concentrate upon the integrated process (thus $u = 0$) in the case without drift, thus $M = 0$. If $\Sigma\mu\Sigma$ is positive semidefinite (or equivalently $\mu\Sigma^2 \in S_d^+$) then there is no explosion for Ψ_μ . Otherwise, by taking the (right handside) limit for β_i tending to 0 in (19), the explosion time T_{exp} of Ψ_μ is easily shown to be finite and it is given by*

$$T_{\text{exp}} = \frac{\pi}{2\sqrt{-\lambda_{\min}}}, \tag{20}$$

where λ_{\min} denotes the minimum (negative) eigenvalue of $2\mu\Sigma^2$. We further remark that in the one-dimensional case where $\Sigma = 1$ and μ is a scalar, the explosion time is coherent with the one of a squared Bessel process with dimension δ starting from x ($BESQ^{(\delta)}(x)$), namely

$$T_{\text{exp}} = \frac{\pi}{2\sqrt{-2\mu}}.$$

This follows immediately from the expression of the Laplace transform of the integrated one-dimensional squared Bessel process with dimension δ , which can be found e.g. in Proposition 6.2.5.5 in Jeanblanc et al. [25].

Note that even in the case of zero drift, $M = 0$, Corollary 2 gives the explosion time in terms of the properties of the matrix $\Sigma\mu\Sigma$. In Theorem 3 below, we give some bounds for the explosion time of the integrated process $X \in \text{WIS}(\delta, \Sigma, d, x)$ in terms of the properties of the matrices μ and Σ^2 . These bounds will be derived by using a result due to Ostrowski [33], which we first recall.

Proposition 2 (Ostrowski). *Let A and B be two Hermitian matrices that are congruent to each other through $A = SBS^\top$ (with S invertible), with eigenvalues resp. $\mu_1 \leq \mu_2 \leq \dots \leq \mu_d$ and $\gamma_1 \leq \gamma_2 \leq \dots \leq \gamma_d$. For all $\gamma_k \neq 0$, it then follows that*

$$\sigma_{\max} \geq \frac{\mu_k}{\gamma_k} \geq \sigma_{\min},$$

where σ_{\max} (resp. σ_{\min}) is the maximum (resp. the minimum) eigenvalue of the (positive definite) matrix SS^\top .

Theorem 3. *The explosion time for the Laplace transform Ψ_μ of the integrated process $X \in \text{WIS}(\delta, \Sigma, d, x)$ (with μ not positive semidefinite) belongs to the following interval:*

$$\frac{\pi}{2\sqrt{-2\gamma_{\min}\sigma_{\max}}} \leq T_{\text{exp}} \leq \frac{\pi}{2\sqrt{-2\gamma_{\min}\sigma_{\min}}}, \quad (21)$$

where γ_{\min} denotes the minimum (negative) eigenvalue of μ and where σ_{\max} (resp. σ_{\min}) is the maximum (resp. minimum) eigenvalue of the (positive definite) matrix Σ^2 .

Proof. We start by using the fact that $\Sigma\mu\Sigma$ and μ share the same inertia. Indeed, if we denote the eigenvalues of $\mu\Sigma^2$ by α_i , $i = 1, \dots, d$ such that

$$\alpha_1 \leq \dots \leq \alpha_k < \alpha_{k+1} = 0 = \dots = 0 = \alpha_h < \alpha_{h+1} \leq \dots \leq \alpha_d,$$

then the eigenvalues γ_i , $i = 1, \dots, d$ of μ have the same structure, that is

$$\gamma_1 \leq \dots \leq \gamma_k < \gamma_{k+1} = 0 = \dots = 0 = \gamma_h < \gamma_{h+1} \leq \dots \leq \gamma_d.$$

From the Proposition of Ostrowski it follows that if the eigenvalue $\alpha_i \neq 0$ (and therefore also $\gamma_i \neq 0$) then

$$\sigma_{\max} \geq \frac{\alpha_i}{\gamma_i} \geq \sigma_{\min},$$

where $\sigma_{\max} \geq \sigma_{\min} > 0$. Hence, for negative eigenvalues γ_i , $i = 1, \dots, k$, this inequality can be rewritten as follows

$$\gamma_i\sigma_{\max} \leq \alpha_i \leq \gamma_i\sigma_{\min},$$

and in particular for the minimum eigenvalue $\gamma_1 = \gamma_{\min}$ the following inequalities hold

$$\gamma_{\min}\sigma_{\max} \leq \alpha_{\min} \leq \gamma_{\min}\sigma_{\min}.$$

Hence, by observing that λ_{\min} in equation (20) denotes the minimum eigenvalue of $2\mu\Sigma^2$ and α_{\min} is as defined above the minimum eigenvalue of $\mu\Sigma^2$, we easily find the expression of α_{\min} in function of the explosion time, so that

$$-\gamma_{\min}\sigma_{\min} \leq \frac{\pi^2}{8T_{\text{exp}}^2} \leq -\gamma_{\min}\sigma_{\max},$$

which yields the result. \square

As a conclusion, this theorem allows to consider separately the impact of the dependence structure (given by matrix Σ^2) and the impact of the different factors (measured by the frequency matrix μ) in the determination of an upper bound and a lower bound of the explosion time. Indeed, the reverse side is that equation (21) no longer gives the explosion time in a closed-form expression, but as contained in an interval with bounds in function of the matrices Σ^2 and μ .

3 Numerical Illustrations

In this section, we study the impact of the off-diagonal parameters of Σ^2 on the explosion time of the joint Laplace transform. In order to understand the role of these factors we focus on the special case where there is no mean-reverting term (i.e. $M = 0$). When Σ is a diagonal matrix, the corresponding Wishart process can be seen as a multidimensional squared Bessel process with independent components, see e.g. Benabid et al. [4]. As a consequence, in that case the problem of determining the explosion time reduces to a one dimensional problem since the Laplace transform splits into a product. Therefore, in this example we investigate the role of the correlation among the positive factors in the behavior of the explosion time. Although the example is chosen to be as simple as possible, we are nevertheless able to show some new features in the story which go far beyond what is possible to describe by using the classical affine framework.

We consider the following matrices for Σ^2 , μ and u

$$\Sigma^2 = \begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix}, \quad \mu = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \quad u = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix},$$

where α, a, b are real numbers, with $\alpha^2 < 1$ in order to grant the positivity of Σ^2 . We notice that in these settings, condition (14) reduces to $u\Sigma^2\mu = \mu\Sigma^2u$ and this is clearly satisfied. Since further $\Sigma\mu\Sigma \in S_d^+$ and $\Sigma u\Sigma \in S_d^+$, assumption (7) is also fulfilled. We take the same frequencies for X^{11} and X^{22} in the Laplace transform since we are especially interested in the impact of the factor X^{12} . It is indeed this factor that causes correlations and dependence between X^{11} and X^{22} which cannot be reproduced in the classical state space domain. Note that the case $\alpha = 0$ corresponds to the independent classical affine case.

3.1 Explosion time as a function of the dependence between the positive factors

The next figure shows the explosion time as a function of α when the values of the parameters of the joint Laplace transform of the Wishart process are fixed. We take for the parameters in this example : $\alpha \in [0, 1[$, $a = 2$ and $b = -3$. In this case, the joint Laplace transform becomes for $t \geq 0$:

$$\Psi_{u,\mu}(t) = \mathbb{E}_x \left[e^{-\text{Tr}(uX_t) - \text{Tr}(\int_0^t \mu X_s ds)} \right] = \mathbb{E}_x \left[e^{3(X_t^{11} + X_t^{22}) - 2 \int_0^t (X_s^{11} + X_s^{22}) ds} \right].$$

Figure 1 shows that the explosion time decreases when α increases. This can be explained in the following way. If α is positive, the two diagonal elements of the Wishart process have tendency

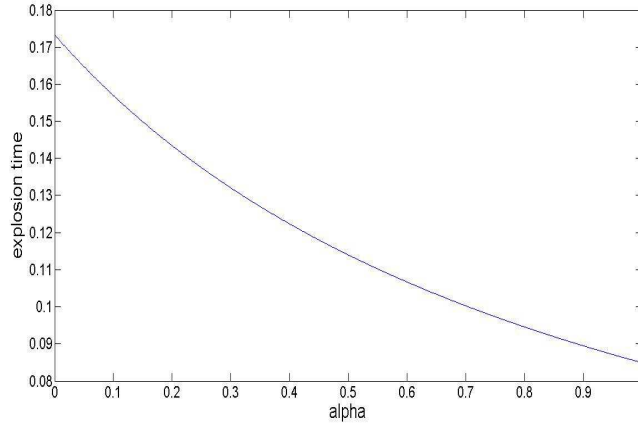


Figure 1: Explosion time for $\Psi_{u,\mu}$ for $a = 2$, $b = -3$

to be positively correlated and then they will contribute in the same way to the explosion. Indeed, it is well-known that the quadratic covariation between the two diagonal elements of the Wishart process is given by

$$d\langle X^{11}, X^{22} \rangle_t = 4X_t^{12}(\Sigma_{11}\Sigma_{12} + \Sigma_{21}\Sigma_{22})dt = 4\alpha X_t^{12}dt$$

and the drift term of X_t^{12} is positive since it equals $\delta\alpha$. Hence the covariation has tendency to be positive.

In this example, the eigenvalues are resp. $\lambda_1 = 4(1 + \alpha)$, $\lambda_2 = 4(1 - \alpha)$, $\beta_1 = -6(1 + \alpha)$ and $\beta_2 = -6(1 - \alpha)$. Since the function $x \operatorname{artanh}(x)$ is increasing on $[0, 1[$, one easily finds that

$$T_{\text{exp}} = \frac{1}{\sqrt{4(1 + \alpha)}} \operatorname{artanh} \left(\frac{2}{3\sqrt{4(1 + \alpha)}} \right).$$

Thanks to the closed-form expression of the explosion time, we can now confirm the intuition that the more the positive factors are positively correlated, the sooner the explosion time will take place. On the contrary, if the positive factors are (close to be) independent then the explosion time is larger.

3.2 Explosion time for α fixed

Now we represent the explosion time for different values of a and b when the parameter α is fixed. We choose two different values for α , namely α equal to 0 or $\frac{1}{2}$.

In the next figures the red surface corresponds to $\alpha = 0$ while the green one corresponds to $\alpha = 1/2$. Figure 2 shows clearly the region where the explosion time is infinite (which is represented on the figure by the level of 50) for the different values of α . Moreover we observe that the maximal domain for the joint Laplace transform is greater when α decreases. In particular, Figure 3 shows for $a \in [0, 5]$ and $b \in [-1.6, 0]$ how the border of the maximal domain moves when α increases. Figure 4 corresponds to Figure 2 where we reduced the height to 1.5. Figure 5 represents a rotated view of Figure 4.

From the Figures 4 and 5 we clearly see that the value of the explosion time is smaller when α increases for a and b fixed, a conclusion which was already made in the setting of Figure 1, namely when the two positive factors have tendency to be positively correlated as explained by and below equation (22).

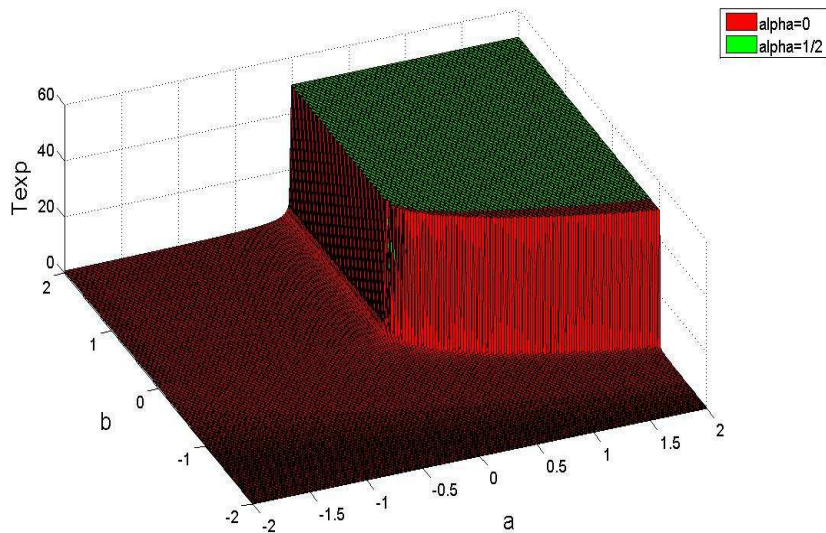


Figure 2: Explosion time for $\Psi_{u,\mu}$ for $\alpha = 0$ and $\alpha = 1/2$

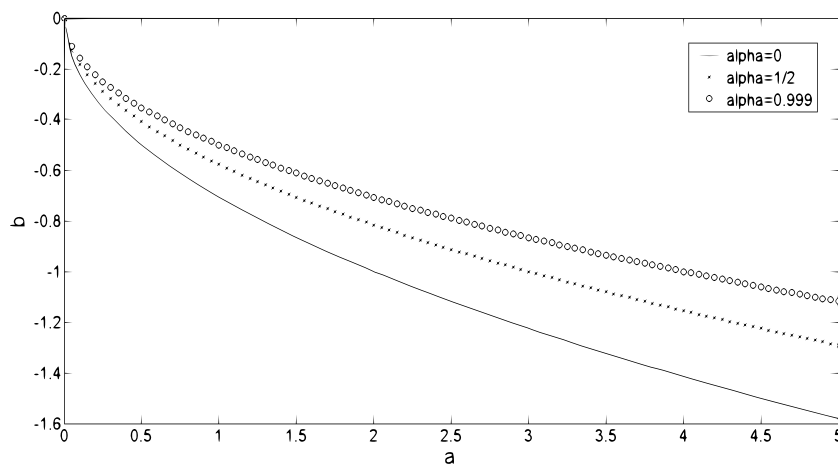


Figure 3: Border of the maximal domain for $\Psi_{u,\mu}$ for $\alpha = 0$, $\alpha = 1/2$, $\alpha = 0.999$

4 Conclusion

In this paper, we have investigated the explosion times of the Laplace transform for a class of Wishart processes. Thanks to the ability of these processes to produce non-trivial dependence between the positive factors, we are able to explain the behavior of the explosion time in terms of the relative importance of the involved factors and their dependence structure. These results extend considerably known properties in the classical affine setting where only one dimensional formulae are available in closed-form for explosion times. We also provided some numerical experiments in order to get more insights into the impact of the analytical results. Of course a complete numerical investigation is beyond the scope of the paper: the interested reader can find more numerical results in the PhD thesis of Van Weverberg [34].

From the theoretical side, there is still interesting work that could be done in order to allow under weaker conditions for the general case where the process has a general mean-reversion.

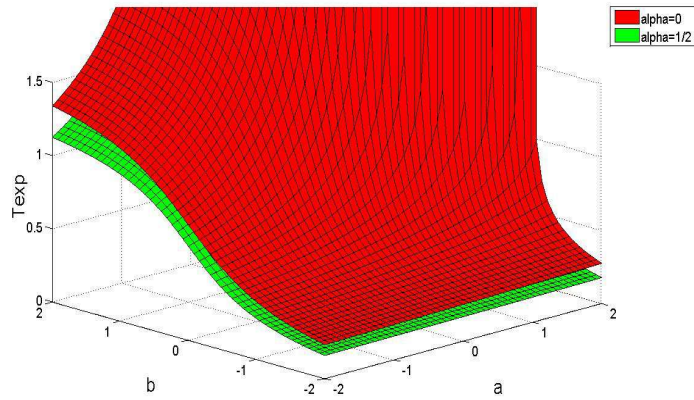


Figure 4: Explosion time for $\Psi_{u,\mu}$ for $\alpha = 0$ and $\alpha = 1/2$

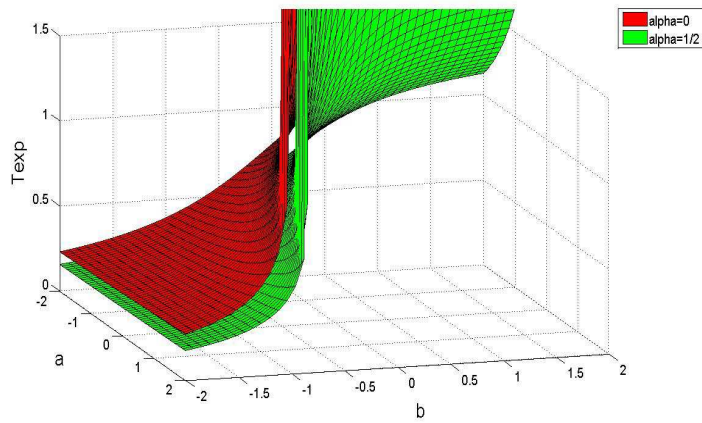


Figure 5: Explosion time $\Psi_{u,\mu}$ for $\alpha = 0$ and $\alpha = 1/2$

Acknowledgments

The authors would like to thank Luigi Salce and the anonymous referees for their constructive suggestions and comments. Griselda Deelstra acknowledges support of the ARC grant IAPAS "Interaction between Analysis, Probability and Actuarial Sciences" 2012-2017, and of the BNB-contract 5D00F000027. Christopher Van Weverberg is supported by a doctoral scholarship FRIA from the Fonds National de la Recherche Scientifique, Communauté française de Belgique.

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