Pricing Variable Annuity Guarantees in a Local Volatility framework

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July 23, 2013

Abstract

In this paper, we study the price of Variable Annuity Guarantees, particularly those of Guaranteed Annuity Options (GAO) and Guaranteed Minimum Income Benefit (GMIB), in the settings of a derivative pricing model where the underlying spot (the fund) is locally governed by a geometric Brownian motion with local volatility, while interest rates follow a Hull-White one-factor Gaussian model. Notwithstanding the fact that in this framework, the local volatility depends on a particularly complex expectation where no closed-form expression exists and it is neither directly related to European call prices or other liquid products, we present in this contribution a method based on Monte Carlo Simulations to calibrate the local volatility model. We further compare Variable Annuity Guarantee prices obtained in three different settings, namely the local volatility, the stochastic volatility and the constant volatility models all combined with stochastic interest rates and show that an appropriate volatility modelling is important for these long-dated derivatives. More precisely, we compare prices of GAO, GMIB Rider and barrier types GAO obtained by using local volatility, stochastic volatility and constant volatility models.

1 Introduction

Variable Annuities are insurance contracts that propose a guaranteed return at retirement often higher than the current market rate and therefore they have become a part of many retirement plans. Variable Annuity products are typically based on an investment in a mutual fund composed of stocks and bonds (see for example Gao [13] and Pelsser and Schrager [27]) and offer a range of options to give minimum guarantees and protect against negative equity movements. One of the most popular type of Variable Annuity Guarantees in Japan and North America is the Guaranteed Minimum Income Benefit (GMIB). At her retirement date, a GMIB policyholder will have the right to choose between the fund value at that time or (life) annuity payments based on the initial fund value at a fixed guarantee rate. Similar products are available in Europe under the name Guaranteed Annuity Options (GAO). Many authors have previously studied the pricing and hedging of GMIBs and GAOs assuming a geometric Brownian motion and a constant volatility for the fund value (see for example Boyle and Hardy [7; 8], Ballotta and Haberman [3], Pelsser [26], Biffis and Millossovich [4], Marshall et al. [25], Chu and Kwok [9]).

GAO and GMIB can be considered as long-dated options since their maturity is based on the retirement date. When pricing long-dated derivatives, it is highly recommended that the pricing model used to evaluate and hedge the products takes into account the stochastic behavior of the interest rates as well as the stochastic behavior of the fund. Furthermore, the volatility of the fund can have a significant impact and should not be neglected. It has been shown in [8] that the value of the fund as well as the interest rates and the mortality assumptions strongly influence the cost of these guarantees. Some authors consider the evolution of mortality stochastic as well (see for example [3] and [4]). In [33], van Haastrecht et al. have studied the impact of the volatility of the fund on the price of GAO by using a stochastic volatility approach.

Another category of models capable of fitting the vanilla market implied volatilities are local volatility models introduced in 1994 by Derman and Kani and by Dupire in resp. [11] and [12] and recently extended by, among others, Atlan [2] in a two-factor local volatility model with stochastic interest rates; and then by Piterbarg [28] and Deelstra and Rayée [10] in a three-factor model for the pricing of long-dated FX derivatives. The main advantage of local volatility models is that the volatility is a deterministic function of the equity spot and time, which avoids the issue in working in incomplete markets in comparison with stochastic volatility models. Therefore local volatility models are more appropriate for hedging strategies. The local volatility function is expressed in terms of implied volatilities or market call prices and the calibration is undertaken on the entire implied volatility surface directly. Consequently, local volatility models usually capture in a more precise manner the surface of implied volatilities compared to stochastic volatility models.

Stochastic volatility models are advantageous in that it is possible to derive closed-form solutions for some European derivatives. In [33], van Haastrecht et al. have derived closed-form formulae for GAO prices in the Schöbel and Zhu stochastic volatility model combined with Hull and White stochastic interest rates. However, the GMIB Rider, one of the popular products traded by insurance companies in North America (see [1]) has a more complicated payoff compared to a pure GAO and therefore no closed-form solution exists for the price of a GMIB Rider, not even in the Schöbel and Zhu stochastic volatility model. The only way to evaluate a GMIB Rider is through the use of numerical approaches, such as Monte Carlo simulations.

In this paper, we study the prices of GAO, GMIB Riders and barrier type GAOs in the settings of a two-factor pricing model where the equity (fund) is locally governed by a geometric Brownian motion with a local volatility, while interest rates follow a Hull-White one-factor Gaussian model. In this framework, the local volatility expression contains an expectation for which no closed-form expression exists and which is unfortunately not directly related to European call prices or other liquid products. Its calculation can be done by numerical integration methods or Monte Carlo simulations. Alternative approaches are to calibrate the local volatility from stochastic volatility models by using links between local and stochastic volatility models or by adjusting the tractable local volatility surface coming from a deterministic interest rates framework (see [10]).

Furthermore, we compare Variable Annuity Guarantee prices obtained in three distinct settings, namely, the local volatility, the stochastic volatility and the constant volatility models, all in the settings of stochastic interest rates. We show that using a non constant volatility for the volatility of the equity fund value can have significant impact on the value of these Variable Annuity Guarantees and that the impact generated by a local volatility model is not equivalent to the one generated by a stochastic volatility model, even if both are calibrated to the same market data.

This paper is organized as follows: Section 2 is a summary of properties of the local volatility model in a constant interest rates framework and of its extension in a stochastic interest rates framework. In Section 3, we present an approach based on Monte Carlo simulations for the calibration of the local volatility function in the stochastic interest rates setting. In Section 4, we present the three types of Variable Annuity Guarantees we study in this paper, namely, the GAO, the GMIB Rider and barrier type GAOs. In Subsection 4.1 we present the GAO, and then in Subsection 4.2 we define a GMIB Rider and finally in Subsection 4.3, we study two types of barrier GAO. Section 5 is devoted to numerical results. In Subsection 5.1, we present the calibration procedure for the Hull and White parameters and the calibration of the local volatility with respect to the vanilla market. Subsections 5.2, 5.3 and 5.4 investigate how the local volatility model behaves when pricing GAO, GMIB Rider and barrier type GAOs (respectively) with respect to the Schöbel-Zhu Hull-White stochastic volatility model and the Black-Scholes Hull-White model. Conclusions are given in Section 6.

2 The local volatility model: from a constant to a stochastic interest rates framework

In a constant interest rates setting, the risk neutral probability density of an underlying asset S can be derived from the market prices of European options. More precisely, the risk neutral price of a European Call with strike K and maturity T is given by

$$C(K,T) = e^{-rT}\mathbf{E}^Q[(S_T - K)^+] = e^{-rT}\int_0^{+\infty}(x-K)^+\phi(x,T)dx \tag{1}$$

where Q denotes the risk neutral measure, r is the constant interest rate and where $\phi(x,T)$ corresponds to the risk neutral probability density of the underlying asset S at time T. Differentiating this equation (1) twice with respect to K one obtains the well-known equality

$$\frac{\partial^2 C(K,T)}{\partial K^2} = e^{-rT}\phi(K,T).$$

Using these results, Dupire [12] and Derman and Kani [11] introduced in 1994, in the setting of constant interest rates, the so-called local volatility models for the underlying assets which have a deterministic time and state-dependent volatility function, consistent with the current European option prices. In a local volatility model with constant interest rate, the underlying asset S (paying a constant dividend yield q) is assumed to be governed by the following risk neutral dynamics

$$dS(t) = (r-q)S(t)dt + \sigma(t,S(t))S(t)dW_S^Q(t), \tag{2}$$

where $W_S^Q(t)$ is a Brownian motion under the risk neutral measure Q and where the diffusion function $\sigma(t,S(t))$ satisfies conditions such that equation (2) has a unique solution.

Dupire [12] and Derman and Kani [11] noted that there is a unique volatility function consistent with European option prices, and called it the local volatility function. Indeed, given a complete set of European option prices for all strikes and expirations, this local volatility function is given uniquely by

$$\sigma(T,K) = \sqrt{\frac{\frac{\partial C(K,T)}{\partial T} + (r-q)K\frac{\partial C(K,T)}{\partial K} + qC(K,T)}{\frac{1}{2}K^2\frac{\partial^2 C(K,T)}{\partial K^2}}}. \tag{3}$$

This local volatility model is very tractable since the local volatility surface can directly be computed from vanilla option market prices.

Another useful property of the local volatility model is its link with stochastic volatility models. More precisely, if the underlying spot is governed by the following risk neutral dynamics with constant interest rate,

$$dS(t) = (r-q)S(t)dt + \gamma(t,\nu(t))S(t)dW_S^Q(t), \tag{4}$$

and applying Gyöngy's mimicking theorem [17], one can show that the local volatility is given by the square root of the conditional expectation under the risk neutral measure[1] Q of the instantaneous equity stochastic spot volatility to the square at the future time t, conditional on the equity spot level $S(t)$ being equal to K:

$$\sigma(t,K) = \sqrt{\mathbf{E}^Q[\gamma^2(t,\nu(t)) \mid S(t) = K]}. \tag{5}$$

Common designs for the function $\gamma(t,\nu(t))$ are $\nu(t)$, $exp(\sqrt{\nu(t)})$ and $\sqrt{\nu(t)}$. The stochastic variable $\nu(t)$ is often modelled by a Cox-Ingersoll-Ross (CIR) process (such as the Heston model [19]) or by an Ornstein-Uhlenbeck process (OU) (such as the Schöbel and Zhu [30] stochastic volatility model, implying that the function $\gamma(t,\nu(t))$ can allow for negative values).

While for short-dated options (less than 1 year), assuming constant interest rates does not lead to significant mispricing, for long-dated derivatives, the effect of interest rate volatility becomes increasingly pronounced with increasing maturity and can become as important as that of the underlying spot volatility. In this paper we use the extended version of the previous model where interest rates are stochastic and where the volatility of the underlying spot S is a local volatility function. More precisely, we consider a model where the underlying S is governed by the following risk neutral dynamics (see [2])

$$dS(t) = (r(t)-q)S(t)dt + \sigma(t,S(t))S(t)dW_S^Q(t), \tag{6}$$

where the local volatility function $(\sigma(t,S(t)))$ satisfies conditions such that equation (6) has a unique solution. As before, the constant parameter q denotes the continuous dividend yield. We assume that interest rates follow a Hull-White one-factor Gaussian model [20] defined by the Ornstein-Uhlenbeck process

$$dr(t) = [\theta(t) - \alpha(t)r(t)]dt + \sigma_r(t)dW_r^Q(t), \tag{7}$$

[1] Assuming that the risk neutral probability measure Q used in the stochastic volatility framework is the same as the one used in the local volatility framework.

where $\theta(t), \sigma_r(t)$ and $\alpha(t)$ are deterministic functions of time and where $W_r^Q(t)$ is a Brownian motion under the risk neutral measure Q, correlated with $W_S^Q(t)$. We assume that the dynamics of the underlying S (i.e. the fund) and the interest rates are linked by a constant correlation structure:

$$E^Q[dW_S^Q(t)dW_r^Q(t)] = \rho_{Sr}dt. \tag{8}$$

We use the popular Hull-White model as it is a tractable nontrivial interest rate model, allowing closed-form solutions for many derivatives, a fact which is useful for the calibration.

In the following we will denote this model by LVHW since it combines a local volatility model with a Hull-White one-factor Gaussian model. When $\sigma(t, S(t))$ equals a constant, this model reduces to the Black-Scholes Hull-White model, denoted by BSHW.

Differentiating the expression of a European call price $(C(K,T))$ with respect to its maturity T and twice with respect to the strike K, we can derive the local volatility expression associated to this two-factor model (see [10]). Indeed the analogue expression of the local volatility function (3) in the framework of stochastic interest rates is given by

$$\sigma(T,K) = \sqrt{\frac{\frac{\partial C(K,T)}{\partial T} + qC(K,T) - qK\frac{\partial C(K,T)}{\partial K} + KP(0,T)\mathbf{E}^{Q_T}[r(T)\mathbf{1}_{\{S(T)>K\}}]}{\frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}}}, \tag{9}$$

where $P(0,T)$ is the price of a zero-coupon bond maturing at time T and where Q_T is the T-forward measure associated to the zero-coupon bond with maturity T as numeraire.

This local volatility function is not easy to calibrate with respect to the vanilla market since there is no immediate way to link the expectation term $(\mathbf{E}^{Q_T}[r(T)\mathbf{1}_{\{S(T)>K\}}])$ with vanilla option prices or other liquid products. However, we present in Section 3 a calibration method based on Monte Carlo simulations.

Note that when assuming constant interest rates,

$$\begin{aligned}\frac{\partial C(K,T)}{\partial K} &= \frac{\partial(P(0,T)\int_0^{+\infty}(x-K)^+\phi_F(x,T)dx)}{\partial K} \\ &= P(0,T)\int_K^{+\infty} -\phi_F(x,T)dx = -P(0,T)\mathbf{E}^{Q_T}[\mathbf{1}_{\{S(T)>K\}}],\end{aligned} \tag{10}$$

where $\phi_F(x,T)$ corresponds to the T-forward probability density of the underlying asset at time T. Substituting (10) in equation (9), we find back the equation (3) in the setting of constant interest rates.

The difference between the tractable local volatility function $\sigma_{1f}(T,K)$ coming from the one-factor Gaussian model (see equation (3)) and the local volatility function which takes into account the effects of stochastic interest rates $\sigma_{2f}(T,K)$ (i.e. equation (9)) can also be derived. Indeed, first remark that the expectation $\mathbf{E}^{Q_T}[r(T)\mathbf{1}_{\{S(T)>K\}}]$ can be written as

$$\begin{aligned}\mathbf{E}^{Q_T}[r(T)\mathbf{1}_{\{S(T)>K\}}] &= \mathbf{E}^{Q_T}[r(T)]\mathbf{E}^{Q_T}[\mathbf{1}_{\{S(T)>K\}}] + \mathbf{Cov}^{Q_T}[r(T),\mathbf{1}_{\{S(T)>K\}}] \\ &= f(0,T)\left(-\frac{1}{P(0,T)}\frac{\partial C(K,T)}{\partial K}\right) + \mathbf{Cov}^{Q_T}[r(T),\mathbf{1}_{\{S(T)>K\}}] \end{aligned} \tag{11}$$

where we have used the well-known result that under the T-forward measure Q_T: $\mathbf{E}^{Q_T}[r(T) \mid \mathcal{F}_t] = f(t,T)$, for $T \geq t$, where $f(t,T)$ is the forward rate at time t for the maturity T. Substituting equation (11) in equation (9) and using the fact that in a constant interest rates framework, the forward rate $f(0,T)$ is given by the interest rate r, one obtains the expression

$$\sigma_{2f}^2(T,K) - \sigma_{1f}^2(T,K) = \frac{P(0,T)\mathbf{Cov}^{Q_T}[r(T),\mathbf{1}_{\{S(T)>K\}}]}{\frac{1}{2}K\frac{\partial^2 C}{\partial K^2}}, \tag{12}$$

where $\mathbf{Cov}^{Q_T}(X,Y)$ represents the covariance between two stochastic variables X and Y under the T-forward measure Q_T.

Since the market often quotes options in terms of implied volatilities σ_{imp} instead of option prices, it is more convenient to express the local volatility in terms of implied volatilities. The implied volatility of an option with price $C(K,T)$, is defined through the Black-Scholes formula ($C^{mkt}(K,T) = C^{BS}(K,T,\sigma_{imp})$) and therefore computing the derivatives of call prices through the chain rule and substituting in equation (3) leads to the following equation (see [34]) in the setting of constant interest rates,

$$\begin{aligned}\sigma(T,K) &= \sqrt{\frac{Vega^{BS}(\frac{\sigma_{imp}}{2T} + \frac{\partial\sigma_{imp}}{\partial T} + (r-q)K\frac{\partial\sigma_{imp}}{\partial K})}{\frac{1}{2}K^2 Vega^{BS}(\frac{1}{\sigma_{imp}K^2T} + \frac{2d_+}{\sigma_{imp}K\sqrt{T}}\frac{\partial\sigma_{imp}}{\partial K} + \frac{\partial^2\sigma_{imp}}{\partial K^2} + \frac{d_+d_-}{\sigma_{imp}}(\frac{\partial\sigma_{imp}}{\partial K})^2)}} \\ &= \sqrt{\frac{\sigma_{imp}^2 + 2T\sigma_{imp}\frac{\partial\sigma_{imp}}{\partial T} + 2(r-q)KT\sigma_{imp}\frac{\partial\sigma_{imp}}{\partial K}}{(1+Kd_+\sqrt{T}\frac{\partial\sigma_{imp}}{\partial K})^2 + K^2T\sigma_{imp}(\frac{\partial^2\sigma_{imp}}{\partial K^2} - d_+(\frac{\partial\sigma_{imp}}{\partial K})^2\sqrt{T})}},\end{aligned} \tag{13}$$

with

$$d_\pm = \frac{log\frac{S(0)}{K} + (r-q\pm\frac{\sigma_{imp}^2}{2})T}{\sigma_{imp}\sqrt{T}},$$

$$\mathcal{N}(x) = \int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}e^{\frac{-z^2}{2}}dz,$$

$$Vega^{BS} = e^{-qT}S(0)\mathcal{N}'(d_+)\sqrt{T},$$

$$\mathcal{N}'(x) = \frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}.$$

Using the same approach in the setting of stochastic interest rates, the local volatility expression (9) can be written in terms of implied volatilities σ_{imp}:

$$\sigma(T,K) = \sqrt{\frac{Vega^{BS}(\frac{\sigma_{imp}}{2T} + \frac{\partial \sigma_{imp}}{\partial T} - qK\frac{\partial \sigma_{imp}}{\partial K}) + Kr(0)e^{-r(0)T}\mathcal{N}(d_-) + KP(0,T)\mathbf{E}^{Q_T}[r(T)\mathbf{1}_{\{S(T)>K\}}]}{\frac{1}{2}K^2 Vega^{BS}(\frac{1}{\sigma_{imp}K^2T} + \frac{2d_+}{\sigma_{imp}K\sqrt{T}}\frac{\partial \sigma_{imp}}{\partial K} + \frac{\partial^2 \sigma_{imp}}{\partial K^2} + \frac{d_+d_-}{\sigma_{imp}}(\frac{\partial \sigma_{imp}}{\partial K})^2)}}. \tag{14}$$

Note that the difference between $\sigma_{2f}^2(T,K)$ and $\sigma_{1f}^2(T,K)$ can be derived in terms of implied volatilities (σ_{imp}) by using equation (14) and (13) and leads to

$$\sigma_{2f}^2(T,K) - \sigma_{1f}^2(T,K) = \frac{Kr(0)e^{-r(0)T}\mathcal{N}(d_-) - Vega^{BS}r(0)K\frac{\partial \sigma_{imp}}{\partial K} + KP(0,T)\mathbf{E}^{Q_T}[r(T)\mathbf{1}_{\{S(T)>K\}}]}{\frac{1}{2}K^2 Vega^{BS}(\frac{1}{\sigma_{imp}K^2T} + \frac{2d_+}{\sigma_{imp}K\sqrt{T}}\frac{\partial \sigma_{imp}}{\partial K} + \frac{\partial^2 \sigma_{imp}}{\partial K^2} + \frac{d_+d_-}{\sigma_{imp}}(\frac{\partial \sigma_{imp}}{\partial K})^2)}. \tag{15}$$

Finally, a link also exists between the local volatility model and a stochastic volatility one under the assumption that interest rates follow the same stochastic model. Consider the following risk neutral dynamics for the equity spot

$$dS(t) = (r(t) - q)S(t)dt + \gamma(t,\nu(t))S(t)dW_S^Q(t), \tag{16}$$

where $r(t)$ and $\nu(t)$ are stochastic processes. In [2] and [10], it is proven that if a local volatility exists, such that the one-dimensional probability distribution of the equity spot modelled by diffusion (6) is the same as the one of the equity spot with dynamics (16) for every time t when assuming that the risk neutral probability measure Q used in the stochastic and the local volatility framework is the same, then the local volatility function $\sigma(t, S(t) = K)$ is given by the square root of the conditional expectation under the t-forward measure of the instantaneous equity stochastic spot volatility to the square at the future time t, conditional on the equity spot level $S(t)$ being equal to K:

$$\sigma(t,K) = \sqrt{\mathbf{E}^{Q_t}[\gamma^2(t,\nu(t)) \mid S(t) = K]}. \tag{17}$$

Local volatility models with stochastic interest rates are often used by practitioners to price long-dated derivatives (see for example [28]). In the following we study the pricing of long-dated insurance derivatives, namely Variable Annuity Guarantees, in this local volatility framework with stochastic interest rates. In Section 3, we present a Monte Carlo approach for the calibration of this local volatility model with respect to the implied volatility surface. Details about the products we study and numerical results are presented in resp. Section 4 and Section 5.

3 Calibration

Before using a model to price any derivatives, practitioners are used to calibrate it on the vanilla market. The calibration consists of determining all parameters present in the model in such a way that all European option prices derived in the model are as consistent as possible with the corresponding market ones. More precisely, practitioners need a model which, after calibration, is able to price vanilla options so that the resulting implied volatilities match the market-quoted ones.

The calibration procedure for the LVHW model can be decomposed in three steps: (i) Parameters present in the Hull-White one-factor dynamics for the interest rates, namely $\theta(t), \alpha(t)$ and $\sigma_r(t)$, are chosen to match European swaptions. Such methods are well developed in the literature (see for example [6]). (ii) The correlation coefficient ρ_{Sr} is estimated from historical data. (iii) After these two steps, one has to find the local volatility function which is consistent with the implied volatility surface.

3.1 The Monte Carlo approach

In this section we present a Monte Carlo approach for the calibration of the local volatility expression (9) or (14). More precisely, we use Monte Carlo simulations to calculate an approximation of the expectation $\mathbf{E}^{Q_T}[r(T)\mathbf{1}_{\{S(T)>K\}}]$. Therefore we have to simulate interest rates $r(t)$ and the fund value $S(t)$ up to time T starting from the actual interest rate $r(0)$ and fund value $S(0)$ respectively. Note that in the remainder of the paper we concentrate on the Hull and White model where $\alpha(t) = \alpha$ and $\sigma_r(t) = \sigma_r$ are positive constants. In that case, one can fit the market term structure of interest rates exactly if the parameter $\theta(t)$ satisfies (see [6] and [21])

$$\theta(t) = \frac{\partial f^{mkt}(0,t)}{\partial T} + \alpha f^{mkt}(0,t) + \frac{\sigma_r^2}{2\alpha^2}(1 - e^{-2\alpha t}),$$

where $f^{mkt}(0,t)$ denotes the market instantaneous forward rate at time 0 for the maturity t and where $\frac{\partial f^{mkt}}{\partial T}$ denotes partial derivatives of f^{mkt} with respect to its second argument.

Since the expectation $\mathbf{E}^{Q_T}[r(T)\mathbf{1}_{\{S(T)>K\}}]$ is expressed under the measure Q_T, we use the dynamics of $S(t)$ and $r(t)$ under that measure, namely,

$$\begin{cases} dS(t) = (r(t) - q - \sigma(t,S(t))\sigma_r b(t,T)\rho_{Sr})S(t)dt + \sigma(t,S(t))S(t)dW_S^{Q_T}(t), & (18) \\ dr(t) = [\theta(t) - \alpha r(t) - \sigma_r^2(t) b(t,T)]dt + \sigma_r dW_r^{Q_T}(t), & (19) \end{cases}$$

where $b(t,T) = \frac{1}{\alpha}(1 - e^{-\alpha(T-t)})$.

Following the Monte Carlo principle, we simulate n times (i.e. n scenarios) the stochastic variables $r(t)$ and $S(t)$ up to time T, using $\Delta t = t_{i+1} - t_i$, $i = 0, ..., p-1$ as time step of discretization, assuming that $t_0 = 0$ and $t_p = T$ and applying an Euler scheme.

Note that with a well-known change of variable, one can remove the need to calculate $\theta(t)$. The idea is to rewrite the stochastic interest rates as a sum of a stochastic and a deterministic part (see [23]):

$$r(t) = x(t) + \overline{x}(t), \tag{20}$$

where the stochastic part obeys the following dynamics:

$$dx(t) = -(\alpha x(t) + \sigma_r^2(t)b(t,T))dt + \sigma_r(t)dW_r^{Q_T}(t) \tag{21}$$

and where the deterministic part obeys the dynamics :

$$d\overline{x}(t) = (\theta(t) - \alpha\overline{x}(t))dt, \tag{22}$$

which yields (see [23])

$$\overline{x}(t) = f^{mkt}(0,t) + \frac{\sigma_r^2}{2\alpha^2}(1-e^{-\alpha t})^2.$$

In the Monte Carlo simulation method, the Euler scheme is used[2] for the discretization of equations (18) and (21):

$$\begin{aligned} S(t_{k+1}) &= S(t_k) + (r(t_k) - q - \sigma(t_k, S(t_k))\sigma_r b(t_k,T)\rho_{Sr})S(t_k)\Delta t + \sigma(t_k,S(t_k))S(t_k)\sqrt{\Delta t}Z_S, \\ x(t_{k+1}) &= x(t_k) - [\alpha x(t_k) + \sigma_r^2 b(t_k,T)]\Delta t + \sigma_r\sqrt{\Delta t}[\rho_{Sr}Z_S + \sqrt{1-\rho_{Sr}^2}Z_r], \\ r(t_{k+1}) &= x(t_{k+1}) + \overline{x}(t_{k+1}), \end{aligned} \tag{23}$$

where

$$\overline{x}(t_k) = f^{mkt}(0,t_k) + \frac{\sigma_r^2}{2\alpha^2}(1-e^{-\alpha t_k})^2 \text{ and } b(t_k,T) = \frac{1}{\alpha}(1-e^{-\alpha(T-t_k)})$$

and Z_r and Z_S are two independent standard normal variables.

As one can see in equation (23), the local volatility function has to be known for the simulation of the path for $S(t)$ and $r(t)$. Consequently, the only possible way to work is forward in time. Let assume that market Call prices are available for p maturities[3] $t_1, t_2, ..., t_p$. Furthermore, we assume that the "initial" local volatility $\sigma(t_0, K)$ is known for all strike K. More precisely, in this paper, the "initial" local volatility we use is obtained by using the local volatility expression given by equation (3). Note that this local volatility is directly obtained by using market data (see equation (13))[4].

[2]We assume that the local volatility function obtained by Monte Carlo simulations satisfies conditions to ensure that the Euler approximation will converge to the solution of equations (18) and (21). For example, the local volatility has to be smooth and positive. For more details about the convergence of the Euler approximation we refer the reader to [16]. However you one can check numerically whether the convergence to the solution of the SDE is satisfied. The convergence is satisfied if one obtains the given market Call prices when using the "calibrated" local volatility.

[3]If the market does not price European Call options for all these maturities, an alternative method is to build first the whole implied volatility surface from available market data by using an interpolation method as for example a bicubic spline (see for example [29]). The implied volatility surface can also be built from a stochastic volatility model or the SABR model (see for example [14]).

[4]We assume that the "initial" local volatility $\sigma(t_0, K)$ is equal in both the stochastic and the constant interest rates framework. At time t_0, the interest rate is given by the fixed market data $r(t_0)$ and in this case, expression (9) is equivalent to expression (3).

The first step is to determine the local volatility function at time step t_1 for all strikes K. Knowing the "initial" local volatility function $\sigma(t_0, K)$, $S(t_1)$ and $r(t_1)$ can be simulated. Then, the expectation $\mathbf{E}^{Q_{t_1}}[r(t_1)\mathbf{1}_{\{S(t_1)>K\}}]$ can be computed for all K by using:

$$\mathbf{E}^{Q_{t_1}}[r(t_1)\mathbf{1}_{\{S(t_1)>K\}}] \cong \frac{1}{n}\sum_{i=1}^{n} r_i(t_1)\mathbf{1}_{\{S_i(t_1)>K\}}. \tag{24}$$

At this step we are able to build the local volatility function $\sigma(t_1, K)$ at time t_1 for all strikes K by using equation (9). Following the same procedure we can easily calibrate the local volatility at time t_2 by using the local volatility obtained at time t_1 and also the simulated paths until time t_2. Following this procedure we are able to generate the local volatility expression up to a final date $T = t_p$.

4 Variable Annuity Guarantees

In this section we present three different Variable Annuity products and in section 5.2, we discuss the price of all these products using local, stochastic and constant volatility models. First, we define the Guaranteed Annuity Option in Subsection 4.1, and in Subsection 4.2 we define the Guaranteed Minimum Income Benefit (Rider). This latter product has the particularity to be strongly dependent upon the path of the fund value. Finally, in Subsection 4.3, we study two barrier type GAOs with a strong dependence upon the path of the interest rates.

4.1 Guaranteed Annuity Options

Consider an x year policyholder who disposes at time T of the payout of his capital policy which corresponds to an amount of money $S(T)$. A Guaranteed Annuity Option gives to the policyholder the right to choose either an annual payment of $S(T)g$ where g is a fixed rate called the Guaranteed Annuity rate or a cash payment equal to the equity fund value $S(T)$ at time T which can be considered as an annual payment of $S(T)r_x(T)$, with $r_x(T)$ being the market annuity payout rate. Assuming that the mortality risk is unsystematic and independent of the financial risk, the market annuity payout rate is defined by $r_x(T) = \frac{1}{\ddot{a}_x(T)}$ with $\ddot{a}_x(T) = \sum_{n=0}^{\omega-(x+T)} {}_np_{x+T}P(T, T+n)$ where ω is the largest survival age, $P(T, T+n)$ is the zero-coupon bond at time T maturing at $T+n$ and ${}_np_{x+T}$ is the probability that the remaining lifetime of the policyholder at time T is strictly greater than n. At time T the value of the GAO is given by

$$\begin{aligned} V(T) &= max(S(T)g\ddot{a}_x(T), S(T)) && (25)\\ &= S(T) + gS(T)max\left(\sum_{n=0}^{\omega-(x+T)} {}_np_{x+T}\ P(T, T+n) - K, 0\right), && (26)\end{aligned}$$

where $K = \frac{1}{g}$.

Applying the risk neutral valuation procedure, we can write the value of a GAO entered by an x-year policyholder at time $t = 0$ as

$$\begin{aligned} V(x,0,T) &= E^Q[e^{-\int_0^T r(s)ds}V(T)1_{(\tau_x>T)}|\mathcal{F}_0] \\ &= E^Q[e^{-\int_0^T r(s)ds}V(T)]E^Q[1_{(\tau_x>T)}] \\ &= E^Q[e^{-\int_0^T r(s)ds}V(T)]\ {}_Tp_x, \end{aligned} \tag{27}$$

where τ_x is a random variable which represents the remaining lifetime of the policyholder. Substituting equation (26) in (27) leads to

$$V(x,0,T) = {}_Tp_x E^Q[e^{-\int_0^T r(s)ds}S(T)] + C(x,0,T). \tag{28}$$

where

$$C(x,0,T) = {}_Tp_x E^Q\left[e^{-\int_0^T r(s)ds}gS(T)\left[max\left(\sum_{n=0}^{\omega-(x+T)} {}_np_{x+T}P(T,T+n) - K, 0\right)\right]\right]. \tag{29}$$

By definition of the risk neutral measure Q, the discounted value of the risky fund $e^{-\int_0^T r(s)ds}S(T)$ is a martingale and therefore the value of the GAO becomes

$$V(x,0,T) = {}_Tp_x S(0) + C(x,0,T). \tag{30}$$

The first term in equation (30) is a constant and therefore, in literature, one generally only studies the second term $C(x,0,T)$. In [3] and [33], the authors define $C(x,0,T)$ as the GAO total value. In this paper we retain the same terminology. More precisely, in section 5.2, when we compare GAO total values obtained in different models, we compare values obtained for $C(x,0,T)$.

To derive analytical expressions for $C(x,0,T)$ in the Black-Scholes Hull-White and the Schöbel-Zhu Hull-White models it is more convenient to work under the measure Q_S (where the numeraire is the fund value S), rather than under the risk neutral measure Q (see [3] and [33]). By the density process $\xi_T = \frac{dQ_S}{dQ}|_{\mathcal{F}_T} = e^{-\int_0^T r(s)ds}\frac{S(T)}{S(0)}$ a new probability measure Q_S equivalent to the measure Q is defined, see e.g. Geman et al. [15]. Under this new measure Q_S, the $C(x,0,T)$ value becomes

$$C(x,0,T) = {}_Tp_x gS(0)E^{Q_S}\left[\left(\sum_{n=0}^{\omega-(x+T)} {}_np_{x+T}P(T,T+n) - K\right)^+\right]. \tag{31}$$

Under Q_S the model dynamics turn out to be

$$dS(t) = [r(t) - q + \sigma^2(t,S(t))]S(t)dt + \sigma(t,S(t))S(t)dW_S^{Q_S}(t), \tag{32}$$
$$dx(t) = [-\alpha x(t) + \rho_{rS}\sigma_r\sigma(t,S(t))]dt + \sigma_r dW_r^{Q_S}(t). \tag{33}$$

The zero-coupon bond $P(T,T+n)$ in the Gaussian Hull and White one-factor model with α and σ_r constant has the following expression (see e.g. [23])

$$P(T, T+n) = A(T, T+n)e^{-b(T,T+n)x(T)}, \quad (34)$$

where

$$\begin{aligned} A(T, T+n) &= \frac{P^{mkt}(0, T+n)}{P^{mkt}(0, T)} e^{-\frac{1}{2}[V(0,T+n)-V(0,T)-V(T,T+n)]}, \\ b(T, T+n) &= \frac{1}{\alpha}(1 - e^{-\alpha n}), \\ V(t_1, t_2) &= \frac{\sigma_r^2}{\alpha^2}[t_2 - t_1 + \frac{2}{\alpha}e^{-\alpha(t_2-t_1)} - \frac{1}{2\alpha}e^{-2\alpha(t_2-t_1)} - \frac{3}{2\alpha}], \end{aligned}$$

where $P^{mkt}(0, T)$ denotes the market price of a zero coupon bond with maturity T. Substituting the expression (34) in equation (31) leads to the following pricing expression for $C(x, 0, T)$ under Q_S

$$C(x, 0, T) = {}_T p_x g S(0) E^{Q_S} \left[\left(\sum_{n=0}^{\omega-(x+T)} {}_n p_{x+T} A(T, T+n) e^{-b(T,T+n)x(T)} - K \right)^+ \right]. \quad (35)$$

Note that when pricing GAO using a local volatility model, one has to use some numerical methods like Monte Carlo simulations. The calculation of the price can be based on the equation (35) but it is also convenient to work under the T-forward measure Q_T, where the dynamics of $S(t)$ and $r(t)$ are given by (18) and (19) respectively and where the GAO value is given by

$$C(x, 0, T) = {}_T p_x g P(0, T) E^{Q_T} \left[S(T) \left(\sum_{n=0}^{\omega-(x+T)} {}_n p_{x+T} A(T, T+n) e^{-b(T,T+n)x(T)} - K \right)^+ \right]. \quad (36)$$

Since in both cases one has to compute the local volatility value at each time step, both methods are equivalent in time machine consumption.

4.2 Guaranteed Minimum Income Benefit (Rider)

Guaranteed Minimum Income Benefit (GMIB) is the term used in North America for an analogous product of a GAO in Europe (see [18]). GAO and GMIB payoffs are usually slightly different, but they are common in offering both maturity guarantees in the form of a guaranteed minimum income on the annuitization of the maturity payout.

Many different guarantee designs exist for GMIB. The policyholders can for example choose between a life annuity or a fixed duration annuity; or choose an annual growth rate guarantee for the fund, or a withdraw option, etc. (for more details see [18] and [1]). In this paper we focus on the valuation of an "exotic GMIB", namely a GMIB Rider.

A GMIB Rider (based on examples given in [1] and in [25]) gives the x year policyholder the right to choose at the date of annuitization T between 3 guarantees: an annual payment of $gS(0)(1 + r_g)^T$

where r_g is a guaranteed annual rate; an annual payment of $g \underset{n}{max}(S(n))$, $n = 1, 2, ..., T$ where $S(n)$, $n = 1, 2, ..., T$ are the anniversary values of the fund or a cash payment equal to the equity fund value $S(T)$ at maturity T. Therefore, at time T the value of the GMIB Rider is given by

$$V(T) = max(S(0)(1 + r_g)^T g\ddot{a}_x(T), \underset{n\in A}{max}(S(n))g\ddot{a}_x(T), S(T)), \tag{37}$$

where A is the set of anniversary dates $A = \{1, 2, ..., T\}$. The valuation of this product in the Black-Scholes Hull-White model has been studied by e.g. Marshall et al. in [25].

Assuming that the mortality risk is unsystematic and independent of the financial risk, we can write the value of a GMIB Rider entered by an x-year policyholder at time $t = 0$ as

$$\begin{aligned} V(x, 0, T) &= E^Q[e^{-\int_0^T r(s)ds} V(T)]\ {}_Tp_x \\ &= P(0, T)E^{Q_T}[V(T)]\ {}_Tp_x. \end{aligned} \tag{38}$$

The GMIB Rider payoff is path-dependent and more complex than a pure GAO. There is no closed-form expression in the BSHW neither in the SZHW model nor in the LVHW model. Consequently one has to use numerical methods to evaluate the expectation in equation (38) in all models we consider. In Subsection 5.3, we compare GMIB Rider values given by using the LVHW model, the SZHW model and the constant volatility BSHW model using a Monte Carlo approach.

4.3 Barrier GAOs

In this section, we introduce path-dependent GAOs which are barrier type options. The first one is a "down-and-in GAO" which becomes activated only if the interest rates reach a downside barrier level. This product is attractive for buyers since it protects against low interest rates at the retirement age and has a smaller price than the pure GAO price. The second exotic GAO we study below is the "down-and-out GAO" which becomes deactivated if the interest rates reach a downside barrier level. This barrier option gives protection to insurers against low market of interest rates. For example it allows to avoid situations as occurred in the UK where insurers have sold products with guarantee rates g of around 11% while the market rates are currently very low[5], resulting in considerable losses for insurance companies. To our knowledge, no closed-form solutions exist for the price of these path-dependent GAOs in the BSHW model nor in the SZHW and LVHW models. Therefore we will also price these path-dependent derivatives by using a Monte Carlo simulation approach.

Under the measure Q_S, the price of the "down-and-out GAO" is given by the following expression

$$C^{DO}(x, 0, T) = {}_Tp_x g S(0) E^{Q_S}\left[\left(\sum_{n=0}^{\omega-(x+T)} {}_np_{x+T} A(T, T+n)e^{-b(T,T+n)x(T)} - K\right)^+ 1_{(x(t)>B,\ 0<t\le T)}\right]. \tag{39}$$

where we use the notation of Section 4.1. The barrier B is defined as a barrier for the homogeneous part $x(t)$ and therefore the interest rates $r(t) = x(t) + \overline{x}(t)$ have a time dependent barrier.

[5]In United Kingdom in the 1970's and 1980's the most popular Guaranteed Annuity rate proposed by UK life insurers was about 11% (see Bolton et al. [5]).

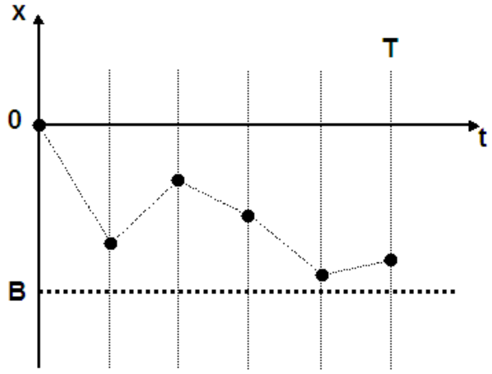


Figure 1: Example of path realization

When pricing barrier options with Monte Carlo simulations it is possible to miss some barrier hitting realization between two time steps. Consider the particular path realization of Figure 1. There are five time steps and none of the five underlying realizations have breached the barrier and therefore the path is going to count in the payoff sum. Following the idea presented in [31], we should weight the actualized payoff by a certain factor accounting for the probability of breaching the barrier between the discrete time points. As a weighting factor we can use the product over all time intervals $\Delta_t^i = t_{i+1} - t_i$ of the survival probability of the option,

$$w = \prod_{i=1}^{n} P_{surv}(\Delta_t^i). \tag{40}$$

Assuming that the volatility of the equity is constant over the interval Δ_t^i, it is possible to derive an analytical expression for $P_{surv}(\Delta_t^i)$ applicable in all different models treated in this paper. More precisely, under this assumption, $x(t+\Delta_t^i)$ is normally distributed with mean μ_x and variance σ_x^2 given by

$$\begin{aligned} \mu_x &= \frac{\rho_{Sr}\sigma_r\sigma_S}{\alpha}(1 - e^{-\alpha\Delta_t^i}), \\ \sigma_x^2 &= \frac{\sigma_r^2}{2\alpha}(1 - e^{-2\alpha\Delta_t^i}). \end{aligned} \tag{41}$$

Using the well-known reflection principle (see [32]), we have the analytical expression for $P_{surv}(\Delta_t^i)$

$$P_{surv}(\Delta_t^i) = P(x(s) > B,\ t < s \le t + \Delta_t^i) \tag{42}$$

$$= \mathcal{N}\left(\frac{-B + \mu_x\Delta_t^i}{\sigma_x\sqrt{\Delta_t^i}}\right) - e^{\left(\frac{-2B\mu_x}{\sigma_x^2}\right)}\mathcal{N}\left(\frac{B + \mu_x\Delta_t^i}{\sigma_x\sqrt{\Delta_t^i}}\right). \tag{43}$$

The last assumption about the volatility of the equity is in contradiction with the nature of the LVHW and the SZHW models. However, as time intervals become smaller, this assumption is more and more justified.

The price of the "down-and-in GAO" can easily be computed from the price of the "down-and-out GAO" and the "pure GAO" by using the following relation

$$C^{DI}(x, 0, T) = C(x, 0, T) - C^{DO}(x, 0, T). \tag{44}$$

Note that for knock-in type options, the survival probability is readily derived from the one coming from the corresponding knock-out type option by complementarity ($P^{KI}_{surv} = 1 - P^{KO}_{surv}$).

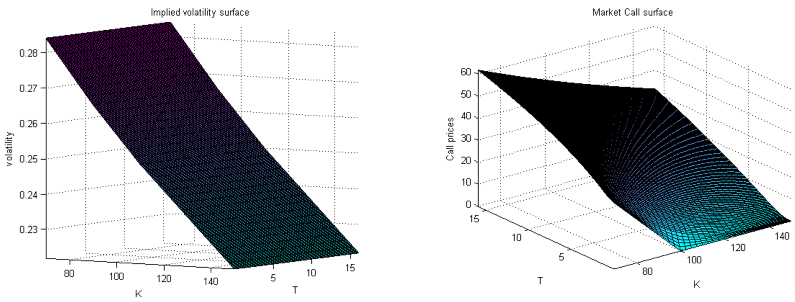
5 Numerical results

In this section we study the contribution of using a local volatility model with Hull and White stochastic interest rates (LVHW) (introduced in Section 2) to the pricing of Variable Annuity Guarantees. More precisely, we compare GAO, GMIB Rider and two barrier type GAO prices obtained by using the LVHW model to those obtained with the Schöbel-Zhu Hull-White (SZHW) stochastic volatility model and the Black-Scholes with Hull and White stochastic interest rates (BSHW). For a fair analysis, first one must calibrate these three models to the same options market data. To this end, we have used the same data as in [33]. More precisely, the equity components (fund) of the Variable Annuity Guarantees considered are on one hand the EuroStoxx50 index (EU) and on the other hand the S&P500 index (US). In the following subsection we explain in detail the calibration of the three models. We summarize the calibration of the interest rate parameters and the calibration of the SZHW and the BSHW made in [33] and explain the calibration of the local volatility surface for the equity component in our LVHW model. In Subsection 5.2 we compare GAO values obtained by using the LVHW model with the SZHW and the BSHW prices studied in [33]. In Subsection 5.3 and Subsection 5.4, we do the same study for path-dependent Variable Annuity Guarantees namely GMIB Riders and barrier type GAOs.

5.1 Calibration to the Vanilla option's Market

In order to compare LVHW GAO results to those obtained in the BSHW and the SZHW models presented in van Haastrecht et al. [33] we are using Hull and White parameters and the implied volatility curve they used[6]. In [33], interest rate parameters are calibrated to EU and US swaption markets using swaption mid prices of the 31st of July 2007. Moreover, the effective 10 years correlation between the log equity returns and the interest rates is determined by time series analysis of the 10-year swap rate and the log returns of the EuroStoxx50 index (EU) and the S&P500 index (US) over the period from February 2002 to July 2007 and turns out to be resp. 34.65% and 14.64% for the EU and the US market. The equity parameters in the SZHW and BSHW models are calibrated by using vanilla option prices on the EuroStoxx50 and S&P500 index obtained from the implied volatility service of

[6]We would like to thank A. van Haastrecht, R. Plat and A. Pelsser for providing us the Hull and White parameters and interest rate curve data they used in [33].



(a) implied volatility surface (b) market call prices

Figure 2: Plot of the implied volatility (built from the 10-years S&P500 implied volatility curve and assuming a constant implied volatility with respect to the maturity ($\frac{\partial \sigma_{imp}}{\partial T} = 0$)) and the corresponding market call prices surface.

MarkIT[7].

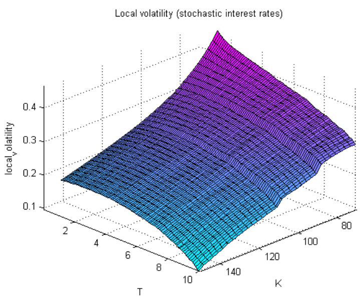
In [33] the authors have calibrated the equity model to market option prices maturing in 10 years time. In the LVHW model, the equity volatility is a local volatility surface and the calibration consists in building this surface using equation (14). This calibration procedure uses the whole implied volatility surface and returns a local volatility for all strikes and all maturities. In this paper we consider three distinct cases. In the first case, we assume that the implied volatility is constant with respect to the maturity ($\frac{\partial \sigma_{imp}}{\partial T} = 0$) and denote the local volatility model by LVHW1. The second case denoted by LVHW2 considers an increasing term structure ($\frac{\partial \sigma_{imp}}{\partial T} = 0.01$) and finally in the third case denoted by LVHW3, the implied volatility is a decreasing function of the maturity ($\frac{\partial \sigma_{imp}}{\partial T} = -0.003$). Note that for the aid of a fair comparison between the models, we have consistently retained the same volatility smile at time $T = 10$. A plot of the "S&P500" implied volatility surface and the resulting market call option prices (in the case ($\frac{\partial \sigma_{imp}}{\partial T} = 0$)) can be found in Figures 2(a) and 2(b). By applying the Monte Carlo approach described in Section 3.1, the corresponding local volatility surface can be derived, see Figure 3(a). In Section 2, we have explained the difference between the local volatility derived in a constant interest rate framework and the local volatility derived in a stochastic interest rates one (see equation (12)). Figure 3(c) is a plot of this difference and Figure 3(b) is a plot of the local volatility derived in a constant interest rates framework (see equation (3)) using the same market data, namely the "S&P500" implied volatility surface in the case of a constant implied volatility with respect to the maturity (Figure 2(a)).

Both Stochastic volatility and local volatility models are able to reproduce the market smile. For example in Table 1, we compare the market implied volatility (for a range of seven different strikes and a fixed maturity $T = 10$) with the calibrated volatility of each of the three models. We observe that the local volatility model (LVHW1) and the stochastic volatility model (SZHW) are both well calibrated since they are able to generate the Smile/Skew quite close to the market one. Note that, the LVHW1 implied volatilities are extracted from the European call values obtained by Monte Carlo simulations. Therefore, in the last column of Table 1, the corresponding ± 95% confidence intervals

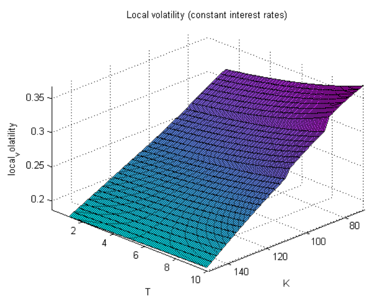
[7] A financial data provider, which provides (mid) implied volatility quotes by averaging quotes from a large number of issuers.

Implied volatility, 10-year call options, US				
strike	**Market**	**BSHW**	**SZHW**	**LVHW1 (± 95% interval)**
80	27.50%	25.80%	27.50%	27.503% (± 0.01778 %)
90	26.60%	25.80%	26.60%	26.601% (± 0.01745 %)
95	26.20%	25.80%	26.20%	26.198% (± 0.01693 %)
100	25.80%	25.80%	25.80%	25.800% (± 0.01631 %)
105	25.40%	25.80%	25.40%	25.397% (± 0.01501 %)
110	25.00%	25.80%	25.00%	24.998% (± 0.01432 %)
120	24.30%	25.80%	24.40%	24.333% (± 0.01325 %)
Implied volatility, 10-year call options, EUR				
strike	**Market**	**BSHW**	**SZHW**	**LVHW1 (± 95% interval)**
80	27.80%	26.40%	27.90%	27.826% (± 0.0214 %)
90	27.10%	26.40%	27.10%	27.103% (± 0.0200 %)
95	26.70%	26.40%	26.70%	26.699% (± 0.0194 %)
100	26.40%	26.40%	26.40%	26.396% (± 0.0189 %)
105	26.00%	26.40%	26.00%	25.999% (± 0.0181 %)
110	25.70%	26.40%	25.70%	25.702% (± 0.0173 %)
120	25.10%	26.40%	25.10%	25.101% (± 0.0165 %)

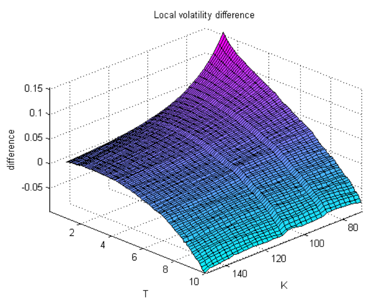
Table 1: Comparison of the implied volatility curve (for 10-year call options) generated by the SZHW, the BSHW and the LVHW1 models after being calibrated to the market implied volatility. US and European market implied volatilities are taken from [33].



(a) local volatility (stochastic interest rates)



(b) local volatility (constant interest rates)



(c) difference

Figure 3: Plot of the local volatility obtained in both stochastic and constant interest rates framework and finally, the difference between these two local volatilities. These local volatility surfaces are built from the S&P500 implied volatility surface assuming a constant implied volatility with respect to the maturity ($\frac{\partial \sigma_{imp}}{\partial T} = 0$)), namely Figure 2(a).

are also given. However, contrarily to the SZHW model, the LVHW model has to be calibrated over the whole implied volatility surface and as we will see in Subsection 5.2, 5.3 and 5.4, this fact has an impact to the price of Variable Annuity Guarantees.

5.2 GAO results

In this section, we study the impact generated to GAO values by using the LVHW model compared to prices obtained by using the BSHW and the SZHW models computed in [33]. We therefore make the same assumptions as in [33], namely that the policyholder is 55 years old and that the retirement age is 65 (i.e. the maturity T of the GAO option is 10 years). The fund value at time 0, $S(0)$ is assumed to be 100. The survival rates are based on the PNMA00[8] table of the Continuous Mortality Investigation (CMI) for male pensioners.

In Table 2 we give prices obtained for the GAO using the LVHW model case 1 (LVHW1), the LVHW model case 2 (LVHW2), the LVHW model case 3 (LVHW3), the SZHW and the BSHW models for different guaranteed rates g. The results for the SZHW and BSHW models are obtained using the closed-form expression derived in resp. [33] and [3]. Note that GAO prices presented in this section derived by the BSHW and SZHW models are slightly different than those presented in [33]. These differences come from the interpolation method used for constructing the zero coupon bond curve. As it was pointed out in [33], GAO prices are sensitive to the interest rate curve and a small change in the zero coupon bond curve induces changes in GAO prices. The results for the LVHW model are obtained by using Monte Carlo simulations (100 000 simulations and 5000 steps). Figure 4 shows the corrections induced by the LVHW and the SZHW models with respect to the BSHW model[9].

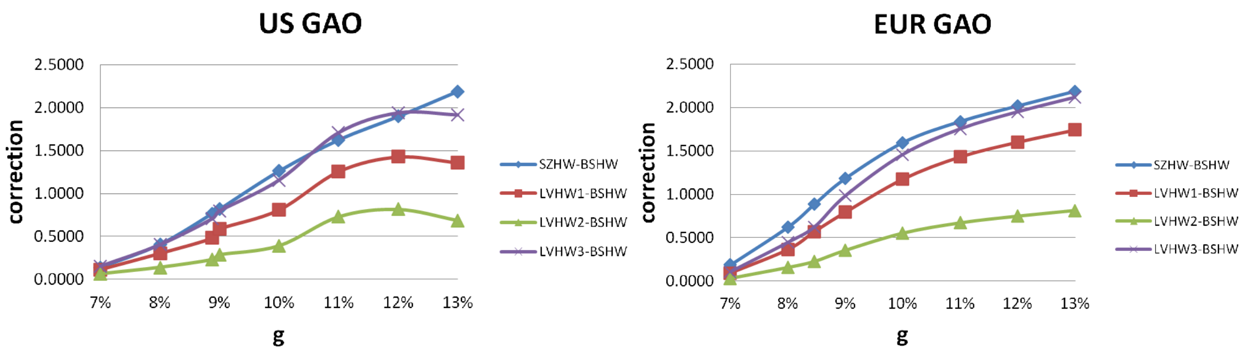


Figure 4: Graphical representation of the LVHW1, LVHW2, LVHW3 and the SZHW model corrections with respect to the BSHW model for different guaranteed annuity rates g.

Table 3 presents the time value given by the difference between the GAO total value and its intrinsic value. The volatility of an option is an important factor for this time value since this value depends

[8]Available at http://www.actuaries.org.uk/research-and-resources/pages/00-series-mortality-tables-assured-lives-annuitants-and-pensioners

[9]More precisely, the LVHW correction is the difference between the LVHW price and the BSHW price. Similarly, the SZHW correction is the difference between the SZHW price and the BSHW price.

GAO Total Value US								
g	**BSHW**	**SZHW**	**LVHW1**	**SE**	**LVHW2**	**SE**	**LVHW3**	**SE**
7%	0.906860	1.045977	1.021760	0.013730	0.970480	0.013374	1.059320	0.014154
8%	3.160037	3.567810	3.463430	0.026539	3.299570	0.025895	3.562870	0.027142
8.88%	7.101917	7.869198	7.584480	0.039751	7.332270	0.038987	7.808470	0.040443
9%	7.738402	8.555375	8.326130	0.041560	8.024770	0.040791	8.533410	0.042259
10%	14.880173	16.141393	15.690000	0.055644	15.271200	0.054907	16.034900	0.056342
11%	23.643769	25.267434	24.900900	0.066981	24.374900	0.066285	25.350600	0.067621
12%	33.689606	35.586279	35.117300	0.075737	34.507000	0.075030	35.626800	0.076333
13%	44.382228	46.570479	45.742500	0.082976	45.070200	0.082227	46.296100	0.083563
GAO Total Value EUR								
g	**BSHW**	**SZHW**	**LVHW1**	**SE**	**LVHW2**	**SE**	**LVHW3**	**SE**
7%	0.395613	0.583057	0.486412	0.007961	0.428838	0.007326	0.502376	0.008006
8%	2.259404	2.882720	2.622240	0.019585	2.417440	0.018578	2.704880	0.019687
8.46%	4.095928	4.981611	4.664650	0.026196	4.323210	0.025098	4.719230	0.026313
9%	7.210501	8.393904	8.004080	0.033921	7.565690	0.032806	8.193780	0.034034
10%	15.421258	17.017295	16.597400	0.045658	15.974700	0.044657	16.880900	0.045707
11%	25.530806	27.369899	26.964500	0.053381	26.202900	0.052448	27.285900	0.053395
12%	36.282533	38.300598	37.885600	0.058946	37.031900	0.057988	38.234200	0.058954
13%	47.164918	49.351551	48.908600	0.063960	47.979600	0.062924	49.284400	0.063964

Table 2: Comparison of GAO total values in the BSHW, the SZHW and the LVHW1, LVHW2 and LVHW3 models for different guaranteed annuity rates g. The rates g of 8.88% and 8.46% correspond to the at-the-money guaranteed annuity rates in the US and EUR market respectively.

on the time until maturity and the volatility of the underlying instrument's price. The time value reflects the probability that the option will gain in intrinsic value or become profitable to exercise before maturity. Figure 5 is a plot of the time value given by all considered models. For a deeper analysis about time value we refer the interested reader to [33].

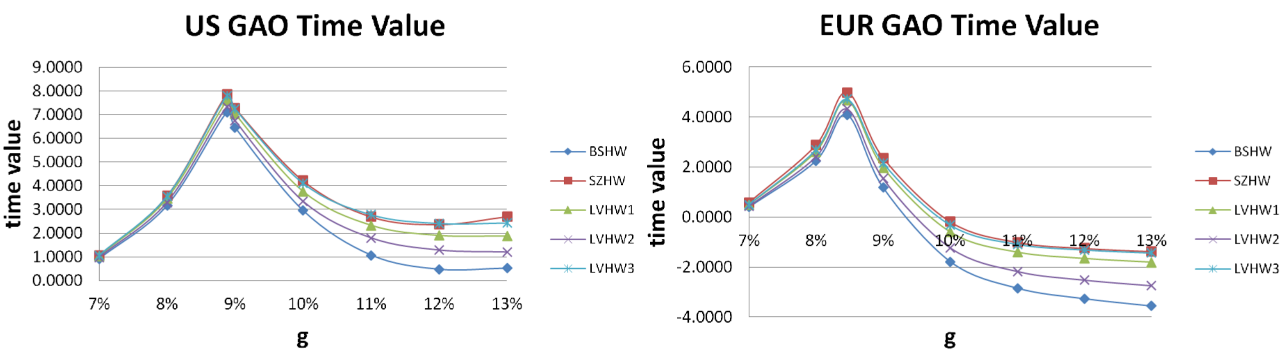


Figure 5: Graphical representation of GAO time values of the BSHW, the SZHW and the LVHW1, LVHW2 and LVHW3 models for different guaranteed annuity rates g.

These results show that the use of a non constant volatility model such as the SZHW model and the LVHW model has a significant impact on the total value and the time value of GAOs. Furthermore, the term structure of the implied volatility surface has also an influence on the total value. The

GAO Time Value US					
g	**BSHW**	**SZHW**	**LVHW1**	**LVHW2**	**LVHW3**
7%	0.9069	1.0460	1.0218	0.9705	1.0593
8%	3.1600	3.5678	3.4634	3.2996	3.5629
8.88%	7.1019	7.8692	7.5845	7.3323	7.8085
9%	6.4575	7.2745	7.0452	6.7439	7.2525
10%	2.9527	4.2140	3.7626	3.3438	4.1075
11%	1.0698	2.6935	2.3269	1.8009	2.7766
12%	0.4691	2.3658	1.8968	1.2865	2.4063
13%	0.5152	2.7034	1.8755	1.2032	2.4291
GAO Time Value EUR					
g	**BSHW**	**SZHW**	**LVHW1**	**LVHW2**	**LVHW3**
7%	0.3956	0.5831	0.4864	0.4288	0.5024
8%	2.2594	2.8827	2.6222	2.4174	2.7049
8.46%	4.0959	4.9816	4.6647	4.3232	4.7192
9%	1.1839	2.3673	1.9775	1.5391	2.1672
10%	-1.7792	-0.1831	-0.6030	-1.2257	-0.3195
11%	-2.8435	-1.0044	-1.4098	-2.1714	-1.0884
12%	-3.2656	-1.2475	-1.6625	-2.5162	-1.3139
13%	-3.5570	-1.3704	-1.8134	-2.7424	-1.4376

Table 3: Comparison of GAO time values of the BSHW, the SZHW and the LVHW1, LVHW2 and LVHW3 models for different guaranteed annuity rates g.

numerical results show that the LVHW3 prices tend to the SZHW prices, whereas the LVHW2 prices are the lowest of the three LVHW models, but still above the BSHW prices. In the LVHW1 model prices remain between BSHW and SZHW values.

The fact that GAO prices obtained in different implied volatility scenarios turn out to be significantly different, underlines that it is most important to always take into account the whole implied volatility surface. This impact in GAO value can be justified by equation (33) where you observe the influence of the level of the equity spot S_t in the dynamics of x_t under the measure Q_S.

In [33], it is pointed out that GAO values are also particularly sensible to three different risk drivers namely the survival probabilities, the fund value and the interest rate market curve. From equation (35), we can easily deduce that an increase of $x\%$ of the equity fund value $S(0)$ will induce an $x\%$ increase of the GAO value. It is also clear that a shift in the mortality table will induce a shift in the GAO value in the same direction. Finally, a shift down applied to the interest rates curve will increase the GAO value. The sensibility of GAO prices with respect to the implied volatility, the survival probabilities, the fund value and the interest rates market curve underline the fact that for practical purposes, the market data used should be considered as important as the model used.

5.3 GMIB Rider

In this section, we analyze how the BSHW, the SZHW and the LVHW models behave when pricing a GMIB Rider. This product has a strong dependence on the path of the equity fund S coming from the anniversary component in the payoff, namely $\underset{n \in A}{max}(S(n))g\ddot{a}_x(T)$ (see equation (37)), where A is the set of anniversary dates $A = \{1, 2, ..., T\}$. We use exactly the same initial data as for the GAO.

More precisely, the policyholder is assumed to be 55 years old with a retirement age of 65 (i.e. the maturity T of the GMIB Rider is 10 years). The fund value at time 0, $S(0)$ is assumed to be 100. The survival rates are based (as before) on the PNMA00 table of the Continuous Mortality Investigation (CMI) for male pensioners. We present numerical results obtained only in the US market because the European market leads to similar behavior in pricing and conclusions. The parameters used in each model are those obtained after calibration as explained in Section 3. In Table 4, we compare the price of a GMIB Rider for eight different guaranteed annuity rates g and three different guaranteed annual rates r_g (computed by using Monte Carlo simulations with 100 000 simulations and 5000 steps). Note that currently, the standard rate r_g offered by insurance companies is around 5% (see [1]).

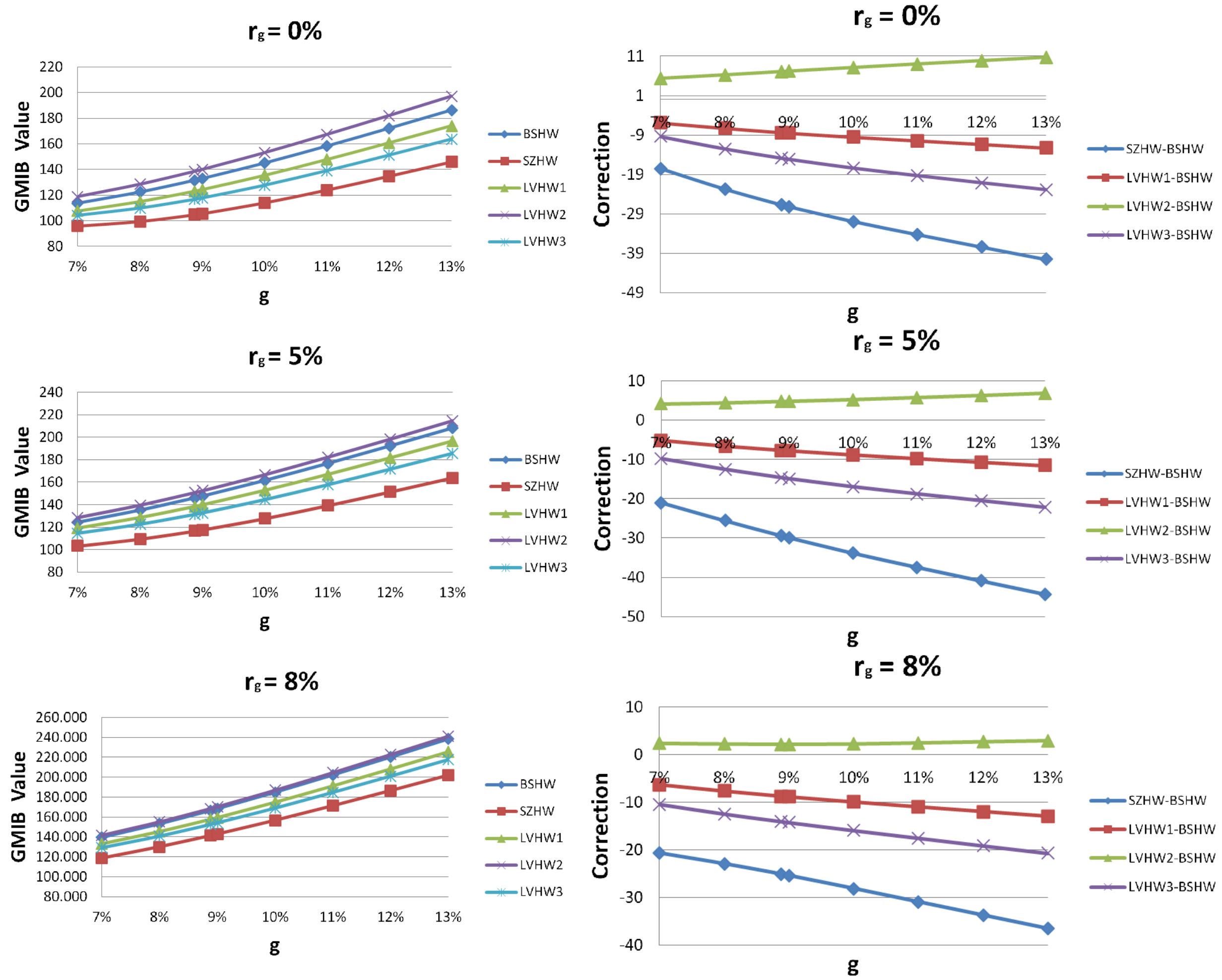


Figure 6: Graphical representation of GMIB Rider total values given by the SZHW, the LVHW1, LVHW2 and LVHW3 models for different guaranteed annuity rates g and for three different fixed guaranteed annual rates r_g. The corrections given by each model with respect to the BSHW model are also presented.

Contrarily to the results obtained for GAOs (in Section 5.2), the corrections given to GMIB values

US GMIB Rider Total Value $r_g = 0\%$										
g	**BSHW**	**SE**	**SZHW**	**SE**	**LVHW1**	**SE**	**LVHW2**	**SE**	**LVHW3**	**SE**
7%	113.5440	0.2892	95.9552	0.0720	107.6490	0.1235	118.8630	0.1497	104.1640	0.1185
8%	122.3090	0.2951	99.5474	0.0776	115.0510	0.1298	128.5280	0.1577	109.8140	0.1233
8.88%	131.4990	0.3063	104.7690	0.0810	123.1420	0.1375	138.5730	0.1675	116.6260	0.1300
9%	132.8540	0.3083	105.6330	0.0896	124.3560	0.1387	140.0450	0.1690	117.6870	0.1310
10%	144.9830	0.3290	114.0400	0.0900	135.4570	0.1503	153.1060	0.1830	127.5870	0.1414
11%	158.2590	0.3557	124.0140	0.0938	147.8150	0.1636	167.3030	0.1990	138.9400	0.1536
12%	172.1850	0.3857	134.7800	0.1001	160.8320	0.1779	182.0900	0.2163	151.0450	0.1669
13%	186.3910	0.4171	145.8680	0.1077	174.1050	0.1925	197.1540	0.2341	163.4870	0.1806
US GMIB Rider Total Value $r_g = 5\%$										
g	**BSHW**	**SE**	**SZHW**	**SE**	**LVHW1**	**SE**	**LVHW2**	**SE**	**LVHW3**	**SE**
7%	124.253	0.2770	103.267	0.0713	119.021	0.1325	128.412	0.1577	114.516	0.1238
8%	135.096	0.2804	109.527	0.0738	128.454	0.1404	139.552	0.1666	122.620	0.1308
8.88%	145.994	0.2897	116.600	0.0768	138.313	0.1495	150.778	0.1767	131.384	0.1392
9%	147.576	0.2914	117.684	0.0818	139.766	0.1509	152.407	0.1783	132.699	0.1405
10%	161.528	0.3105	127.717	0.0830	152.701	0.1637	166.769	0.1927	144.587	0.1526
11%	176.537	0.3354	139.129	0.0859	166.782	0.1783	182.269	0.2093	157.761	0.1663
12%	192.154	0.3637	151.289	0.0877	181.503	0.1938	198.384	0.2272	171.643	0.1809
13%	208.030	0.3934	163.754	0.0944	196.494	0.2097	214.782	0.2459	185.806	0.1959
US GMIB Rider Total Value $r_g = 8\%$										
g	**BSHW**	**SE**	**SZHW**	**SE**	**LVHW1**	**SE**	**LVHW2**	**SE**	**LVHW3**	**SE**
7%	139.3960	0.2579	118.7200	0.0759	133.1170	0.1441	141.7610	0.1600	128.9560	0.1339
8%	153.0990	0.2589	130.1780	0.0775	145.4630	0.1542	155.3320	0.1710	140.6420	0.1439
8.88%	166.4050	0.2661	141.3230	0.0820	157.6800	0.1651	168.5760	0.1831	152.3680	0.1547
9%	168.3040	0.2676	142.9100	0.0828	159.4420	0.1667	170.4710	0.1849	154.0570	0.1563
10%	184.7890	0.2834	156.6750	0.0904	174.8370	0.1812	187.0320	0.2013	168.8660	0.1705
11%	202.2100	0.3051	171.2920	0.0988	191.2250	0.1972	204.6760	0.2197	184.6850	0.1863
12%	220.1720	0.3304	186.4520	0.1075	208.1920	0.2141	222.8780	0.2389	201.0500	0.2027
13%	238.3830	0.3571	201.8650	0.1163	225.4030	0.2314	241.3100	0.2586	217.6730	0.2194

Table 4: Comparison of GMIB Rider total values obtained in the BSHW, the SZHW and the LVHW1, LVHW2 and LVHW3 models for different guaranteed annuity rates g and guaranteed annual rates r_g of 0%, 5% and of 8%.

by the SZHW model with respect to the BSHW ones are always negative. For GMIB Riders the highest values are always given by the LVHW2 model while the smallest values are coming from the SZHW model. In the GAO case, the SZHW and the LVHW3 GAO values were quite close and leading to the highest values among the observed models. For GMIB Riders, however, the smallest prices are given by the SZHW and the LVHW3 models, as illustrated in Figure 6.

5.4 Barrier GAOs

In this subsection, we compare the LVHW price, the SZHW price and the price obtained by using the three different cases of the LVHW model for "down-and-in GAOs" given by equation (44) and computed by using Monte Carlo simulations (100 000 simulations and 5000 steps). While GMIB Riders are path-dependent especially in $S(t)$, these barrier options are particularly dependent on the path of the interest rates $r(t)$. In Table 5, total values of "down-and-in GAOs" with five different barriers are presented. The first barrier is taken to be equal to $B = -0.015$ and corresponds to a market annuity rate $r_x(T)$ equal to 8%. More precisely, when $x^* = -0.015$ and $r_x(T) = 8\%$, the definition of market annuity rate holds, namely,

$$\sum_{n=0}^{\omega-(x+T)} {}_n p_{x+T} A(T, T+n) e^{-b(T,T+n)x^*} = 1/r_x(T).$$

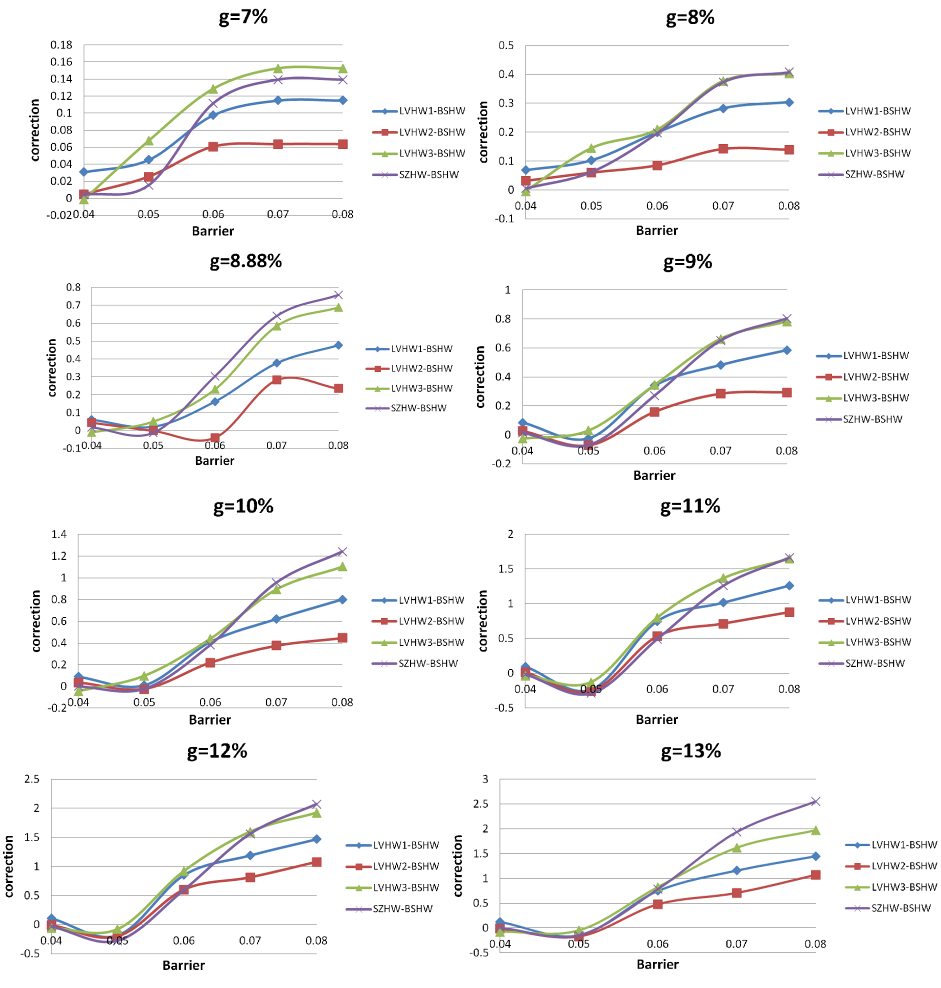


Figure 7: Graphical representation of DI GAO corrections of the SZHW, the LVHW1, LVHW2 and LVHW3 models for different guaranteed annuity rates g with respect to the BSHW model.

BSHW DI GAO Total Value (US)						**BSHW GAO**
Barrier / g	8.00%	7.00%	6.00%	5.00%	4.00%	
7%	0.9069	0.9069	0.6754	0.0931	0.0023	0.9069
8%	3.1600	2.8189	1.3813	0.1053	0.0086	3.1600
8.88%	6.9412	5.0746	2.0215	0.2096	0.0198	7.1019
9%	7.5111	5.3759	2.0906	0.2878	0.0385	7.7384
10%	13.3689	8.2064	2.8525	0.2979	0.0581	14.8802
11%	19.6122	10.8281	3.3454	0.5995	0.0829	23.6438
12%	26.3137	13.6888	4.0752	0.6122	0.1110	33.6896
13%	33.2845	16.7626	5.0114	0.6437	0.1297	44.3822
SZHW DI GAO Total Value (US)						**SZHW GAO**
7%	1.0460	1.0460	0.7869	0.1085	0.0064	1.0460
8%	3.5678	3.1925	1.5783	0.1680	0.0145	3.5678
8.88%	7.6988	5.7147	2.3250	0.1969	0.0372	7.8692
9%	8.3141	6.0276	2.3623	0.2168	0.0518	8.5554
10%	14.6116	9.1621	3.2343	0.2759	0.0586	16.1414
11%	21.2745	12.0856	3.8351	0.3093	0.0679	25.2674
12%	28.3855	15.2518	4.6729	0.3367	0.0811	35.5863
13%	35.8365	18.6996	5.7971	0.5005	0.1373	46.5705
LVHW1 DI GAO Total Value (US)						**LVHW1 GAO**
7%	1.0218	1.0218	0.7731	0.1383	0.0331	1.0218
8%	3.4634	3.1009	1.5804	0.2079	0.0772	3.4634
8.88%	7.4179	5.4514	2.1841	0.2320	0.0817	7.5845
9%	8.0950	5.8578	2.4303	0.2630	0.1223	8.3261
10%	14.1694	8.8259	3.2620	0.3096	0.1485	15.6900
11%	20.8737	11.8444	4.0955	0.3627	0.1801	24.9009
12%	27.7819	14.8798	4.9278	0.4283	0.2190	35.1173
13%	34.7312	17.9215	5.7595	0.4941	0.2492	45.7425
LVHW2 DI GAO Total Value (US)						**LVHW2 GAO**
7%	0.9705	0.9705	0.7361	0.1184	0.0068	0.9705
8%	3.2996	2.9612	1.4667	0.1657	0.0401	3.2996
8.88%	7.1769	5.3578	1.9822	0.2087	0.0651	7.3323
9%	7.8041	5.6591	2.2500	0.2157	0.0657	8.0248
10%	13.8150	8.5819	3.0712	0.2736	0.0936	15.2712
11%	20.4883	11.5410	3.8766	0.3429	0.0996	24.3749
12%	27.3924	14.5022	4.6793	0.4150	0.1048	34.5070
13%	34.3552	17.4711	5.4916	0.4765	0.1133	45.0702
LVHW3 DI GAO Total Value (US)						**LVHW3 GAO**
7%	1.0593	1.0593	0.8039	0.1605	0.0008	1.0593
8%	3.5629	3.1964	1.5906	0.2499	0.0045	3.5629
8.88%	7.6300	5.6593	2.2524	0.2608	0.0104	7.8085
9%	8.2923	6.0394	2.4369	0.3192	0.0131	8.5334
10%	14.4721	9.1002	3.2908	0.3953	0.0140	16.0349
11%	21.2630	12.1972	4.1453	0.4700	0.0487	25.3506
12%	28.2351	15.2863	4.9954	0.5330	0.0497	35.6268
13%	35.2509	18.3785	5.8336	0.5942	0.0516	46.2961

Table 5: Comparison of "down-and-in GAO" total values given by the BSHW and the SZHW and the LVHW models for eight different guaranteed annuity rates g and for five different barriers. The pure GAO values for the eight different guaranteed annuity rates g are also given in the last column.

The other barriers correspond to rates $r_x(T)$ of 7%, 6%, 5% and 4% respectively or equivalently to barrier levels B equal to -0.033, -0.05225, -0.0791 and -0.1019 respectively. Note that since the initial value of $x(0)$ is equal to 0, the barrier level B has to be smaller and consequently strictly negative. The value of a "pure GAO" is given in the last column of Table 5. A graphical representation of the corrections given by each model with respect to the BSHW model can be found in Figure 7, and this for eight different guaranteed annuity rates g. In the case of barrier GAOs the corrections given by each model are more complicated to analyze in the sense that there is no general conclusion with respect to the correction behavior. More precisely, we are not able to answer the question which model gives the highest values or the smallest ones because the prices depend on both the barrier level and the guaranteed annuity rate g.

When the barrier level is equal to 4%, the "down-and-in GAO" value is close to zero. In that case, the survival probability of the "down-and-in GAO" is close to zero. The "down-and-out GAO" can easily be computed from the price of the "down-and-in GAO" and the "pure GAO" by using the relation $C^{DO}(x, 0, T) = C(x, 0, T) - C^{DI}(x, 0, T)$. Consequently, when the barrier level is equal to 4%, the "down-and-out GAO" value is close to the GAO value. In that case, the survival probability of the "down-and-out GAO" is close to one.

6 Conclusion

In this paper we used a local volatility model with stochastic interest rates to price and hedge long maturities life insurance contracts. This model takes into account the stochastic behavior of the interest rates as well as the vanilla market smile effects. Practitioners often price and hedge derivatives with a local volatility model since it has the property to be able to capture the whole implied volatility surface and moreover it has the advantage of implying a complete market.

A first contribution of the paper is the calibration of a local volatility surface in a stochastic interest rates framework. We have developed a Monte Carlo approach for the calibration and this method has successfully been tested on US and European market call data.

The second contribution is the analysis of the impact of using a local volatility model to the pricing of long-dated insurance products as Variable Annuity Guarantees. More precisely, we have compared prices of GAO, GMIB Rider and barrier GAO obtained by using the local volatility model with stochastic interest rates to the prices given by a constant volatility and a stochastic volatility model, all calibrated to the same data. The particularity of the GMIB Rider is the strong dependence on the path of the equity fund; whereas, the interest rate barrier type options have a strong dependence on the path of interest rates. The results confirm that calibrating such models to the vanilla market is by no means a guarantee that derivatives will be priced identically in the different models.

This study shows that using a non constant volatility has a significant impact on the price of Variable Annuity Guarantees. The constant volatility Black Scholes model turns out to underestimate the value of GAO compared to the Schöbel and Zhu stochastic volatility model and the local volatility model in a stochastic interest rates setting. While the constant volatility Black Scholes model turns out to overestimate the GMIB Riders.

Furthermore, we show that the price of Variable Annuity Guarantees depends on the whole option's implied volatility surface and that the calibration of the model used has to be done on the whole implied volatility surface and not only with respect to the implied volatility at maturity of the Variable Annuity Guarantee considered.

This paper underlines the fact that due to the sensibilities of Variable Annuity Guarantee prices with respect to the model used (after calibration to the Vanilla market), and also the sensibilities with respect to data, namely, the survival probability table and the yield curve, practitioners should be cautious in their model choice as well as in the choice of the market data and the calibration of the model.

The results presented in this paper demonstrate that stochastic and local volatility models, perfectly calibrated on the same market implied volatility surface, do not imply the same prices for Variable Annuity Guarantees. In [24], the authors underline that the market dynamics could be better approximated by a hybrid volatility model that contains both stochastic volatility dynamics and local volatility ones. The study of a pure local volatility model is crucial for the calibration of such hybrid volatility models (see [10]). For future research we plan to calibrate hybrid volatility models, based on the results obtained for the pure local volatility model. We will further study the impact of the hybrid volatility models to GAOs, GMIB Riders and barrier GAOs. The hedging performance of all these models is another area for future work.

Acknowledgments

We would like to thank A. van Haastrecht, R. Plat and A. Pelsser for providing us the Hull and White parameters and interest rate curve data they have used in [33]. We would like to thank the anonymous referees for their constructive suggestions and comments. Griselda Deelstra acknowledges support of the ARC grant IAPAS "Interaction between Analysis, Probability and Actuarial Sciences" 2012-2017.

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