Minimizing the risk of a financial product using a put option

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Abstract

In this paper, we elaborate a method for determining the optimal strike price for a put option, used to hedge a position in a financial product such as a basket of shares and a bond. This strike price is optimal in the sense that it minimizes, for a given budget, a class of risk measures satisfying certain properties. Formulas are derived for one single underlying as well as for a weighted sum of underlyings. For the latter we will consider two cases depending on the dependence structure of the components in this weighted sum. Applications and numerical results are presented.

Keywords: risk measures, hedging, constrained optimization, coupon-bearing bond, basket of assets

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1 INTRODUCTION

The importance of a sound risk management system can hardly be underestimated. The advent of new capital requirements for both the banking (Basel II) and insurance (Solvency II) industry are two recent examples of the growing concern of regulators for the financial health of firms in the economy. This paper adds to this goal. In particular, we consider the problem of determining the optimal strike price for a put option, which is used to hedge the risk of an investment in a risky financial product, such as a share, a basket of assets, a zero-coupon or a coupon-bearing bond. In order to measure risk, we consider a general risk measure satisfying well-known properties such as monotonicity, positive homogeneity, translation invariance and comonotonic additivity. This class of risk measures includes besides Value-at-Risk the following coherent risk measures (Artzner et al., 1999) which will

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be introduced in section 2: Tail Value-at-Risk, Conditional Tail Expectation, Conditional Left Tail Expectation and, more general, distorted risk measures and Weighted Value-at-Risk. The optimization is constrained by a maximum hedging budget. Alternatively, our approach can also be used to determine the minimal budget that a firm needs to spend in order to achieve a predetermined absolute risk level. For an investment in a portfolio of risky assets we are dealing with a sum of risks as well as with a put option on multiple underlyings of which the price processes have cumulative distribution functions that do not have to be *strictly* increasing. When the risks are comonotone the risk minimizing problem can easily be dealt with. When the risks are not comonotone we propose an approximation of the problem by replacing the non-comonotonic sum by comonotonic sums that are smaller or larger in convex order, see Dhaene et al. (2002a) for the theory and, e.g., Dhaene et al. (2002b) and Darkiewicz et al. (2009) for possible applications. The lower comonotonic sum is based on a conditioning random variable that has to be chosen in an appropriate way. We illustrate our method for a basket of assets in a Black-Scholes framework and for an investment in a coupon-bearing bond where the instantaneous short-rate is modeled by the two-additive-factor Gaussian model G2++ (Brigo and Mercurio, 2001). This boils down to lognormal distribution functions for the price processes of the underlyings. In this lognormal case, there are various choices (global and local) for the conditioning random variable in the comonotonic lower sum proposed in literature (Dhaene et al., 2002b; Deelstra et al., 2004; Vanduffel et al., 2005, 2008). For the risk measures we focus on Value-at-Risk and Tail Value-at-Risk which is in the lognormal case equal to the Conditional Tail Expectation.

The current paper can be seen as an extension of the results of Ahn et al. (1999), Deelstra et al. (2007) and Annaert et al. (2007), who consider the particular problems for an investment in one share and in a bond for which the instantaneous short-rate model satisfies the assumption to apply the Jamshidian decomposition (Jamshidian, 1989).

The paper is composed as follows: In section 2 we introduce some well-known risk measures and their properties. The hedging problem is formulated in section 3 as a constrained risk minimization problem. In this section, we deal with the case of one risky asset. In section 4 we recall the basic notions of comonotonicity and convex ordering in order to apply it in section 5 where we look at the generalization of the hedging problem to the case of a portfolio of risky assets. We finish with some applications and numerical results in section 6.

Unless stated otherwise, all random variables are assumed to be defined on a common probability space $(\Omega, \mathcal{F}, P)$.

2 RISK MEASURES

2.1 Risk measure

Consider a set Γ of real valued random variables defined on this given probability space $(\Omega, \mathcal{F}, P)$. We assume that $Y_1, Y_2 \in \Gamma$ implies that $Y_1 + Y_2 \in \Gamma$, and also $aY_1 \in \Gamma$ for any $a > 0$ and $Y_1 + b \in \Gamma$ for any b. Any function $\rho : \Gamma \to \mathbb{R}$ that assigns a real number to any element of Γ is called a *risk measure* (with domain Γ).

Properties of risk measures have been investigated extensively (e.g. Artzner et al., 1999; Dowd and Blake, 2006; Dhaene et al., 2008). Some well-known properties that risk measures may or may not satisfy are monotonicity, positive homogeneity, translation invariance,

subadditivity and additivity for comonotonic risks. They are defined as follows:

(P1) *Monotonicity*: for any $Y_1, Y_2 \in \Gamma$, one has that $Y_1 \leq Y_2$ implies $\rho[Y_1] \leq \rho[Y_2]$.

(P2) *Positive homogeneity*: for any $Y \in \Gamma$ and $a > 0$, one has that $\rho[aY] = a\rho[Y]$.

(P3) *Translation invariance*: for any $Y \in \Gamma$ and $b \in \mathbb{R}$, one has that $\rho[Y + b] = \rho[Y] + b$.

(P4) *Subadditivity*: for any $Y_1, Y_2 \in \Gamma$, one has that $\rho[Y_1 + Y_2] \leq \rho[Y_1] + \rho[Y_2]$.

(P5) *Additivity of comonotonic risks*: for any $Y_1, Y_2 \in \Gamma$ which are comonotonic, one has that $\rho[Y_1 + Y_2] = \rho[Y_1] + \rho[Y_2]$.

In Artzner et al. (1999) a risk measure that satisfies the properties of monotonicity, positive homogeneity, translation invariance and (most noticeably) subadditivity is called a *coherent* risk measure.

Note that for the translation invariance property, we have that adding a positive (negative) amount increases (decreases) the risk since we consider in what follows Y as a loss when in a particular state ω of the world $Y(\omega) > 0$. A negative outcome will be considered as a gain.

2.2 Some well-known risk measures

With the notations of Dhaene et al. (2006) we recall some well-known risk measures frequently used in financial and actuarial literature.

A very popular risk measure is the p-quantile risk measure, often called the **Value-at-Risk** (VaR) at level p. For any p in $(0, 1)$, the p-quantile risk measure for a random variable Y, which will be denoted by $Q_p[Y]$, is defined by

$$Q_p[Y] = \inf\{x \in \mathbb{R} \mid F_Y(x) \geq p\}, \qquad p \in (0, 1), \tag{1}$$

where $F_Y(x) = \Pr[Y \leq x]$ is the cumulative distribution function (cdf) of Y taken under the same measure as in which the risk measure is considered. We also introduce the risk measure $Q_p^+[Y]$ which is defined by

$$Q_p^+[Y] = \sup\{x \in \mathbb{R} \mid F_Y(x) \leq p\}, \qquad p \in (0, 1). \tag{2}$$

Note that only values of p corresponding to a horizontal segment of F_Y lead to different values of $Q_p[Y]$ and $Q_p^+[Y]$, a case that can occur when the cdf F_Y is not strictly increasing.

Artzner et al. (1999) pointed out several drawbacks of VaR and therefore suggested the use of the coherent risk measure Conditional VaR (CVaR). CVaR is also known as TVaR, or **Tail Value-at-Risk** and is defined for a level p, element of $(0, 1)$, as follows:

$$\text{TVaR}_p[Y] = \frac{1}{1-p} \int_p^1 Q_q[Y] dq. \tag{3}$$

This formula boils down to taking the arithmetic average of the quantiles of Y, from the threshold p on, and hence $\text{TVaR}_p[Y]$ will always be larger than the corresponding quantile.

A third risk measure is the **Conditional Tail Expectation** (CTE) at level p denoted by $\text{CTE}_p[Y]$ and defined as

$$\text{CTE}_p[Y] = \text{E}[Y \mid Y > Q_p[Y]], \qquad p \in (0,1). \tag{4}$$

The conditional tail expectation at level p can be seen as the mean of the top $(1-p)\%$ losses. It can also be interpreted as the VaR at level p augmented by the average exceeding of the claims Y over that quantile, given that such an exceeding occurs.

If the cumulative distribution function of Y is continuous, $\text{TVaR}_p[Y]$ equals $\text{CTE}_p[Y]$ otherwise it holds that

$$\text{CTE}_p[Y] = \text{TVaR}_{F_Y(Q_p[Y])}[Y], \qquad p \in (0,1).$$

We also introduce the **Conditional Left Tail Expectation**, denoted by $\text{CLTE}_p[Y]$, which for continuous random variables is defined as

$$\text{CLTE}_p[Y] = \text{E}[Y \mid Y < Q_p[Y]], \qquad p \in (0,1). \tag{5}$$

Another class of risk measures is formed by the concave **distorted risk measures** (DRM) defined as

$$\text{DRM}[Y] = \int_{-\infty}^{0} \Psi(F_Y(x))dx + \int_{0}^{\infty} (\Psi(F_Y(x)) - 1)dx,$$

where $\Psi : [0,1] \to [0,1]$ is an increasing concave function with properties $\Psi(0) = 0$ and $\Psi(1) = 1$ (Wang, 1996). This risk measure can be viewed as an expectation under a distortion of the probability distribution effected by the function Ψ. It can be shown (Wirch and Hardy, 1999) that the TVaR is a special case obtained by a bilinear distortion. Distortion risk measures can be viewed as Choquet integrals (Denneberg, 1990, 1994). It turns out that this class of functionals (with different Ψ) is exactly the class of **Weighted Value-at-Risk**s (WVaR) (with different μ) defined as

$$\text{WVaR}_\mu[Y] = \int_0^1 \text{TVaR}_p[Y]\mu(dp),$$

where μ is a probability measure on $[0,1]$ (Cherny, 2006).
All these measures mentioned above satisfy the properties (P1) up to (P5) except of VaR which is well-known not to be subadditive.

In the sequel we will be confronted with the quantiles of a function of a random variable. The following lemma (Dhaene et al., 2002a) will allow us to express those as a function in terms of quantiles of the random variable.

Lemma 2.1 (Quantiles transformed random variables) *Let Y be a real-valued random variable, and $0 < p < 1$. For any non-decreasing and left continuous function g, it holds that*

$$Q_p[g(Y)] = g(Q_p[Y]). \tag{6}$$

On the other hand, for any non-increasing and right continuous function g, one has

$$Q_p[g(Y)] = g(Q^+_{1-p}[Y]). \tag{7}$$

From equation (7) it easily follows that for any non-increasing linear function function g, one also has

$$\text{TVaR}_p[g(Y)] = g\left(\frac{1}{1-p}\int_p^1 Q^+_{1-q}[Y]dq\right). \tag{8}$$

3 THE HEDGING PROBLEM

We will show that the results of Ahn et al. (1999), Annaert et al. (2007) and Deelstra et al. (2007) can be extended to a whole class of risk measures. In this section, we assume that we have, at time zero, one financial asset and we will sell this asset at time T. When this financial asset has a maturity S, we assume that T is prior to S. In case of a decrease of its value, not hedging can lead to severe losses. Therefore, the company decides to spend an amount C on hedging. This amount will be used to buy one or part of a put option with the asset as underlying, so that, in case of a substantial decrease in the asset price, the put option can be exercised in order to prevent large losses. The remaining question now is how to choose the strike price of the option.

We will find the optimal strike price which minimizes a given risk measure for a given hedging cost. An alternative interpretation of our set up is that it can be used to calculate the minimal hedging budget the firm has to spend in order to achieve a specified risk measure level. The latter setup was followed by Miyazaki (2001) for the case of a bond and VaR.

3.1 The loss function

Let us assume that the institution has an exposure to a risky financial asset $X(> 0)$, which may have a maturity S as in the case of a bond, and that the company has decided to hedge the value by using a percentage h ($0 < h < 1$) of one put option $P(0, T, K)$ with strike price K and exercise date T (with $T \leq S$).

We look at the future value of the hedged portfolio that is composed of the asset X and the put option $P(0, T, K)$ at time T as a function of the form

$$H(T) = \max(hK + (1-h)X(T), X(T)).$$

Taking into account the cost of setting up our hedged portfolio, which is given by the sum of the asset price $X(0)$ and the cost C of the position in the put option, we get for the value of the loss at time zero:

$$L = X(0) + C - \max(hK + (1-h)X(T), X(T)), \tag{9}$$

where we work with the nominal values instead of the discounted ones. Discounting by means of a deterministic risk free interest rate or by means of a bond will only introduce an additional constant factor in front of $H(T)$. The case of a stochastic interest rate has to be treated separately. Omitting the discount factor will not introduce a large error in case of a short time period. Moreover the goal of the paper is to test the quality of the in this paper proposed approximation methods compared to Monte-Carlo approximation methods.

We will replace the real loss L by a higher loss, corresponding to the worst case scenario, namely when the put option finishes in-the-money (i.e. $X(T) < K$). The corresponding future value of the portfolio equals

$$H_{ITM}(T) = (1-h)X(T) + hK,$$

and the related loss function is:

$$L_{ITM} = X(0) + C - ((1-h)X(T) + hK). \tag{10}$$

Obviously, $L_{ITM} \geq L$ and by the monotonicity of a risk measure also $\rho[L_{ITM}] \geq \rho[L]$.

We can state the following result.

Proposition 3.1 *Consider a risk measure which satisfies the properties (P1), (P2) and (P3), then the risk of the loss function L, equation* (9)*, satisfies*

$$\rho[L] \leq \rho[L_{ITM}]$$

with L_{ITM} given by equation (10) *and*

$$\rho[L_{ITM}] = \rho[X(0)+C-((1-h)X(T)+hK)] = X(0)+C-hK+(1-h)\rho[-X(T)]. \tag{11}$$

3.2 Risk minimization

We study the case of determining the optimal strike K when minimizing the risk of L in the worst case scenario, thus of the risk L_{ITM}, under a constraint on the hedging cost.

The risk measure of a portfolio $H = \{X, h, P\}$ consisting of a risky asset X and h put options P (which are assumed to be in-the-money at expiration) with a strike price K and an expiry date T is given by equation (11)[1].

We would like to find the optimal strike K^* of the put options by minimizing the risk of the future value $H_{ITM}(T)$ of the hedged portfolio, given a maximum hedging expenditure C. More precisely, we consider the constrained minimization problem

$$\min_{K,h} \rho[L_{ITM}] \tag{12}$$

$$\text{subject to } C = hP(0,T,K) \text{ and } h \in (0,1), \tag{13}$$

with $\rho[L_{ITM}]$ given by equation (11).

Theorem 3.2 *Let $\rho[\cdot]$ be a risk measure that satisfies the properties (P1), (P2) and (P3). Then the strike K^* of the put option P in the hedged portfolio $H = \{X, h, P\}$ which is optimal for the minimization problem* (12)-(13)*, solves the following implicit equation:*

$$P(0,T,K) - (K + \rho[-X(T)])\frac{\partial P}{\partial K}(0,T,K) = 0. \tag{14}$$

The optimal fraction h^* is given by $C/P(0,T,K^*)$ and the minimal $\rho[L_{ITM}]$ by $X(0)+C-h^*K^*+(1-h^*)\rho[-X(T)]$.

[1] In case of an unhedged portfolio, take $C = h = 0$ in equations (10) and (11) to obtain the loss function $L(= L_{ITM})$ with corresponding risk measure $\rho[L](= \rho[L_{ITM}])$.

Proof. The constrained optimization problem (12)-(13) has the Lagrange function

$$\mathcal{L}(K, h, \lambda) = \rho[L_{ITM}] - \lambda(C - hP(0, T, K)),$$

containing one multiplicator λ. Note that the multiplicators to include the inequalities $0 < h$ and $h < 1$ are zero since these constraints are not binding. Taking into account that the optimal strike K^* will differ from zero, we find from the Kuhn-Tucker conditions

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial K} = -h + h\lambda \dfrac{\partial P}{\partial K}(0, T, K) = 0 \\ \dfrac{\partial \mathcal{L}}{\partial h} = -(K + \rho[-X(T)]) + \lambda P(0, T, K) = 0 \\ \dfrac{\partial \mathcal{L}}{\partial \lambda} = -C + hP(0, T, K) = 0 \\ 0 < h < 1 \qquad \text{and} \qquad \lambda > 0 \end{cases}$$

that this optimal strike K^* should satisfy equation (14). The optimal values for h and $\rho[L_{ITM}]$ follow immediately from the condition (13) and relation (11). □

Corollary 3.3 *When we in addition assume that the put option is given by a discounted expectation under an appropriate measure, i.e.*

$$P(0, T, K) = \textit{disc} \cdot \boldsymbol{E}[(K - X(T))_+], \tag{15}$$

where disc is a short hand notation for discount factor[2] *and that the cdf $F_{X(T)}$ of $X(T)$ is continuous then the optimal strike K^* solves*

$$P(0, T, K) - \textit{disc} \cdot (K + \rho[-X(T)])F_{X(T)}(K) = 0. \tag{16}$$

Important remarks

1. We note that the optimal strike price is independent of the hedging cost C. This independence implies that for the optimal strike K^*, $\rho[L_{ITM}]$ in equation (11) is a linear function of h or C:

$$\rho[L_{ITM}] = X(0) + \rho[-X(T)] + h(P(0, T, K^*) - \rho[-X(T)] - K^*) \tag{17}$$

$$= X(0) + \rho[-X(T)] + C\left(1 - \frac{\rho[-X(T)] + K^*}{P(0, T, K^*)}\right). \tag{18}$$

   So, there is a linear trade-off between the hedging expenditure and the risk measure level. Although the set up of the paper is determining the strike price which minimizes a certain risk criterion, given a predetermined hedging budget, this trade-off shows that the analysis and the resulting optimal strike price can evidently also be used in the case where a firm is fixing a nominal value for the risk criterion and seeks the minimal hedging expenditure needed to achieve this risk level. It is clear that, once the optimal strike price is known, we can determine, in both approaches, the remaining unknown variable (either $\rho[L_{ITM}]$ or C).

[2]The discount factor can be e^{-rT} with r the risk free interest rate or $\mathrm{ZB}(0, T)$ a zero-coupon bond which pays 1 at time T.

2. We also note that the optimal strike price is higher than the risk measure level $-\rho[-X(T)]$. This has to be the case since $P(0,T,K)$ is always positive and the change in the price of a put option due to an increase in the strike is also positive. This result is also quite intuitive since there is no point in taking a strike price which is situated below the asset price you expect in a worst case scenario.

3. When we would like to work with only option prices received from a financial institution or the market, we could use the following approximate form instead of equation (14):

$$P(0,T,K) - (K + \rho[-X(T)])\frac{\triangle P}{\triangle K}(0,T,K) = 0. \tag{19}$$

Of course this will lead to an approximate value for the optimal strike. On the other hand one works with the true option prices and not with the theoretical ones.

VaR, TVaR and CTE minimization

In this section, we focus on three particular model dependent risk measures VaR, TVaR and CTE that satisfy the properties of monotonicity, translation invariance and positive homogeneity. Recall that TVaR and CTE coincide when the cdf of the random variable of which the risk is measured is continuous.

The VaR at a level p of the loss L_{ITM} is minimized under the given budget constraint for the optimal strike K^* that satisfies equation (14) with

$$\rho[-X(T)] = Q_p[-X(T)] = -Q^+_{1-p}[X(T)], \tag{20}$$

where we used equation (7) from Lemma 2.1 in the last equality.

For the corresponding TVaR at level p of the loss L_{ITM} the optimal strike is a solution to equation (14) with

$$\rho[-X(T)] = \text{TVaR}_p[-X(T)] = \frac{-1}{1-p}\int_p^1 Q^+_{1-q}[X(T)]dq, \tag{21}$$

where in the last equality we invoked equation (8).

Finally, for the Conditional Tail Expectation at level p of the loss L_{ITM}, the optimal strike is a solution to equation (14) with

$$\rho[-X(T)] = \text{CTE}_p[-X(T)] = -\text{E}[X(T) \mid X(T) < Q^+_{1-p}[X(T)]], \tag{22}$$

where we applied Lemma 2.1.

When the cdf of $X(T)$ is strictly increasing, then $Q^+_{1-p}[X(T)] = Q_{1-p}[X(T)]$ in equations (20) and (21), and according to definition (5) relation (22) becomes

$$\rho[-X(T)] = \text{CTE}_p[-X(T)] = -\text{CLTE}_{1-p}[X(T)]. \tag{23}$$

4 COMONOTONICITY

We shortly introduce the concepts of convex order and comonotonicity which will be needed for the generalization of the hedging problem to the case of a portfolio of risky assets in the next section. For more details and proofs of the reported results on convex order and comonotonicity we refer the reader to the overview papers by Dhaene et al. (2002a,b) and the references therein.

Definition 4.1 *A random variable X is said to precede a random variable Y in the convex order sense, notation $X \leq_{cx} Y$, if for any convex function v*

$$E[v(X)] \leq E[v(Y)].$$

The value $F_X^{-1}(p)$ ($p \in [0,1]$) of the inverse of a cumulative distribution function F_X of a random variable X is usually defined as the p-quantile $Q_p[X]$, cfr. equation (1), while the inverse $F_X^{-1+}(p)$ equals $Q_p^+[X]$ defined in equation (2).

Next, we define comonotonicity of a random vector.

Definition 4.2 *A random vector $(Y_1, \ldots, Y_n)$ with marginal cdfs $F_{Y_i}(x) = \Pr[Y_i \leq x]$ is said to be comonotonic if it has the same distribution as $\left(F_{Y_1}^{-1}(U), F_{Y_2}^{-1}(U), \ldots, F_{Y_n}^{-1}(U)\right)$, with U a random variable which is uniformly distributed on the unit interval $(0,1)$.*

The components of the comonotonic random vector $\left(F_{Y_1}^{-1}(U), F_{Y_2}^{-1}(U), \ldots, F_{Y_n}^{-1}(U)\right)$ are maximally dependent in the sense that all of them are non-decreasing functions of the same random variable. The following characterisation also holds:

Property 4.3 *A random vector $(Y_1, \ldots, Y_n)$ is said to be comonotonic, if there exist a random variable Z and non-decreasing (or either non-increasing) functions $g_1, g_2, \ldots, g_n$: $\mathbb{R} \to \mathbb{R}$ such that*

$$(Y_1, Y_2, \ldots, Y_n) \stackrel{d}{=} (g_1(Z), g_2(Z), \ldots, g_n(Z)),$$

where the notation $\stackrel{d}{=}$ stands for 'equality in distribution'.

Consider a random vector $(Y_1, \ldots, Y_n)$. Its comonotonic counterpart $(Y_1^c, \ldots, Y_n^c)$ is a comonotonic random vector with the same marginal distributions:

$$(Y_1^c, \ldots, Y_n^c) \stackrel{d}{=} \left(F_{Y_1}^{-1}(U), F_{Y_2}^{-1}(U), \ldots, F_{Y_n}^{-1}(U)\right). \tag{24}$$

The sum of the components of $(Y_1^c, \ldots, Y_n^c)$ is denoted by S^c,

$$S^c = Y_1^c + \cdots + Y_n^c. \tag{25}$$

The distribution function of S^c is completely specified when the marginals F_{Y_i} are given. The probabilities $F_{S^c}(x)$ follow from

$$F_{S^c}(x) = \sup\left\{ p \in [0,1] \mid \sum_{i=1}^{n} F_{Y_i}^{-1}(p) \leq x \right\}. \tag{26}$$

The sum S^c in equation (25) is in convex order sense larger than the sum $S = Y_1 + \cdots + Y_n$ while the sum S^ℓ defined as

$$S^\ell = \mathrm{E}[S|\Lambda] = \sum_{i=1}^{n} \mathrm{E}[Y_i|\Lambda] \tag{27}$$

for any conditioning random variable Λ is smaller than S in convex order sense, i.e.

$$S^\ell \leq_{cx} S \leq_{cx} S^c. \tag{28}$$

The vector

$$(\mathrm{E}[Y_1|\Lambda], \ldots, \mathrm{E}[Y_n|\Lambda]) \tag{29}$$

is not necessarily comonotonic and hence the sum S^ℓ is not necessarily a comonotonic sum. To be comonotonic, the conditioning random variable should be carefully chosen such that Property 4.3 is satisfied.

5 MULTIPLE RISKS

We consider now the case that X is not one risky asset but a linear combination of several risky assets as for example a basket of asset prices or a coupon-bearing bond. Thus, let us assume that

$$X = \sum_{i=1}^{n} a_i X_i \tag{30}$$

for some real positive constants a_i, $i = 1, \ldots, n$. Then the reasoning and results of section 3 remain valid. We can further elaborate the formulas under additional assumptions that are satisfied in some practical cases. In comparison with our previous papers we allow the marginal cumulative distribution functions F_{X_i} to be not strictly increasing.

5.1 Comonotonic sum

When we assume that the components X_i in equation (30) are non-independent random variables and form a comonotonic vector $(X_1, \ldots, X_n)$, then X is a comonotonic sum and we can apply the additivity of comonotonic risks and the positive homogeneity property to a risk measure $\rho[\cdot]$ satisfying the required properties:

$$\rho[-X(T)] = \rho\left[-\sum_{i=1}^{n} a_i X_i(T)\right] = \sum_{i=1}^{n} a_i \rho[-X_i(T)]. \tag{31}$$

When we further assume that the put option is given by a discounted expectation under an appropriate measure, see equation (15), we have a similar decomposition of the put option price:

$$P(0,T,K) = \sum_{i=1}^{n} a_i P_i(0,T,K_i) \quad \text{with} \sum_{i=1}^{n} a_i K_i = K, \tag{32}$$

where $P_i(0,T,K_i)$ is the put option with X_i as underlying and with maturity T and strike K_i. Kaas et al. (2000) provide a characterisation of the components K_i in the decomposition of the strike K:

$$K_i = \alpha F^{-1}_{X_i(T)}(F_{X(T)}(K)) + (1-\alpha)F^{-1+}_{X_i(T)}(F_{X(T)}(K)) := F^{-1(\alpha)}_{X_i(T)}(F_{X(T)}(K)), \tag{33}$$

where $\alpha \in (0,1)$ follows from $\sum_{i=1}^n a_i F^{-1(\alpha)}_{X_i(T)}(F_{X(T)}(K)) = K$, namely

$$\alpha = \frac{K - \sum_{i=1}^n a_i F^{-1+}_{X_i(T)}(F_{X(T)}(K))}{\sum_{i=1}^n a_i (F^{-1}_{X_i(T)}(F_{X(T)}(K)) - F^{-1+}_{X_i(T)}(F_{X(T)}(K)))} \tag{34}$$

when $F^{-1}_{X_i(T)}(F_{X(T)}(K)) \neq F^{-1+}_{X_i(T)}(F_{X(T)}(K))$ and where without loss of generality we take $\alpha = 1$ otherwise. We call the inverse in equation (33) the alpha-inverse cdf.

The probability measure under which the cdfs of the $X_i(T)$ are considered, is the same as the one of the expectation in the put option pricing rule but may differ from the probability measure in which the risk is considered.

The equation (14) for the optimal strike K^* of the hedging problem under consideration is now equivalent to the following set of equations:

$$\sum_{i=1}^n a_i P_i(0,T,K_i) - \left(K + \sum_{i=1}^n a_i \rho[-X_i(T)]\right) \sum_{i=1}^n a_i \frac{\partial P_i}{\partial K_i}(0,T,K_i)\frac{\partial K_i}{\partial K} = 0 \tag{35}$$

$$\sum_{i=1}^n a_i K_i = K \tag{36}$$

$$\sum_{i=1}^n a_i \frac{\partial K_i}{\partial K} = 1. \tag{37}$$

We can further simplify equation (35) when the marginal cdfs $F_{X_i(T)}$ are continuous since according to Breeden and Litzenberger (1978)

$$\frac{\partial P_i}{\partial K_i}(0,T,K_i) = \text{disc} \cdot F_{X_i(T)}(K_i). \tag{38}$$

Plugging in the expression (33) for K_i we find that this first order derivative is independent of i:

$$\frac{\partial P_i}{\partial K_i}(0,T,K_i) = \text{disc} \cdot F_{X_i(T)}(F^{-1(\alpha)}_{X_i(T)}(F_{X(T)}(K))) = \text{disc} \cdot F_{X(T)}(K). \tag{39}$$

Hence, using equations (37) and (39) we obtain

$$\sum_{i=1}^n a_i \frac{\partial P_i}{\partial K_i}(0,T,K_i)\frac{\partial K_i}{\partial K} = \text{disc} \cdot F_{X(T)}(K). \tag{40}$$

We substitute equations (33), (36) and (40) in equation (35). Then, to find the optimal solution of the constrained minimization problem (12)-(13) under the conditions that the risk measure $\rho[\cdot]$ satisfies properties (P1), (P2), (P3) and (P5), and that the marginal cdfs $F_{X_i(T)}$ are continuous, one has to proceed as follows:

Step 1 Denote

$$A_K := F_{X(T)}(K) \tag{41}$$

and solve the following equation for A_K:

$$\sum_{i=1}^{n} a_i P_i(0,T,F^{-1(\alpha)}_{X_i(T)}(A_K)) - \text{disc} \cdot A_K \sum_{i=1}^{n} a_i (F^{-1(\alpha)}_{X_i(T)}(A_K) + \rho[-X_i(T)]) = 0. \tag{42}$$

Step 2 Plug the found value for A_K in equation (33) and substitute the result in equation (36):

$$K^* = \sum_{i=1}^{n} a_i F^{-1(\alpha)}_{X_i(T)}(A_K). \tag{43}$$

Step 3 Plug this value for K^* in equation (13) and solve for h^*. Substitute finally both values K^* and h^* in equation (11) to find the corresponding minimal $\rho[L_{ITM}]$.

Remarks

1. As in the case of a single underlying, the optimal strike price is independent of the hedging cost and one can look at the trade-off between the hedging expenditure and the risk measure level cfr. equations (17)-(18).

2. In relation (31) we decomposed the risk in its components. However the derivation of equation (42) could also be done with $\rho[-X(T)]$ itself, leading to

$$\sum_{i=1}^{n} a_i P_i(0,T,F^{-1(\alpha)}_{X_i(T)}(A_K)) - \text{disc} \cdot A_K \left(\sum_{i=1}^{n} a_i F^{-1(\alpha)}_{X_i(T)}(A_K) + \rho[-X(T)] \right) = 0 \tag{44}$$

   instead of equation (42).

3. Relations (42)-(43) or (43)-(44) require the knowledge of the cdf of the $X_i(T)$. Another approach is to work with the approximate formula (19) in the multiple underlyings settings and use option prices received from a financial institution.

   An application of this can be found in Annaert et al. (2007) where $X(T)$ is a coupon-bearing bond and a short-rate model with an affine term structure is assumed. In the numerical illustration Annaert et al. (2007) focus on the one-factor Hull-White model for the instantaneous interest rate, which is calibrated to a set of cap prices. Both VaR and TVaR are considered as risk measures.

4. When we decompose the total cost C in a similar way as the risk $X(T)$ and the option $P(0,T,K)$, i.e.

$$C = \sum_{i=1}^{n} a_i C_i,$$

   then $C_i = hP_i(0,T,K_i)$ can be interpreted as the cost of the position in the put option $P_i(0,T,K_i)$ to hedge the loss incurred by $X_i(T)$.

5.2 Non-comonotonic sum

We study the case that the components X_i in equation (30) are non-independent random variables but do not form a comonotonic vector $(X_1, \ldots, X_n)$. In that case the decomposition property (31) for the risk does not longer hold, nor does the decomposition (32) for the put option price. One could of course make use of numerical/simulation methods to solve equation (14) or equation (16) for K both in the multiple risks setting.

Another approach consists in approximating the put option price by replacing the vector $(X_1, \ldots, X_n)$ by a comonotonic one, such as $(F^{-1}_{X_1(T)}(U), \ldots, F^{-1}_{X_n(T)}(U))$, see equation (24), with U being a uniform random variable on $(0, 1)$ or $(\mathrm{E}[X_1(T)|\Lambda], \ldots, \mathrm{E}[X_n(T)|\Lambda])$, see equation (29), with Λ a carefully chosen conditioning variable, and then applying the method of the previous section to it.

Let us denote

$$X_i^{\ell}(T) := \mathrm{E}[X_i(T)|\Lambda] \quad \text{and} \quad X_i^{c}(T) := F^{-1}_{X_i(T)}(U) \tag{45}$$

and

$$X^{\ell}(T) := \sum_{i=1}^{n} a_i X_i^{\ell}(T) \quad \text{and} \quad X^{c}(T) := \sum_{i=1}^{n} a_i X_i^{c}(T), \tag{46}$$

then $X^{\ell}(T)$ is a lower bound in convex order sense of $X(T)$ while $X^{c}(T)$ is an upper bound in convex order sense (see equation (28)) and by Definition 4.1

$$\mathrm{E}[(K - X^{\ell}(T))_+] \leq \mathrm{E}[(K - X(T))_+] \leq \mathrm{E}[(K - X^{c}(T))_+]. \tag{47}$$

Let us further denote the corresponding put option prices for $\nu = \ell$ and $\nu = c$ as:

$$P^{\nu}(0, T, K) = \text{disc} \cdot \mathrm{E}[(K - X^{\nu}(T))_+]. \tag{48}$$

$X^{c}(T)$ is by its definition a comonotonic sum, however one has to choose the conditioning variable Λ very carefully such that $X^{\ell}(T)$ is also a comonotonic sum. We give more details about this choice in section 6.

Assuming that $X^{\ell}(T)$ is a comonotonic sum, the following decomposition of the put option price holds in view of equations (32)-(34):

$$P^{\nu}(0, T, K) = \text{disc} \cdot \sum_{i=1}^{n} a_i \mathrm{E}[(K_i^{\nu} - X_i^{\nu}(T))_+] := \sum_{i=1}^{n} a_i P_i^{\nu}(0, T, K_i^{\nu}), \quad \nu = \ell, c, \tag{49}$$

with

$$K_i^{\nu} = F^{-1(\alpha)}_{X_i^{\nu}(T)}(F_{X^{\nu}(T)}(K))$$

satisfying $\sum_{i=1}^{n} a_i K_i^{\nu} = K$ and with $\alpha \in (0, 1)$ given by

$$\alpha = \frac{K - \sum_{i=1}^{n} a_i F^{-1+}_{X_i^{\nu}(T)}(F_{X^{\nu}(T)}(K))}{\sum_{i=1}^{n} a_i (F^{-1}_{X_i^{\nu}(T)}(F_{X^{\nu}(T)}(K)) - F^{-1+}_{X_i^{\nu}(T)}(F_{X^{\nu}(T)}(K)))}$$

when $F^{-1}_{X_i^{\nu}(T)}(F_{X^{\nu}(T)}(K)) \neq F^{-1+}_{X_i^{\nu}(T)}(F_{X^{\nu}(T)}(K))$ and with $\alpha = 1$ otherwise.

Case 1: $\rho[-X(t)]$ **can be calculated**
When it is possible to compute $\rho[-X(T)]$, we study the approximate constrained minimization problems for $\nu = \ell$ and $\nu = c$:

$$\begin{aligned} &\min_{K,h} X(0) + C - hK + (1-h)\rho[-X(T)] \\ &\text{subject to } C = hP^{\nu}(0,T,K) \text{ and } h \in (0,1). \end{aligned} \tag{50}$$

The corresponding optimal strikes K^*_ν, $\nu = \ell, c$, are found in two steps:

Step 1 Denote

$$A^{\nu}_K := F_{X^{\nu}(T)}(K) \tag{51}$$

and solve the following equation for A^{ν}_K:

$$\sum_{i=1}^{n} a_i P^{\nu}_i(0,T,F^{-1(\alpha)}_{X^{\nu}_i(T)}(A^{\nu}_K)) - \text{disc} \cdot A^{\nu}_K \left(\sum_{i=1}^{n} a_i F^{-1(\alpha)}_{X^{\nu}_i(T)}(A^{\nu}_K) + \rho[-X(T)] \right) = 0. \tag{52}$$

Step 2 Plug the found value for A^{ν}_K in the decomposition formula of K:

$$K^*_\nu = \sum_{i=1}^{n} a_i F^{-1(\alpha)}_{X^{\nu}_i(T)}(A^{\nu}_K). \tag{53}$$

Case 2: $\rho[-X(t)]$ **cannot be calculated**
When it is not possible to find $\rho[-X(T)]$ we could also approximate it by $\rho[-X^{\nu}(T)]$, $\nu = \ell, c$ which can be decomposed according to the additivity property (31) of comonotonic risks:

$$\rho[-X^{\nu}(T)] = \sum_{i=1}^{n} a_i \rho[-X^{\nu}_i(T)]. \tag{54}$$

Then, we solve the approximate constrained minimization problems for $\nu = \ell$ and $\nu = c$:

$$\begin{aligned} &\min_{K,h} X(0) + C - hK + (1-h)\rho[-X^{\nu}(T)] \\ &\text{subject to } C = hP^{\nu}(0,T,K) \text{ and } h \in (0,1). \end{aligned} \tag{55}$$

The corresponding optimal strikes K^*_ν, $\nu = \ell, c$, are found in two steps:

Step 1 Denote

$$A^{\nu}_K := F_{X^{\nu}(T)}(K) \tag{56}$$

and solve the following equation for A^{ν}_K:

$$\sum_{i=1}^{n} a_i P^{\nu}_i(0,T,F^{-1(\alpha)}_{X^{\nu}_i(T)}(A^{\nu}_K)) - \text{disc} \cdot A^{\nu}_K \sum_{i=1}^{n} a_i (F^{-1(\alpha)}_{X^{\nu}_i(T)}(A^{\nu}_K) + \rho[-X^{\nu}_i(T)]) = 0. \tag{57}$$

Step 2 Plug the found value for A_K^ν in the decomposition formula of K:

$$K_\nu^* = \sum_{i=1}^{n} a_i F_{X_i^\nu(T)}^{-1(\alpha)}(A_K^\nu). \tag{58}$$

For both cases the optimal fraction h_ν^*, $\nu = \ell, c$ is given by $C/P^\nu(0,T,K_\nu^*)$. With these optimal h_ν^* and K_ν^* we compute the value of the function in the minimisation problem (50) respectively (55) to find an approximation for the risk measure $\rho[L_{ITM}]$ in those cases.

Based on some ordering (stochastic dominance, stop-loss order, convex order) of the risks, an ordering of the risk measures has been studied (e.g. Dhaene et al., 2000). The mutual relation of the approximate option prices and the exact one is implied by equation (47). However in our study we are dealing with nonlinear constrained optimization problems for which we cannot say a priori something about the mutual position of the reached constrained minima. So, it is hard to derive from the theory which approximate problem will provide a good estimate for the real optimal strike K^*. From literature we know that when the $X_i(T)$ are lognormal and their weighted sum is 'nearly comonotonic', the cdf $F_{X^c(T)}$ will be close to the cdf $F_{X(T)}$ and also the put option price $P^c(0,T,K)$ will be a good approximation for $P(0,T,K)$. Even better approximations are obtained when working with $X^\ell(T)$. Moreover in that case one has one parameter to play with, namely the conditioning variable Λ. Several conditioning variables are given at the end of section 6.1.

We will do some numerical experiments in which we will compare the computed optimal strike K_ν^* with the optimal strike K^* obtained through simulations. First, we study the case of a basket of assets in a Black-Scholes setting. Further, we consider a coupon-bearing bond where a two-additive-factor Gaussian model (Brigo and Mercurio, 2001) is assumed for the instantaneous interest rate. For this model an analytical expression for the put option price on the coupon-bearing bond is available, see equation (86) below.

6 APPLICATION

6.1 VaR, TVaR and CTE minimization for a basket of assets

We consider a basket of n assets $X_1, \ldots, X_n$ with the corresponding weights $a_1, \ldots, a_n$ satisfying $\sum_{i=1}^n a_i = 1$. Let r denote the risk-free short rate and suppose that the risk-neutral price process $X_i(t)$ satisfies

$$dX_i(t) = (r - q_i)X_i(t)dt + \sigma_i X_i(t)dW_i(t), \qquad i = 1, \ldots, n$$

where q_i and σ_i are the dividend rate and the volatility of asset i respectively. $W_i(t)$ is a standard Brownian motion associated with the price process of asset i and we assume that the different asset prices are instantaneously correlated:

$$\text{cov}(dW_i(t), dW_j(t)) = \rho_{ij}dt, \qquad i,j = 1, \ldots, n. \tag{59}$$

Given the above dynamics the ith asset price at time t equals

$$X_i(t) = X_i(0)e^{(r - q_i - \frac{1}{2}\sigma_i^2)t + \sigma_i W_i(t)}. \tag{60}$$

Hence the asset price $X_i(t)$ has a lognormal distribution under the risk-neutral probability measure Q and we may write:

$$X_i(t) \stackrel{d}{=} e^{\Pi^Q(0,t,i)+\Sigma(0,t,i)Z_i}, \qquad Z_i \sim N(0,1), \tag{61}$$

with

$$\Pi^Q(0,t,i) = \ln X_i(0) + (r - q_i - \frac{1}{2}\sigma_i^2)t \qquad \Sigma(0,t,i) = \sigma_i\sqrt{t}. \tag{62}$$

Thus the price $X(t)$ of the basket itself is a sum of correlated lognormals which is however not a comonotonic sum. By the risk-neutral pricing method, the initial price of a European basket put option maturing at time T with strike price K is

$$\text{BP}(0,n,T,K) = e^{-rT}\text{E}^Q\left[\left(K - \sum_{i=1}^{n} a_i X_i(T)\right)_+\right],$$

where the expectation is considered w.r.t. the risk-neutral probability measure Q. There is no simple closed form expression for this basket option price. In a practical example we will compute it through simulation and try to obtain the optimal strike K^* by solving numerically the with equation (16) equivalent formula for multiple assets:

$$\text{BP}(0,n,T,K) - e^{-rT}(K + \rho[-X(T)])F_{X(T)}(K) = 0.$$

In here $\rho[-X(T)]$, with $\rho[\cdot]$ being VaR or TVaR(=CTE), will also be obtained by simulating the cdf $F_{X(T)}$ of a sum of lognormals for which the correlations are known. The in this way found optimal K^* will be used as a benchmark for the strikes that solve the approximate minimization problems (50) or (55) with $\nu = \ell, c$. For these last two methods we need explicit expressions for the cdfs and the put option prices in terms of the model parameters. Hereto we note that for $\nu = c$ and $\nu = \ell$, $X_i^\nu(T)$ is a continuous lognormal random variable of which the cdf $F_{X_i^\nu(T)}$ is strictly increasing, hence the alpha-inverse in equation (33) coincides with the standard inverse, and for $x \in [0, +\infty[$

$$F_{X_i^\nu(T)}(x) = \begin{cases} F_{X_i(T)}(x) & \text{if } \nu = c \quad (63a) \\ F_{\text{E}^Q[X_i(T)|\Lambda]}(x) & \text{if } \nu = \ell. \quad (63b) \end{cases}$$

It is well-known that for a lognormal distributed random variable with parameters $\Pi^Q(0,T,i)$ and $\Sigma(0,T,i)$ the inverse cdf is given by:

$$F^{-1}_{X_i(T)}(p) = e^{\Pi^Q(0,T,i)+\Sigma(0,T,i)\Phi^{-1}(p)}, \qquad p \in [0,1], \tag{64}$$

with $\Phi(\cdot)$ the cdf of the standard normal random variable.

Further, the following property based on Lemma 2.1 holds (Dhaene et al., 2002a): for $p \in (0,1)$

$$F^{-1}_{\text{E}^Q[X_i(T)|\Lambda]}(p) = \text{E}^Q[X_i(T)|\Lambda = F^{-1}_\Lambda(q)], \tag{65}$$

with

$$q = \begin{cases} p & \text{if } \text{E}^Q[X_i(T)|\Lambda] \text{ non-decreasing in } \Lambda \quad (66a) \\ 1-p & \text{if } \text{E}^Q[X_i(T)|\Lambda] \text{ non-increasing in } \Lambda. \quad (66b) \end{cases}$$

In what follows we will focus on the case that $\mathrm{E}^Q[X_i(T)|\Lambda]$ is a non-decreasing function of Λ, since for the model under consideration and the choices of Λ this will be the case. The formulas for the other case are completely analogous. The different choices for the conditioning variable Λ (see below) will depend on the measure Q. Therefore we already make this dependence clear in the notation from here on.

When the conditioning random variable Λ^Q is normally distributed with mean μ_{Λ^Q} and variance $\sigma^2_{\Lambda^Q}$ such that $(X_i(T), \Lambda^Q)$ is bivariate normally distributed, the random variable $X_i(T)$ conditionally on $\Lambda^Q = \lambda$ remains a lognormal random variable and the conditional expectation is:

$$\mathrm{E}^Q[X_i(T)|\Lambda^Q = F^{-1}_{\Lambda^Q}(p)] = e^{\Pi^Q(0,T,i)+r_i^Q\Sigma(0,T,i)\Phi^{-1}(p)+\frac{1}{2}(1-(r_i^Q)^2)\Sigma(0,T,i)^2}, \quad p \in (0,1), \quad (67)$$

where we used $F^{-1}_{\Lambda^Q}(p) = \mu_{\Lambda^Q} + \sigma_{\Lambda^Q}\Phi^{-1}(p)$ and with r_i^Q the correlation between Λ^Q and $\ln X_i(T)$ (see equation (61)). According to Kaas et al. (2000), when all correlations r_i^Q are positive, equation (67) implies that for all i $\mathrm{E}^Q[X_i(T)|\Lambda^Q]$ will be non-decreasing in Λ^Q and that hence by Property 4.3 the sum $X^\ell(T) = \sum_{i=1}^n a_i X_i^\ell(T)$ will be comonotonic.

The price of a put option with as underlying a lognormal distributed random variable with parameters μ and σ^2, maturing at T and with strike K is well-known:

$$e^{-rT}\mathrm{E}^Q[(K - e^{\mu+\sigma Z})_+] = e^{-rT}[K\Phi(-d_2) - e^{\mu+\frac{1}{2}\sigma^2}\Phi(-d_1)], \qquad Z \sim N(0,1), \quad (68)$$

with

$$d_1 = \frac{\mu + \sigma^2 - \ln K}{\sigma} \quad \text{and} \quad d_2 = d_1 - \sigma.$$

The price of the put option $P_i^c(0,T,K_i^c)$ written on $X_i^c(T)$ follows from this formula with

$$\mu = \Pi^Q(0,T,i), \quad \sigma = \Sigma(0,T,i), \quad \text{and} \quad K = K_i^c.$$

Similarly the price of the put option $P_i^\ell(0,T,K_i^\ell)$ written on $X_i^\ell(T)$ is given by equation (68) with

$$\mu = \Pi^Q(0,T,i) + \frac{1}{2}(1-(r_i^Q)^2)\Sigma(0,T,i)^2, \quad \sigma = r_i^Q\Sigma(0,T,i), \quad \text{and} \quad K = K_i^\ell.$$

For the case that $\rho[-X(T)]$ is not calculated/simulated, we have to compute in addition the risk measures VaR and TVaR — CTE equals in this case TVaR — of (the minus of) the lognormal random variable $X_i^\nu(T)$ as appearing in relation (57). General expressions for it can be found in several papers (see, e.g., Dhaene et al., 2006). We state them here explicitly in terms of the model parameters.

Combining the definition (1) of the Value-at-Risk at level p with equation (7) of Lemma 2.1 and expressing the resulting quantile in terms of the inverse cdf while recalling that the cdf is strictly increasing for a lognormal random variable, we successively obtain for $p \in (0,1)$:

$$\rho[-X_i^\nu(T)] = \mathrm{VaR}_p[-X_i^\nu(T)] = Q_p[-X_i^\nu(T)] = -Q^+_{1-p}[X_i^\nu(T)] = -F^{-1}_{X_i^\nu(T)}(1-p).$$

Finally, for $\nu = c$ invoke equation (64) while for $\nu = \ell$ apply equation (67).

The Tail Value-at-Risk for a level p is defined in equation (3). We have to combine it with equation (8) and write the integrand in terms of the cdf of $X_i^\nu(T)$:

$$\begin{aligned}\rho[-X_i^\nu(T)] = \text{TVaR}_p[-X_i^\nu(T)] &= \frac{1}{1-p}\int_p^1 Q_q[-X_i^\nu(T)]dq \\ &= -\frac{1}{1-p}\int_p^1 Q^+_{1-q}[X_i^\nu(T)]dq \\ &= -\frac{1}{1-p}\int_p^1 F^{-1}_{X_i^\nu(T)}(1-q)dq.\end{aligned}$$

We now invoke the expression (64) respectively (67) for the cdf, and evaluate the resulting integral:

$$\begin{aligned}\text{TVaR}_p[-X_i^c(T)] &= -\frac{e^{\Pi^Q(0,T,i)}}{1-p}\int_p^1 e^{\Sigma(0,T,i)\Phi^{-1}(1-q)}dq \\ &= -\frac{e^{\Pi^Q(0,T,i)+\frac{1}{2}\Sigma(0,T,i)^2}}{1-p}\Phi(\Phi^{-1}(1-p)-\Sigma(0,T,i)),\end{aligned}$$

respectively

$$\begin{aligned}\text{TVaR}_p[-X_i^\ell(T)] &= -\frac{e^{\Pi^Q(0,T,i)+\frac{1}{2}(1-(r_i^Q)^2)\Sigma(0,T,i)^2}}{1-p}\int_p^1 e^{r_i^Q\Sigma(0,T,i)\Phi^{-1}(1-q)}dq \\ &= -\frac{e^{\Pi^Q(0,T,i)+\frac{1}{2}\Sigma(0,T,i)^2}}{1-p}\Phi(\Phi^{-1}(1-p)-r_i^Q\Sigma(0,T,i)).\end{aligned}$$

Now we will discuss different choices for the (normally distributed) conditioning random variable Λ^Q for the case where $\nu = \ell$. Note that Λ^Q enters the formulas via the correlations r_i^Q. We will report the corresponding expressions for these correlations.

We differentiate between conditioning variables that are global and that are local. Global here means that they lead to lower bounds $X^\ell(T)$ (in convex order) for which the cdf presents a global goodness-of-fit when compared to the cdf of the original sum $X(T)$. When only a part of the distribution has to be well-fitted as for example the tails of the cdf, we can work with lower bound approximations which are locally optimal. Various choices have been proposed in literature, see below.

We define the conditioning random variable Λ^Q as a linear combination of $Z_1, \ldots, Z_n$, with Z_i showing up in the distribution of $X_i(T)$ in equation (61) and with the random vector $(Z_1, \ldots, Z_n)$ having a multivariate standard normal distribution:

$$\Lambda^Q = \sum_{i=1}^n \gamma_i^Q \Sigma(0,T,i) Z_i. \tag{69}$$

Then its correlation with $\ln X_i(T)$ is given by:

$$r_i^Q = \text{corr}(\ln X_i(T), \Lambda^Q) = \frac{\sum_{j=1}^n \gamma_j^Q \Sigma(0,T,j)\text{cov}(Z_i, Z_j)}{\sqrt{\sum_{i=1}^n\sum_{j=1}^n \gamma_i^Q\gamma_j^Q\Sigma(0,T,i)\Sigma(0,T,j)\rho_{ij}}}, \tag{70}$$

with in view of equations (59)-(62)

$$\mathrm{cov}(Z_i, Z_j) = \frac{\mathrm{cov}(\ln X_i(T), \ln X_j(T))}{\Sigma(0,T,i)\Sigma(0,T,j)} = \rho_{ij}. \tag{71}$$

By specifying the coefficients γ_i^Q in equation (69) we get different choices for Λ^Q and by substituting those expressions for the γ_i^Q's in equation (70) we get the corresponding correlations:

1. Taylor-based (Kaas et al., 2000): notation TB and $\gamma_i^Q = a_i e^{\Pi^Q(0,T,i)}$
2. Geometric average based (Nielsen and Sandmann, 2003): notation GA and
$$\gamma_i^Q = \frac{a_i}{\sqrt{\sum_{i=1}^n \sum_{j=1}^n a_i a_j \Sigma(0,T,i)\Sigma(0,T,j)\rho_{ij}}}$$
3. Maximal variance (Vanduffel et al., 2005): notation MV and $\gamma_i^Q = a_i e^{\Pi^Q(0,T,i)+\frac{1}{2}\Sigma(0,T,i)^2}$
4. Maximal CTE and minimal CLTE (Vanduffel et al., 2008): notation MCTE and
$$\gamma_i^Q = a_i e^{\Pi^Q(0,T,i)+\frac{1}{2}\Sigma(0,T,i)^2} \cdot e^{\frac{1}{2}(\Phi^{-1}(1-p)-r_i^{Q,\mathrm{MV}}\Sigma(0,T,i))^2},$$
where $r_i^{Q,\mathrm{MV}}$ denotes the correlation (70) between $\ln X_i(T)$ and MV and with p the level of the risk measure $\mathrm{CTE}_p[-X(T)] = \mathrm{TVaR}_p[-X(T)]$.

TB, GA and MV are global conditioning variables while MCTE is a local conditioning variable.

6.2 The two-additive-factor Gaussian model G2++

In this section we consider as an application coupon bonds in the two-additive-factor Gaussian model, shortly the G2++ model, which is an interest-rate model where the instantaneous short-rate process is given by the sum of two correlated Gaussian factors plus a deterministic function that is properly chosen so as to exactly fit the current term structure of discount factors. The model is quite analytically tractable in that explicit formulas for discount bonds, European options on pure discount bonds, hence caps and floors, can be readily derived.

Another consequence of the presence of two factors is that the actual variability of market rates is described in a better way: among other improvements, a non-perfect correlation between rates of different maturities is introduced.

The Gaussian model of this section is naturally related to the Hull and White (1994) two-factor model in that one can actually prove the equivalence between these two approaches. However, according to Brigo and Mercurio (2001), the formulation with two additive factors leads to less complicated formulas and is easier to implement in practice, even though we may lose some insight and intuition on the nature and the interpretation of the two factors.

The Short-Rate Dynamics

We assume that the dynamics of the instantaneous short-rate process under the risk-adjusted measure Q is given by

$$r(t) = x(t) + y(t) + \varphi(t), \quad r(0) = r_0, \tag{72}$$

where the processes $\{x(t) : t \geq 0\}$ and $\{y(t) : t \geq 0\}$ satisfy

$$dx(t) = -ax(t)dt + \sigma d\hat{W}_1(t), \quad x(0) = 0, \tag{73}$$

$$dy(t) = -by(t)dt + \eta d\hat{W}_2(t), \quad y(0) = 0, \tag{74}$$

where $(\hat{W}_1, \hat{W}_2)$ is a two-dimensional Brownian motion with instantaneous correlation ρ:

$$d\hat{W}_1(t)d\hat{W}_2(t) = \rho dt, \tag{75}$$

where r_0, a, b, σ, η are positive constants, and where $-1 \leq \rho \leq 1$. The function φ is deterministic and well defined in the time interval $[0, T^*]$, with T^* a given time horizon, typically 10, 30 or 50 (years). In particular, $\varphi(0) = r_0$. We denote by $\mathcal{F}_t$ the sigma-field generated by the pair (x, y) up to time t.

Simple integration implies that for each $t > 0$

$$r(t) = \sigma \int_0^t e^{-a(t-u)} d\hat{W}_1(u) + \eta \int_0^t e^{-b(t-u)} d\hat{W}_2(u) + \varphi(t), \tag{76}$$

meaning that $r(t)$ (conditional on $\mathcal{F}_0$) is normally distributed.

When the term structure of discount factors that is currently observed in the market is given by the sufficiently smooth function $S \mapsto \mathrm{ZB}^M(0, S)$, then the function φ is determined in function of the current observed bond prices $\mathrm{ZB}^M(0, S)$ and the price at time t of a zero-coupon bond maturing at S can be written as:

$$\mathrm{ZB}(t, S) = A(t, S) \exp[-B(a, t, S)x(t) - B(b, t, S)y(t)], \tag{77}$$

where

$$A(t, S) = \frac{\mathrm{ZB}^M(0, S)}{\mathrm{ZB}^M(0, t)} \exp\left\{\frac{1}{2}[V(t, S) - V(0, S) + V(0, t)]\right\} \tag{78}$$

$$B(z, t, S) = \frac{1 - e^{-z(S-t)}}{z} \tag{79}$$

and

$$\begin{aligned} V(t, S) = &\frac{\sigma^2}{a^2}\left[S - t - 2B(a, t, S) + B(2a, t, S)\right] \\ &+ \frac{\eta^2}{b^2}\left[S - t - 2B(b, t, S) + B(2b, t, S)\right] \\ &+ 2\rho\frac{\sigma\eta}{ab}\left[S - t - B(a, t, S) - B(b, t, S) + B(a + b, t, S)\right]. \end{aligned}$$

Hence a zero-coupon bond $\mathrm{ZB}(t,S)$ has a lognormal distribution under Q:

$$\mathrm{ZB}(t,S) \stackrel{d}{=} e^{\Pi^Q(0,t,S)+\Sigma(0,t,S)Z}, \quad Z \sim N(0,1), \tag{80}$$

with

$$\Pi^Q(0,t,S) = \ln A(t,S) = \ln\left(\frac{\mathrm{ZB}^M(0,S)}{\mathrm{ZB}^M(0,t)}\right) + \frac{1}{2}\left[V(t,S) - V(0,S) + V(0,t)\right], \tag{81}$$

and with

$$\begin{aligned}\Sigma(0,t,S)^2 =& \sigma^2 B(a,t,S)^2 B(2a,0,t) + \eta^2 B(b,t,S)^2 B(2b,0,t) \\ &+ 2\rho\sigma\eta B(a,t,S)B(b,t,S)B(a+b,0,t).\end{aligned} \tag{82}$$

Besides, a zero-coupon bond $\mathrm{ZB}(t,S)$ also has a lognormal distribution of the form (80) under the forward-neutral measure Q^T with the same variance parameter $\Sigma(0,t,S)$ but with a different mean parameter $\Pi^{Q^T}(0,t,S)$:

$$\Pi^{Q^T}(0,t,S) = \ln\left(\frac{\mathrm{ZB}^M(0,S)}{\mathrm{ZB}^M(0,t)}\right) - \frac{1}{2}\Sigma(0,t,S)^2, \tag{83}$$

The time zero price of a European put option with maturity T and strike K written on a zero-coupon bond with unit face value and maturity S is given by (see Brigo and Mercurio, 2001):

$$\begin{aligned}\mathrm{ZBP}(0,T,S,K) =& \mathrm{ZB}(0,T)K\Phi\left(\frac{\ln\frac{K\mathrm{ZB}(0,T)}{\mathrm{ZB}(0,S)}}{\Sigma(0,T,S)} + \frac{1}{2}\Sigma(0,T,S)\right) \\ &- \mathrm{ZB}(0,S)\Phi\left(\frac{\ln\frac{K\mathrm{ZB}(0,T)}{\mathrm{ZB}(0,S)}}{\Sigma(0,T,S)} - \frac{1}{2}\Sigma(0,T,S)\right).\end{aligned} \tag{84}$$

The arbitrage-free time zero price of a put option with maturity date T and strike K upon a coupon-bearing bond with price at time T

$$\mathrm{CB}(T,\mathcal{T},\mathcal{C}) = \sum_{i=1}^{n} c_i \mathrm{ZB}(T,S_i) \tag{85}$$

with coupons $\mathcal{C} = \{c_1,\ldots,c_n\}$ paid out at times $\mathcal{T} = \{S_1,\ldots,S_n\}$ larger than T, can be derived by a similar proof as in Brigo and Mercurio (2001, chapter 4, appendix D) and is given by numerically computing the following one-dimensional integral:

$$\begin{aligned}&\mathrm{CBP}\left(0,T,\mathcal{T},\mathcal{C},K\right) \\ &= \mathrm{ZB}(0,T)\int_{-\infty}^{+\infty} \frac{e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}}{\sigma_x\sqrt{2\pi}}\left[K\Phi(-h_1(x,K)) - \sum_{i=1}^{n}\lambda_i(x)e^{\kappa_i(x)}\Phi(-h_{2i}(x,K))\right]dx,\end{aligned} \tag{86}$$

where

$$h_1(x,K) := \frac{\hat{y}(x,K)-\mu_y}{\sigma_y\sqrt{1-\rho_{xy}^2}} - \frac{\rho_{xy}(x-\mu_x)}{\sigma_x\sqrt{1-\rho_{xy}^2}}$$
$$h_{2i}(x,K) := h_1(x,K) + B(b,T,S_i)\sigma_y\sqrt{1-\rho_{xy}^2}$$
$$\lambda_i(x) := c_i A(T,S_i)e^{-B(a,T,S_i)x}$$
$$\kappa_i(x) := -B(b,T,S_i)\left[\mu_y - \frac{1}{2}(1-\rho_{xy}^2)\sigma_y^2 B(b,T,S_i) + \rho_{xy}\sigma_y\frac{x-\mu_x}{\sigma_x}\right],$$

with $\hat{y} = \hat{y}(x,K)$ the solution of the following equation for each fixed x

$$\sum_{i=1}^{n} c_i A(T,S_i)e^{-B(a,T,S_i)x - B(b,T,S_i)\hat{y}} = K \tag{87}$$

and

$$\sigma_x := \sigma\sqrt{B(2a,0,T)}, \qquad \sigma_y := \eta\sqrt{B(2b,0,T)}, \tag{88}$$
$$\rho_{xy} := \frac{\rho\sigma\eta}{\sigma_x\sigma_y}B(a+b,0,T) \tag{89}$$
$$\mu_x := -\left(\frac{\sigma^2}{a} + \rho\frac{\sigma\eta}{b}\right)B(a,0,T) + \frac{\sigma_x^2}{a} + \frac{\rho_{xy}\sigma_x\sigma_y}{b}$$
$$\mu_y := -\left(\frac{\eta^2}{b} + \rho\frac{\sigma\eta}{a}\right)B(b,0,T) + \frac{\sigma_y^2}{b} + \frac{\rho_{xy}\sigma_x\sigma_y}{a}.$$

VaR, TVaR and CTE minimization

For the interest rate model under consideration, $X(T)$ in equation (30) equals $\mathrm{CB}(T,\mathcal{T},\mathcal{C})$, see equation (85), and is in view of equation (77) a sum of lognormal random variables.

As for a basket of assets we first obtain the optimal strike K^* by solving numerically the with equation (14) equivalent formula for multiple risks:

$$\mathrm{CBP}(0,T,\mathcal{T},\mathcal{C},K) - (K + \rho[-\mathrm{CB}(T,\mathcal{T},\mathcal{C})])\frac{\partial \mathrm{CBP}}{\partial K}(0,T,\mathcal{T},\mathcal{C},K) = 0, \tag{90}$$

where the first order derivative w.r.t. K is via standard calculations found to be:

$$\frac{\partial \mathrm{CBP}}{\partial K}(0,T,\mathcal{T},\mathcal{C},K) = \mathrm{ZB}(0,T)\int_{-\infty}^{+\infty}\frac{e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}}{\sigma_x\sqrt{2\pi}}\Phi(-h_1(x,K))dx. \tag{91}$$

The put option price (86) and its derivative (91) have a common part, by which equation (90) can be simplified to

$$\mathrm{ZB}(0,T)\int_{-\infty}^{+\infty}\frac{e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}}{\sigma_x\sqrt{2\pi}}\left[\rho[-\mathrm{CB}(T,\mathcal{T},\mathcal{C})]\Phi(-h_1(x,K)) + \sum_{i=1}^{n}\lambda_i(x)e^{\kappa_i(x)}\Phi(-h_{2i}(x,K))\right]dx = 0.$$

To calculate $\rho[-\mathrm{CB}(T,\mathcal{T},\mathcal{C})]$ for VaR or TVaR(=CTE), we simulate the cdf of the coupon-bearing bond as sum of lognormals for which the correlations can be computed, see equations (80) and (92) below.

This optimal strike K^* will be used as benchmark for strikes found as solution to the approximated problems (50) respectively (55). For these last two methods we need explicit expressions for the cdfs and the put option prices in terms of the model parameters. Hereto we note that $\mathrm{ZB}(T,S_i)$ in equation (80) has a similar form as $X_i(T)$ in equation (61). Also $\mathrm{ZB}^\nu(T,S_i)$ with $\nu = c$ or $\nu = \ell$, will be a continuous lognormal random variable with a cdf similar to the one of $X_i^\nu(T)$, see equations (64)-(66). Hence most formulas of section 6.1 can be taken over mutatis mutandis. A special attention has to be paid to the parameter $\Pi^{\mathbb{Q}}(0,T,S_i)$ where $\mathbb{Q} = Q$ or Q^T. Hence now the conditioning variable Λ and its coefficients γ as well as the correlation coefficient r_i depend on $\mathbb{Q}$ with $\mathbb{Q} = Q$ or Q^T. In section 6.1 we had only $\mathbb{Q} = Q$.

In the expression (70) of these correlation coefficients we further elaborate the factor $\mathrm{cov}(Z_i, Z_j)$ in equation (71) by means of relations (77), (79), (88) and (89):

$$\begin{aligned}
&\mathrm{cov}(\ln \mathrm{ZB}(T,S_i), \ln \mathrm{ZB}(T,S_j)) \\
&= B(a,T,S_i)B(a,T,S_j)\sigma_x^2 + B(b,T,S_i)B(b,T,S_j)\sigma_y^2 \\
&\quad + [B(a,T,S_i)B(b,T,S_j) + B(a,T,S_j)B(b,T,S_i)]\sigma_x\sigma_y\rho_{xy}. \qquad (92)
\end{aligned}$$

In particular for $i = j$, formula (92) equals $\Sigma(0,T,S_i)^2$ or $\Sigma(0,T,S_j)^2$, which is in accordance with equation (82). The price of a put option with the lognormal $\mathrm{ZB}^\nu(T,S_i)$ as underlying and with maturity T and strike K can be derived from equation (84). For $\nu = \ell$, $\Sigma(0,T,S_i)$ has to be replaced by $r_i^{\mathbb{Q}}\Sigma(0,T,S_i)$.

6.3 Numerical results

Basket of assets

We consider a basket of seven stock indices underlying the G-7 index-linked guaranteed investment certificates offered by Canada Trust Co (Milevsky and Posner, 1998a,b). The initial value of each index is normalized to be 100 while the other influence parameters are given in Tables 1 and 2. The risk free interest rate r equals 6.3%.

country	index	weight (in %)	volatility (in %)	dividend yield (in %)
Canada	TSE 100	10	11.55	1.69
Germany	DAX	15	14.53	1.36
France	CAC 40	15	20.68	2.39
U.K.	FSTE 100	10	14.62	3.62
Italy	MIB 30	5	17.99	1.92
Japan	Nikkei 225	20	15.59	0.81
U.S.	S&P 500	25	15.68	1.66

Table 1: G-7 index linked guaranteed investment certificate weightings

	Canada	Germany	France	U.K.	Italy	Japan	U.S.
Canada	1.00	0.35	0.10	0.27	0.04	0.17	0.71
Germany	0.35	1.00	0.39	0.27	0.50	-0.08	0.15
France	0.10	0.39	1.00	0.53	0.70	-0.23	0.09
U.K.	0.27	0.27	0.53	1.00	0.45	-0.22	0.32
Italy	0.04	0.50	0.70	0.46	1.00	-0.29	0.13
Japan	0.17	-0.08	-0.23	-0.22	-0.29	1.00	-0.03
U.S.	0.71	0.15	0.09	0.32	0.13	-0.03	1.00

Table 2: Correlation structure

In a first scenario we set $T = 1$ and consider the VaR and TVaR risk measures with $p = 0.95$ and $p = 0.99$. To find the optimal strike price we generate 10,000,000 sample paths of each stock index and calculate the corresponding value of the basket. The results of the optimization routine are given in Table 3. Next to the simulated optimal strike price K^*, we calculate the (sub)optimal strike prices K_ν^* based on the approximations $X^\nu(T)$. Note that, although several negative correlations are involved, all r_i are positive and hence $X^\ell(T)$ is a comonotonic sum.

The suboptimal strike K_c^* (UB) appears to be rather far off. This was to be expected because it assumes a comonotonic dependence structure and neglects the real (smaller) dependence between the stock indices. The suboptimal strikes K_ℓ^* all perform quite well as they capture both the marginal information and the dependence through the conditioning variable. Surprisingly, the choice of a particular conditioning variable is of minor importance: the global conditioning variables TB, GA and MV and the local conditioning variable MCTE give almost identical results. As this phenomenon persists in all scenarios, we conclude that the Taylor based conditioning variable already captures most of the relevant information, so from now on we report only the TB values.

Tables 4 and 5 give the price of the put option and the (Tail) Value-at-Risk it is protecting. For the calculation of the TB and UB option prices, we used the suboptimal strikes K_ν^* and the approximating price formula P^ν. Again the Taylor based approximations are very close to the simulated values.

risk measure	MC (s.e.)	TB	GA	MV	MCTE	UB
VaR(0.95)	94.44 (0.0049)	94.46	94.46	94.46	94.48	85.95
VaR(0.99)	88.32 (0.0087)	88.37	88.37	88.37	88.44	75.88
TVaR(0.95)	90.62 (0.0052)	90.68	90.68	90.68	90.71	79.70
TVaR(0.99)	85.59 (0.0082)	85.66	85.67	85.66	85.76	71.61

Table 3: Optimal strike prices, $T = 1$ year

risk measure	MC (s.e.)	TB	UB
VaR(0.95)	0.4411 (0.00043)	0.4386	0.7158
VaR(0.99)	0.0652 (0.00015)	0.0646	0.1009
TVaR(0.95)	0.1448 (0.00018)	0.1448	0.2318
TVaR(0.99)	0.0224 (0.00006)	0.0220	0.0340

Table 4: Put option price with optimal strike price, $T = 1$ year

risk measure	MC (s.e.)	TB	UB
VaR(0.95)	-90.63 (0.005)	-90.68	-79.70
VaR(0.99)	-85.60 (0.009)	-85.66	-71.61
TVaR(0.95)	-87.54 (0.005)	-87.61	-74.76
TVaR(0.99)	-83.22 (0.011)	-83.31	-67.99

Table 5: (Tail) Value-at-Risk $\rho[-X(T)]$, $T = 1$ year

Note that the optimization routine using simulated values takes about three minutes, while the approximations are available in less than one second. Of course this comes at the cost of a smaller accuracy. In case one needs a high accuracy, the suboptimal strikes however are still very valuable as they can be used in conjunction with the simulation routine. Indeed, as the suboptimal strike K_ℓ^* is a good approximation for K^*, we can use it as a starting point in the simulated optimization routine. On the other hand, the comonotonic approximation does not change the marginal distributions, so the suboptimal strike K_c^* can be easily used as a control variate (Vyncke and Albrecher, 2007).

To assess the influence of the time of maturity of the option, we increase the exercise date to $T = 10$ while keeping the other parameters constant. As the uncertainty grows with time, we expect to find less accurate values in both simulation and approximations. This is indeed confirmed in Table 6. The standard error of the simulated optimal strike price clearly increases and also the suboptimal strikes are less accurate. The Taylor based approximation K_ℓ^* stays however in an acceptable range of K^*. An analoguous conclusion holds for the results in Tables 7 and 8.

risk measure	MC (s.e.)	TB	UB
VaR(0.95)	110.36 (0.018)	111.69	77.04
VaR(0.99)	89.69 (0.024)	91.45	52.66
TVaR(0.95)	97.47 (0.016)	99.13	61.89
TVaR(0.99)	81.52 (0.026)	83.54	44.45

Table 6: Optimal strike prices, $T = 10$ years

risk measure	MC (s.e.)	TB	UB
VaR(0.95)	0.820 (0.00084)	0.800	0.933
VaR(0.99)	0.107 (0.00023)	0.104	0.105
TVaR(0.95)	0.259 (0.00026)	0.253	0.283
TVaR(0.99)	0.039 (0.00042)	0.034	0.034

Table 7: Put option price with optimal strike price, $T = 10$ years

risk measure	MC (s.e.)	TB	UB
VaR(0.95)	-97.49 (0.018)	-99.13	-61.89
VaR(0.99)	-81.56 (0.026)	-83.54	-44.45
TVaR(0.95)	-87.76 (0.016)	-89.63	-51.28
TVaR(0.99)	-74.61 (0.028)	-77.07	-38.14

Table 8: (Tail) Value-at-Risk $\rho[-X(T)]$, $T = 10$ years

Coupon-bearing bond

Brigo and Mercurio (2001) calibrate the two-factor Gaussian model to real-market volatility data. At-the-money Euro cap-volatility data of February 13, 2001 at 5 p.m. is used. The calibration is performed by minimizing the sum of the squares of the percentage differences between model and market cap prices, and leads to the parameters of set 1 in Table 9. Minimization of the sum of the squares of the percentage differences between model and market swaption prices produces the calibration parameters in set 2.

parameter	set 1	set 2
a	0.543009105	0.773511777
b	0.075716774	0.082013014
σ	0.005837408	0.022284644
η	0.011657837	0.010382461
ρ	-0.991401219	-0.701985206

Table 9: Parameters of the two-factor model calibrated to real-market volatility data

risk measure	MC (s.e.)	TB	UB
VaR(0.95)	0.993412 (3.0E-05)	0.993422	0.993419
VaR(0.99)	0.954343 (5.8E-05)	0.954364	0.954359
TVaR(0.95)	0.969101 (3.6E-05)	0.969155	0.969151
TVaR(0.99)	0.936504 (8.1E-05)	0.936560	0.936555

Table 10: Optimal strike prices, parameter set 1

risk measure	MC (s.e.)	TB	UB	TB exact	UB exact
VaR(0.95)	2.8599E-03 (3.6E-06)	2.8610E-03	2.8612E-03	2.8610E-03	2.8607E-03
VaR(0.99)	4.3038E-04 (1.4E-06)	4.3088E-04	4.3090E-04	4.3088E-04	4.3076E-04
TVaR(0.95)	9.4330E-04 (1.7E-06)	9.4584E-04	9.4589E-04	9.4584E-04	9.4565E-04
TVaR(0.99)	1.4726E-04 (7.6E-07)	1.4777E-04	1.4778E-04	1.4777E-04	1.4772E-04

Table 11: Put option price with optimal strike price, parameter set 1

risk measure	MC (s.e.)	TB	UB
VaR(0.95)	-0.969172 (2.4E-05)	-0.969180	-0.969176
VaR(0.99)	-0.936557 (5.0E-05)	-0.936576	-0.936570
TVaR(0.95)	-0.949179 (3.0E-05)	-0.949224	-0.949220
TVaR(0.99)	-0.920914 (7.2E-05)	-0.920963	-0.920957

Table 12: (Tail) Value-at-Risk $\rho[-X(T)]$, parameter set 1

For our numerical illustration, we assume that the firm has an OLO 35 (Annaert et al., 2007). OLO (which stands for Obligation lineair/lineaire Obligatie) are debt instruments issued by the Belgian government, and as such, believed to be risk-free. OLOs have a fixed coupon. The OLO we consider was issued on 28 Sept 2000 and will mature on 28 Sept 2010, so the maturity is 10 years. It pays a yearly coupon of 5.75 %, on 28 Sept of each year. In order to be able to use the term structure data in Brigo and Mercurio (2001) of February 13, 2001 at 5 p.m. and the calibrations above of Brigo and Mercurio (2001) of the G2++ model on the same day, we place ourselves on February 13, 2001. In order to protect this bond at that date, we will buy a percentage of a put option with a maturity which is exactly one year, i.e. $T = 1$ upon it and we calculate its price from formula (86) and the data of Brigo and Mercurio (2001) of February 13, 2001 at 5 p.m. mentioned above. At the maturity date of $T = 1$ of the option, the bond has a remaining life time of 8.62 years and is evaluated by formulae (85) and (77) with coupons c_i to be paid out at $S_1 = 1.62$, $S_2 = 2.62$, and so forth until $S_9 = 9.62$ or in general at $S_i = (i + 0.62)$ years for $i = 1, ..., 9$ with $c_i = 0.0575$ for all $i < 9$ and $c_9 = 1.0575$. If we had chosen to hedge with a put option with maturity date $T = j$ with $j > 1$, then formulae (85) and (77) are applied with only the coupon payments during the remaining life time of the bond after $T = j$.

Note that VaR and TVaR have to be calculated under the true probability measure. Since we have calibrated our interest rate model using option prices, the parameters we obtained are under the risk-neutral measure. So, in order to know the parameters under the true probability measure, we would need to estimate the market price of risk. However, as quite often done (Stanton, 1997), we assumed the market price of risk to be zero.

Table 10 shows the optimal strike prices for parameter set 1. The simulated strike prices are based on 10,000,000 sample paths to obtain an acceptable standard error. The approximations perform extremely well in all cases, since K_ℓ^* and K_c^* are almost equal. This indicates that the variables Z_i are almost comonotonic and hence that the two-factor model essentially reduces to a one-factor model. Brigo and Mercurio (2001) confirm this behaviour by stating that for ρ close to minus one, the two-factor Gaussian model tends to degenerate into a one-factor model.

In Table 11 we compare the different option prices. The values in 'TB' and 'UB' are calculated using approximation (49) with $\nu = \ell$ and c respectively, while 'TB exact' and 'UB exact' are calculated with formula (86) and the suboptimal strike prices K_ℓ^* and K_c^*. We notice very little difference between these option prices and can again conclude that the approximations perform very well. The same conclusion holds for the values in Table 12.

risk measure	MC (s.e.)	TB	UB
VaR(0.95)	1.006724 (2.7E-05)	1.006770	1.006098
VaR(0.99)	0.975481 (3.9E-05)	0.975484	0.974366
TVaR(0.95)	0.987324 (2.0E-05)	0.987328	0.986379
TVaR(0.99)	0.961042 (3.5E-05)	0.961104	0.959777

Table 13: Optimal strike prices, parameter set 2

risk measure	MC (s.e.)	TB	UB	TB exact	UB exact
VaR(0.95)	2.3046E-03 (3.2E-06)	2.3100E-03	2.3432E-03	2.3100E-03	2.2313E-03
VaR(0.99)	3.5106E-04 (9.4E-07)	3.5112E-04	3.5622E-04	3.5113E-04	3.2472E-04
TVaR(0.95)	7.6604E-04 (9.5E-07)	7.6620E-04	7.7725E-04	7.6621E-04	7.2203E-04
TVaR(0.99)	1.2016E-04 (3.4E-07)	1.2074E-04	1.2250E-04	1.2075E-04	1.0867E-04

Table 14: Put option price with optimal strike price, parameter set 2

risk measure	MC (s.e.)	TB	UB
VaR(0.95)	-0.987296 (2.2E-05)	-0.987333	-0.986384
VaR(0.99)	-0.961104 (3.4E-05)	-0.961107	-0.959781
TVaR(0.95)	-0.971273 (1.7E-05)	-0.971276	-0.970096
TVaR(0.99)	-0.948395 (3.1E-05)	-0.948450	-0.946940

Table 15: (Tail) Value-at-Risk $\rho[-X(T)]$, parameter set 2

In parameter set 2 the instantaneous correlation ρ is smaller than in parameter set 1, so we expect to see a less comonotonic behaviour. This is indeed confirmed in Table 13. The suboptimal strike prices K_ℓ^* still closely follow the optimal strike prices K^* with relative errors of order 10^{-5}, but the suboptimal strike prices K_c^* are a little less accurate with relative errors of order 10^{-3}. Note that the option price 'UB' (equation (49) with $\nu = c$) in Table 14 is a better approximation than 'UB exact' (equation (86)) while there is no significant difference between 'TB' and 'TB exact'. The error made in approximating the optimal strike with formula (58) thus appears to be partially compensated by the error in calculating the option price with formula (49).

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