How They Can Jump Together :
Multivariate Lévy Processes and Option pricing

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Abstract

The objectives of this paper are twofold. First it gives an overview of the different techniques that can be used to build multivariate Lévy processes. Second it provides new results on multivariate Lévy processes with stochastic volatility in particular the bivariate counter-monotonic Lévy copula is derived.

Keyword 1. *Multivariate Lévy Processes, dependence, stochastic clock.*

1   Introduction

The traditional approach in the option pricing literature is to assume the following dynamics for stock prices

$$dS(t) = \mu S(t)dt + \sigma S(t)d\tilde{B}(t),$$

where μ represents the deterministic component in stock returns, σ is the return volatility and $\tilde{B}(t)$ is a standard Brownian motion. This stochastic differential equation is easily solved and the solution is given by

$$S(t) = S(0)e^{(\mu - \frac{\sigma^2}{2})t + \sigma\tilde{B}(t)},$$

implying that stock prices are following a log-normal distribution. The arbitrage free price at time 0 of a vanilla call option with maturity T and strike price K has been derived by Black and Scholes (1973) and can be expressed as

$$S(0)\Phi\left(\frac{\ln(\frac{S(0)}{K}) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) - Ke^{-rT}\Phi\left(\frac{\ln(\frac{S(0)}{K}) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right), \tag{1}$$

where r is the risk free rate of return and Φ is the standard normal cumulative distribution function (cdf).

The above model is probably the most famous model in the option pricing literature. People often refer to this model as the Black and Scholes model although the idea of modeling the stock price via a log-normal distribution is due to Samuelson (1965). The reasons for this success are both historical and technical. Historically the Black and Scholes model is the first one in which the arbitrage free pricing problem was solved.

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On the technical side the advantages stem from the nice analytical properties of the continuous time Brownian motion. These analytical properties have two implications. First, they allow for a better understanding of the concepts involved, improving our economic understanding of the model. Second, they allow for considerable simplifications in the computation of option prices, leading either to a closed form expression of the price or to closed form approximations.

Despite all its advantages the Black and Scholes model has been increasingly criticized. These critics have two sources: properties of the observed stock returns, and properties of the observed option prices. The critics focus on two aspects of the model: the Brownian motion $\tilde{B}(t)$ and the volatility parameter σ.

Objections against the log-normal assumption based on observed stock prices are quite old. They already appeared in 1963 and were formulated by Mandelbrot (1963). Mandelbrot studied the price of wool and found considerable evidence against the log-normal distribution hypothesis. He reported that the empirical log-returns distribution is considerably more concentrated in the tail and around the origin than what the normal distribution would suggest. To capture these properties Mandelbrot proposed to replace the continuous standard Brownian motion by a pure jump alpha-stable process. Another critic was formulated against the hypothesis of independent returns with constant volatility and lead to the introduction of GARCH models (see for example Engle (2002)).

Objections based on observed option prices are more recent. They appeared in the late eigthies after the financial crisis of 1987 (Alexander (2002)). They are related to the implied volatility and are best known under the name volatility smile. The implied volatility is computed from option prices by choosing the volatility parameter σ that minimizes the distance between the observed option prices and those implied by the Black and Scholes model. What makes this procedure feasible is the fact that all the other quantities appearing in the option price given by (1) are known at time $t = 0$. The Black and Scholes model assumes that the volatility parameter σ is constant. This hypothesis runs into considerable trouble when one computes the implied volatility. At a given maturity the volatility seems to be a non-linear function of the strike price K, a phenomena referred to as the volatility smile. If we compute the implied volatility for a range of maturities and strike prices, not only do we observe the smile effect at each maturity but we also find that the volatility parameter σ depends on the option maturity. To correct for the presence of volatility smile, for a given maturity, one strategy is to introduce jumps in our model by replacing the continuous Brownian motion by a jump process. While to correct for the non-constant term structure of volatility we need to introduce stochastic volatility into the model.

From the previous paragraphs we see that empirical evidence from the stock and the option market suggests to correct the Black and Scholes model by introducing jumps and stochastic volatility. The need to introduce jumps in the price process led to the introduction of the so called Lévy processes. While the need to fit the non constant volatility term led to the development of stochastic volatility for (Lévy) processes. These lines of research have been extensively exploited in the literature since the beginning of the nineties, see for example Madan and Senata (1990), Eberlein and Keller (1995), Barndorff-Nielsen (1997), Madan et al. (1998), Prause (1999), Barndorff-Nielsen and Shepard (1999), Carr et al. (2003), Carr and Wu (2004).

There is already a large, and still growing, literature on option pricing with Lévy processes. However most of the existing papers deal with the problem in a univariate set-up. This paper aims to fill the gap. The objective of this paper is twofold, first it has a pedagogical purpose. In this paper we expose the basic properties of Lévy processes and provide a discussion of the various methods that can be used to build a multivariate Lévy process. Second, we extend some of the existing univariate Lévy processes with stochastic volatility to a multivariate set-up.

The paper is organized as follows: in section 2 we define Lévy processes and provide a short review of their basic mathematical properties, in section 3 we present the existing multivariate Lévy processes and discuss their construction. In section 4 we discuss univariate stochastic volatility Lévy processes and propose some possible multivariate extensions. Finally section 5 concludes.

2 Mathematical Aspects of Lévy processes

In this section we briefly expose some mathematical properties of Lévy processes. The treatment here is by no mean exhaustive, it only aims at giving some basic intuition to the reader unfamiliar with this class of processes.[1] Good references on the topic are the books of Sato (1999), Bertoin (1996) and Applebaum (2004). The books of Cont and Tankov (2004) and Schoutens (2003) are good suggestions for the reader interested in financial applications.

Strictly speaking a Lévy process is defined as follows (see Sato (1999)):

Definition 1: A process $\{X(t)\}_{t \geq 0}$ defined on $(\Omega, \mathbb{F}, (\mathbb{F})_{t \geq 0}, P)$ is a Lévy process if the following properties are satisfied:

1. For any choice of $n \geq 1$ and $0 = t_0 < t_1 < \ldots < t_n$ the random variables $X(t_1) - X(t_0), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$ are independent.

2. $X(t_0) = 0$ almost surely (a.s.).

3. The distribution of $X(s+t) - X(s)$ does not depend on s (this is the temporal homogeneity property).

4. It is stochastically continuous (i.e. for every $t \geq 0$ and $\epsilon > 0$, $\lim_{s \to t} P\left[|X(s) - X(t)| > \epsilon\right] = 0$).

5. There is $\Omega_0 \in \mathbb{F}$ with $P(\Omega_0) = 1$ such that, for every $\omega \in \Omega_0$, $X(t)$ is right-continuous in $t \geq 0$ and has left limits in $t > 0$.

These properties deserve a number of comments. Property 1 has two implications: First, since the increments over disjoint time interval are independent we should not expect our basic Lévy process to capture volatility clustering phenomenas. Second, and more important, it means that the distribution of our process can be disaggregated. In other words, assume that we choose to model the behavior of the weekly returns using a certain Lévy process, then property 1 tells us that the distribution of the daily returns exists and can be evaluated from the model we estimated using property 3. The conditions imposed on the jumps by properties 4 and 5 imply that the process is allowed to jump and that the jumps are non predictable.

One of the nicest properties of Lévy processes is that the general form of their characteristic function is known. The Lévy-Khintchine Theorem tells us that if $\{X(t)\}_{t \geq 0}$ is a Lévy process on $\mathbb{R}^d$ then its characteristic function can be written as (see Sato (1999)):

$$E(e^{iu'X(t)}) = \exp\left\{t(iu'\gamma - \frac{1}{2}u'\mathbf{A}u + \int_{\mathbb{R}^d \setminus \{0\}} (e^{iu'x} - 1 - iu'x 1_{|x| \leq 1})\nu(dx))\right\}, \quad u \in \mathbb{R}^d,$$

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[1]Another short and excellent introduction to Lévy processes can be found in Eberlein (2007). However the approach we follow here is conceptually different. We will introduce Lévy processes via the Lévy-Khintchine representation Theorem while Eberlein (2007) uses the Ito-Lévy decomposition.

where $\gamma \in \mathbb{R}^d$, $\mathbf{A}$ is a positive-definite matrix of size $d \times d$ and $\nu(dx)$ is called the Lévy measure. It is a positive measure that satisfies:

$$\int_{\mathbb{R}^d \setminus \{0\}} (1 \wedge |x|^2)\nu(dx) < +\infty. \tag{2}$$

From the Lévy-Khintchine decomposition we see that any Lévy process can be represented as the sum of two independent processes. The first one is a standard Brownian motion, this is the continuous part of the process. The second one can be interpreted as an infinite sum of compound Poisson processes, this part represents the jumps of the processes. Remark that the only integrability condition imposed on the measure $\nu(dx)$ is (2). This condition is quite flexible, in particular it allows $\nu(dx)$ to be an infinite measure with an asymptote at the origin.

From the characteristic function it is obvious that the properties of the jumps will be related to the behavior of the measure $\nu(dx)$. When the Lévy measure $\nu(dx)$ is identically equal to zero the process is a standard Brownian motion, and has continuous sample paths. When the matrix $\mathbf{A}$ is equal to zero we have a pure jump process with discontinuous sample paths (but not necessarily purely discontinuous ones). Indeed if $\int_{\mathbb{R}^d/\{0\}} \nu(dx) < \infty$ then the process is said to have finite activity rate. For such processes the jumps are modelled as compound Poisson processes. They will have a finite number of jumps in any compact time interval, so their sample path will be discontinuous. If $\int_{\mathbb{R}^d/\{0\}} \nu(dx) = \infty$ we say that the process has an infinite activity rate. Such a process will display an infinite number of jumps in any compact time interval; their sample path will be purely discontinuous.

It can also be shown that at any time t the rate of arrival of jumps in an interval is given by:

$$\int_E \nu(dx) < \infty,$$

where E is a compact subset of $\mathbb{R}^d \setminus \{0\}$. Putting this together with the discussion of the previous paragraph we see that Lévy processes with an infinite activity will display an infinite number of small jumps in any time interval (this should be put in parallel to the stochastic continuity property of Lévy processes).

Differentiating the characteristic function we obtain an expression for the moments of $X(t)$ as a function of $\nu(dx)$. There is also a relation between the measure $\nu(dx)$ and the functional moments of the process. It can be shown that for any positive measurable function g satisfying some regularity conditions, $E(g(X(t))) < \infty$ if and only if $\int_{|x|>1} g(x)dx < \infty$ (Sato (1999) pp. 159-168). Thus it is the behavior of the big jumps that determines the finiteness of the moments. Finally it should be noted that there is an immediate link between the measure $\nu(dx)$ and the probability density function of the process. Indeed under some regularity assumptions the probability density function of a Lévy process can be computed as the limit of the Lévy measure for $t \to 0$ (Ruschendorf and Woerner (2002)).

We now give three examples of univariate Lévy processes commonly encountered in the financial literature.
Example 1: Merton's jump diffusion model
This is one of the first jump models that have been introduced in the financial literature. It is due to Merton (1976). The return process is assumed to be of the form

$$X(t) = \gamma t + \sigma B(t) + \sum_{i=1}^{N(t)} Y_i,$$

where γ is the drift, σ the volatility, $B(t)$ is a standard Brownian motion, $N(t)$ a Poisson process with intensity rate λ and Y_i a sequence of independent normals with common mean and variance and independent of $N(t)$. The characteristic function of this process is

$$E(e^{iuX(t)}) = \exp\left\{iu\gamma t - \frac{u^2 t^2 \sigma^2}{2} + t \int_{\mathbb{R}\backslash\{0\}} (e^{iux} - 1)\nu(dx)\right\},$$

with

$$\nu(dx) = \lambda \frac{1}{(2\pi)^{1/2}\tilde{\sigma}} e^{-\frac{1}{2\tilde{\sigma}^2}(x-\mu)^2} dx,$$

where μ and $\tilde{\sigma}$ are the common mean and variance of the normal r.v.s Y_i. Since this is a compound Poisson process it has finite activity. The finiteness of the Lévy measure implies that the arrival rate of jumps in a finite interval can also be interpreted as a probability (after being normalized by λ).

Several compound Poisson process models can be found in the literature. They differ by the probability distribution function that they choose for the jumps. For example a popular alternative to the Merton model is the Kou model (Kou (2002)) where the jumps are assumed to follow an exponential distribution. Some multivariate jump diffusion models have also been proposed (see Lindskog and McNeil (2001), Cont and Tankov (2004)) and methods allowing for stochastic volatility have been developed (Duffie et al. (1999)). However in this paper we will not discuss these models any further. Instead we will focus on jump processes with an infinite activity rate and no Brownian motion component.

Example 2: The Variance Gamma Model
This process was originally introduced by Madan and Senata (1990) and further studied in Madan et al. (1998). It is a pure jump process that is obtained by *changing the clock* of a standard Brownian motion by a Gamma process. The Variance Gamma process is defined as

$$\text{VG}(t) = \theta G(t) + \sigma B(G(t)),$$

where $G(t)$ is a gamma process, it is a positive and strictly increasing and purely discontinuous Lévy process. In the literature the process $G(t)$ is also called the stochastic clock in contrast to the calendar clock t. Generally the condition $E(G(t)) = t$ is imposed, this normalization condition simply means that the stochastic clock is not expected to run faster than the real one. Using this normalization condition the density of $G(t)$ can be written as

$$f_{G(t)}\left(x; \frac{t}{\nu}, \frac{1}{\nu}\right) = x^{\frac{t}{\nu}-1} \frac{e^{-\frac{x}{\nu}}}{\Gamma(\frac{t}{\nu})\nu^{\frac{t}{\nu}}}, \quad \nu > 0,$$

where $\Gamma(\cdot)$ stands for the gamma function. Positive and strictly increasing Lévy processes like $G(t)$ are also called subordinators. It can be shown that a Lévy process that is time changed by a subordinator remains a Lévy process (see Sato (1999) pp. 197-202).

The density and the characteristic function of a Brownian motion time changed by a subordinator, $G(t)$, can be derived exploiting the fact that $\text{VG}(t) \mid G(t) = g$ has a Gaussian distribution with mean θg and variance $\sigma^2 g$. Since we will frequently use the Variance Gamma model in this paper we report the characteristic and the density function of the univariate Variance Gamma

$$\varphi_{\text{VG}(t)}(u) = (1 - iu\nu\theta + \frac{1}{2}\nu u^2 \sigma^2)^{-\frac{t}{\nu}}, \tag{3}$$

$$f_{\text{VG}(t)}(x) = \frac{2e^{\frac{\theta x}{\sigma^2}}}{\nu^{\frac{t}{\nu}}\sqrt{2\pi}\sigma\Gamma(\frac{t}{\nu})} \left(\frac{x^2}{\frac{2\sigma^2}{\nu}+\theta^2}\right)^{\frac{t}{2\nu}-\frac{1}{4}} K_{\frac{t}{\nu}-\frac{1}{2}}\left(\frac{1}{\sigma^2}\sqrt{x^2(\frac{2\sigma^2}{\nu}+\theta^2)}\right),$$

where $\Gamma(\cdot)$ is the gamma function and $K_k(\cdot)$ denotes the modified Bessel function of the second kind of order k, see Madan et al. (1998) for a detailed derivation.

Example 3: The Generalized Hyperbolic Model

The generalized hyperbolic distribution (GH) is a pure jump process that has been introduced in finance by the work of Eberlein and Keller (1995). Its construction is similar to the one of the variance gamma process. However, this time the stochastic clock follows a generalized inverse Gaussian distribution. The univariate generalized hyperbolic distribution has the following probability distribution function at time $t = 1$:

$$
\begin{aligned}
f_{\mathrm{GH}(1)}(x) &= a(\lambda, \delta, \beta, \alpha) \frac{K_{\lambda-\frac{1}{2}}(\alpha\sqrt{\delta^2 + (x-\mu)^2})}{(\delta^2 + (x-\mu)^2)^{\frac{1}{4}-\frac{\lambda}{2}}} e^{\beta(x-\mu)}, \\
a(\lambda, \delta, \beta, \alpha) &= \frac{(\alpha^2 - \beta^2)^{\frac{\lambda}{2}}}{\sqrt{2\pi}\alpha^{\lambda-\frac{1}{2}}\delta^\lambda K_\lambda(\delta\sqrt{\alpha^2 - \beta^2})}, \\
\varphi_{\mathrm{VG}(1)}(u) &= e^{iu\mu} \left(\frac{\alpha^2 - \beta^2}{\alpha^2 - (\beta+iu)^2}\right)^{\frac{\lambda}{2}} \frac{K_\lambda(\delta\sqrt{\alpha^2 - (\beta+iu)^2})}{K_\lambda(\delta\sqrt{\alpha^2 - \beta^2})}.
\end{aligned}
\tag{4}
$$

Derivation of the Lévy measure and the characteristic function of the GH distribution can be found in Prause (1999).

Remark When building market models using Lévy processes we generally model asset prices using the standard or the Doleans Dade exponential of a Lévy process (remark that under some conditions these approaches are equivalent, see Tankov (2004)). It is then natural to ask if the Lévy market models are arbitrage free, and if so are they complete? Under some relatively weak conditions market models based on Lévy processes will be arbitrage free (meaning that we will be able to find an equivalent martingale measure). But except in some special cases, the most notorious being the Brownian motion market model, these models will never be complete (implying that the equivalent martingale measure will not be unique). These points were clarified by Cherny (2001). The non uniqueness of the equivalent martingale measure has two implications. First, it means that the pay off of a derivative cannot be perfectly replicated. This implies that the arbitrage free price in these models will not be unique (since each equivalent martingale measure gives us a different price). Second, when pricing a claim we will need to choose between one of those martingale measures. Several approaches based on utility indifference pricing have been proposed to choose between the equivalent martingale measures (see Cont and Tankov (2004) for a discussion). However it is a common practice in the option pricing literature to use observed option prices to estimate the parameters of the model after adjusting the drift of the process to obtain a martingale. This method could be seen as deriving the measure from the market.

3 Multivariate processes

There are three different ways to build multivariate Lévy processes: multivariate subordination, affine transformation and Lévy copulas. In this section we present those three methods.

3.1 Multivariate processes via subordination

One of the easiest ways to build multivariate Lévy processes is to subordinate a multivariate Brownian motion. The main advantage of this method is that we can use our economic intuition when building the model. Various multivariate Lévy processes exist in the literature, however in what follows we will focus on multivariate extensions of the variance gamma model.

3.1.1 Common Clock Variance Gamma

In their seminal paper Madan and Senata (1990) propose the following generalization of the univariate variance gamma model. Let $B(t)$ be a d-dimensional Brownian motion process with covariance rate Σt and let $G(t)$ denote a univariate gamma process. Then we call the process

$$X(t) = \theta G(t) + B(G(t)),$$

a multivariate gamma process, where θ is a d-dimensional drift vector. In this model there are two sources of dependence. The first one comes from the stochastic clock, all the Brownian motions are subordinated by the same gamma process. Since the discontinuities of the sample path are due to the subordinator and the same subordinator is acting on all the processes this implies that all the processes will jump at the same time, in other words they will jump together. The second one comes from the correlation between the Brownian motions. The correlation implies that the amplitude of the jumps will be correlated across assets. In what follows we call this model the common clock variance gamma (CCVG). Remark that since the processes are subordinated by the same stochastic clock the components of the CCVG process will remain dependent even if we set $\Sigma = I_d$.

After some computations we can also derive the characteristic function of the CCVG at time t (see Leoni and Schoutens (2008)):

$$\varphi_{\text{CCVG}}(u) = (1 - iu'\nu\theta + \frac{1}{2}\nu u'\Sigma u)^{-\frac{t}{\nu}}.$$

The correlation coefficient between two components of the CCVG process can easily be computed and it is given by

$$
\begin{aligned}
\rho_{ij}^{\text{CCVG}} &= \frac{E(X_1^i X_2^j) - E(X_1^i)E(X_2^j)}{\sqrt{Var(X_1^i)}\sqrt{Var(X_2^i)}} \\
&= \frac{\theta_i\theta_j\nu + \sigma_{ij}}{\sqrt{\sigma_i^2 + \theta_i^2\nu}\sqrt{\sigma_j^2 + \theta_j^2\nu}},
\end{aligned}
$$

where $\sigma_{ij} = \Sigma_{ij}$. It is also possible to derive the probability density function of the CCVG. Since we did not find this result in the literature we state it as a Theorem.

Theorem 1. *The probability density function of the CCVG model is given by*

$$f(x; \nu, \Sigma, \theta) = \frac{2e^{x'\Sigma^{-1}\theta}}{(2\pi)^{\frac{N}{2}}|\Sigma|^{\frac{1}{2}}\nu^{\frac{t}{\nu}}\Gamma(\frac{t}{\nu})}\left(\frac{x'\Sigma^{-1}x}{(\frac{2}{\nu} + \theta'\Sigma^{-1}\theta)}\right)^{\frac{t}{2\nu}-\frac{N}{4}}K_{\frac{t}{\nu}-\frac{N}{2}}\left(\sqrt{(x'\Sigma^{-1}x)(\frac{2}{\nu} + \theta'\Sigma^{-1}\theta)}\right),$$

where $K_{\frac{t}{\nu}-\frac{N}{2}}$ is the modified Bessel function of order $\frac{t}{\nu} - \frac{N}{2}$.

Proof
The proof uses the conditional gaussianity of the CCVG process. We know that $X(t) \mid G_t = g$ is a gaussian random vector with mean $g\theta$ and variance covariance matrix $g\Sigma$. This means that the density of $X(t)$ can be obtained from the integral

$$f(x; \nu, \Sigma, \theta) = \int_0^{+\infty} \phi(x; \theta g, g\Sigma) f_{\text{gamma}}(g; \frac{t}{\nu}, \frac{1}{\nu})dg,$$

where $\phi(x; \theta g, g\Sigma)$ is a multivariate normal density with mean θg and variance covariance $g\Sigma$. After some straightforward manipulations this integral can be solved analytically if we use formula 3.479 from Gradshteyn and Ryzhik (1973)[2]. □

Note that for $N = 1$ and $\Sigma = \sigma^2$ the density of the CCVG reduces to $f_{\text{VG}(t)}(x)$ in (3).

Let us mention two papers studying the calibration of the above model to option prices. The first calibration of this model was carried out by Luciano and Schoutens (2006) in the particular case of $\Sigma = I_d$. As they point out, this specification has the advantage that one can calibrate the model using data on vanilla options only. A second calibration can be found in the paper of Leoni and Schoutens (2008). This time the authors did not restrict the matrix Σ and explained how stock price time series can be used to compute it.

3.1.2 Correlated Clock and Independent Brownian Motions (IBCCVG)

The above generalization of the univariate variance gamma model is not unique. Recently a new multivariate version has been developed by Semeraro (2006) and studied extensively in Luciano and Semeraro (2007) and Fiorani et al. (2007). The construction is in this case sensibly different from the one above. Here we will follow the work of Fiorani et al. (2007).

Consider the following multivariate gamma random vector,

$$G(t) = (g_1(t) + \alpha_1 g_{d+1}(t), g_2(t) + \alpha_2 g_{d+1}(t), \ldots, g_d(t) + \alpha_d g_{d+1}(t))',$$

where $\alpha_i > 0$, $g_i(t)$ are d independent gamma processes with parameter $((\frac{1}{\alpha_i} - a)t, \frac{1}{\alpha_i})$ and $g_{d+1}(t)$ is an independent gamma process with parameter $(at, 1)$[3]. Now taking a d-dimensional Brownian motion vector $B(t)$ with diagonal variance covariance matrix Σt we define the following process

$$X(t) = \theta * G(t) + B(G(t)),$$

where $\theta \in \mathbb{R}^d$, $*$ is the componentwise multiplication, and the ith element of the vector $B(G(t))$ is $B_i(g_i(t) + \alpha_i g_{d+1}(t))$. The characteristic function of this process is given by (see Fiorani et al. (2007)):

$$\varphi_{\text{IBCCVG}}(u) = \prod_{j=1}^{d} (1 - \alpha_j(i\theta_j u_j - \frac{1}{2}\sigma_j^2 u_j^2))^{-t(\frac{1}{\alpha_j} - a)} (1 - \sum_{s=1}^{d} \alpha_s(i\theta_s u_s - \frac{1}{2}\sigma_s^2 u_s^2))^{-ta}, \quad (5)$$

where $\sigma_j^2 = (\Sigma)_{jj}$. Using the characteristic function it is straightforward to see that the marginal distributions of this process are variance gamma. Some straightforward computations show that the correlation coefficient between the different components of this process are given by:

$$\rho_{ij}^{\text{IBCCVG}} = \frac{\theta_j \theta_i \alpha_j \alpha_i a}{\sqrt{(\sigma_i^2 + \theta_i^2 \alpha_i)}\sqrt{(\sigma_j^2 + \theta_j^2 \alpha_j)}}.$$

In this multivariate version of the variance gamma distribution, the dependence between the components of the vector is due to the correlation between the stochastic clocks.

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[2] $\int_0^{+\infty} x^{\nu-1} e^{-\frac{\beta}{x} - \lambda x} dx = 2 \left(\frac{\beta}{\lambda}\right)^{\frac{\nu}{2}} K_\nu(2\sqrt{\lambda\beta})$.
[3] With the constraint that $0 < a < \frac{1}{\alpha_j}$.

This model has a nice economic interpretation. Here, asset prices are affected by two sources of uncertainty, one is common to all the assets and can be seen as a global uncertainty, the other is an idiosyncratic uncertainty. The presence of the idiosyncratic clock implies that our assets will not jump together. Unfortunately no closed form expression for the probability density function of this model is known. This poses some serious problems when one is trying to perform estimations under the historical probability measure.

3.1.3 Other Multivariate Processes Obtained by Subordination

Of course in the above constructions we could have used any subordinator. For example, if we follow the same steps as for the construction of the process in 3.1.1 but with a generalized inverse Gaussian subordinator we obtain the following pdf for common clock generalized hyperbolic distribution (CCGH):

$$
gh_d(x) \;=\; a_d \frac{K_{\lambda - \frac{d}{2}}(\alpha\sqrt{\delta^2 + x'\Delta^{-1}x})}{(\alpha^{-1}\sqrt{\delta^2 + x'\Delta^{-1}x})^{\frac{d}{2}-\lambda}} e^{\beta' x},
$$

$$
a_d \;=\; \frac{(\sqrt{\alpha^2 - \beta'\Delta\beta})^\lambda}{\delta^\lambda (2\pi)^{\frac{d}{2}} K_\lambda(\delta\sqrt{\alpha^2 - \beta'\Delta\beta})},
$$

where $\lambda \in \mathbb{R}$, $\beta \in \mathbb{R}^d$, $\delta > 0$, $\alpha^2 - \beta'\Delta\beta \geq 0$ and $\Delta \in \mathbb{R}^{d\times d}$ is positive definite and has a determinant equal to one. This distribution has been extensively studied in the literature, see for example Blaesild (1981) and Prause (1999). Finally, remark that if we set $\lambda = -\frac{1}{2}$ we obtain a process with negative inverse Gaussian margins.

We could also have used a set of generalized inverse Gaussian subordinators when building the multivariate model of section 3.1.2. This approach has been followed by Luciano and Semeraro (2007) who develop a multivariate process with negative inverse Gaussian margins. The construction goes as follows; let $g_i \sim IG(1 - a\gamma_i, \frac{b}{\gamma_i})$ for $i = 1, \ldots, d$ and $g_{d+1} \sim IG(a, b)$. Then the process defined as

$$
G(t) = (g_1(t) + \gamma_1^2 g_{d+1}(t), g_2(t) + \gamma_2^2 g_{d+1}(t), \ldots, g_d(t) + \gamma_d^2 g_{d+1}(t))',
$$

has inverse Gaussian margins. Let $B(t)$ is a d-dimensional standard Brownian motion, then the multivariate generalized hyperbolic process is defined as:

$$
X(t) = \beta * \delta^2 * G(t) + \delta * B(G(t)),
$$

where β and δ are d-dimensional vectors and δ^2 refers to the componentwise square. The characteristic function of $X(t)$ is:

$$
\varphi_{\text{IBCCNIG}}(u) \;=\; \exp\left\{ -\sum_{j=1}^n (1 - a\gamma_j)\left(\sqrt{-2i(i\beta_j\delta_j^2 u_j - \frac{1}{2}\delta_j^2 u_j^2) + \frac{b^2}{\gamma_j^2}} - \frac{b}{\gamma_j}\right)\right.
$$
$$
\left. -\sum_{j=1}^n a\gamma_j\left(\sqrt{-2i\sum_{s=1}^n \gamma_s(i\beta_s\delta_s^2 u_s - \frac{1}{2}\delta_s^2 u_s^2) + \frac{b^2}{\gamma_j^2}} - \frac{b}{\gamma_j}\right)\right\}.
$$

3.2 Multivariate processes via Linear combination

Another way of modeling the dependence structure of multivariate Lévy processes is through linear combination of independent Lévy processes. This method relies on the following idea, consider an n-dimensional

Lévy process $X(t)$ with independent components. Then for any matrix $\mathbf{A} \in \mathbb{R}^{d \times n}$ and vector $b \in \mathbb{R}^{d \times 1}$ the process $Y(t) = \mathbf{A}X(t) + bt$ is a d-dimensional Lévy process (Sato (1999) pp. 65-66). In the next Proposition we derive the characteristic function of such process $Y(t)$.

Proposition 1. *Let $X(t)$ be an n-dimensional Lévy process with independent components. Then the characteristic function of $Y(t) = \mathbf{A}X(t) + bt$ is given by*

$$\varphi_{Y(t)}(u) = e^{iu'bt} \prod_{s=1}^{n} \left[\varphi_{X_s}(\sum_{j=1}^{d} u_j(\mathbf{A})_{js}) \right]^t ,$$

where $\varphi_{X_s}(u)$ is the characteristic function of the marginal $X_s(1)$ and $(\mathbf{A})_{js}$ is the element js of the matrix $\mathbf{A} \in \mathbb{R}^{d \times n}$.

Proof
By definition we have

$$\varphi_{Y(t)}(u) = E(e^{iu'Y(t)}) = E(e^{iu'(\mathbf{A}X(t)+bt)}).$$

After some straightforward rearrangement and using the independence of the X_s, we can rewrite

$$E(e^{iu'(\mathbf{A}X(t)+bt)}) = e^{iu'bt} \prod_{s=1}^{n} E(e^{i(\sum_{j=1}^{d} u_j(\mathbf{A})_{js})X_s(t)}).$$

The result then follows after straightforward computations. $\qquad \square$

Let us now discuss two examples of multivariate affine Lévy process.

Example 4: Schmidt et al. (2006) study what they call a multivariate affine generalized hyperbolic (MAGH) process. They proceed as follows: consider the d-dimensional random vector $X = (X_1, \ldots, X_d)'$ such that all the X_i are independent with distribution function (4) with $\mu = 0$ and $\delta = 1$. Then $Y = \mathbf{A}X + b$ is a MAGH process, where $\mathbf{A} \in \mathbb{R}^{d \times d}$ is a lower triangular matrix and $b \in \mathbb{R}^{d \times 1}$.

Schmidt et al. (2006) find that the (MAGH) processes have several advantages compared to the CCGH process. First, the MAGH process can produce multivariate distributions with independent margins (simply take a diagonal matrix $\mathbf{A}$). Second, Schmidt et al. (2006) proved that the MAGH probability distribution function can capture tail dependence for some parameter combinations. This is an interesting property since it allows us to capture dependence of extreme events such as crashes. It should be noted that since the CCVG and CCGH distribution are tail independent. Third, we see that the components of a MAGH vector can be made independent by a simple linear transformation. Using this property it is possible to set-up a two stages estimation procedure based on a Cholesky decomposition that can be used to perform estimation under the historical probability measure (see Schmidt et al. (2006)).

However let us note two potential drawbacks of the MAGH model. First, the marginal distributions of the MAGH distribution are not GH (except the first component of the vector of course). Indeed the marginal distributions will generally be unknown. Second, the MAGH distribution will not be closed under convolution (except if all the components of the X vector have negative inverse Gaussian distribution).

Example 5: In a recent paper Garcia et al. (2009) developed a structural model to price a CDO. Their model can be rewritten as an affine Lévy process. In their model the value of firm i at time $t = 1$ is given by the exponential of the following Lévy process

$$Y^{(i)}(1) = X(\rho) + X^{(i)}(1 - \rho) \quad 1 \leq i \leq d \quad 0 \leq \rho \leq 1,$$

where $X(1)$ and $X^{(i)}(1)$ are $d + 1$ independent and identically distributed Lévy processes with mean zero and variance one. Simple computations show that the correlation between the Lévy processes $Y^{(i)}(1)$ is constant and equal to ρ. This model can be rewritten in the form of an affine process. Indeed we have

$$Y(t) = \mathbf{A}\hat{X}(t) = \begin{bmatrix} e & I_d \end{bmatrix} \hat{X}(t) \quad t \in [0, T],$$

where e is a d-dimensional vector of ones, I_d a d-dimensional identity matrix and $\hat{X}(t)$ a $(d+1)$-dimensional Lévy process whose expression at time t is given by

$$\hat{X}(t) = (X(\rho t), X^{(1)}((1 - \rho)t), \ldots, X^{(d)}((1 - \rho)t))'.$$

In their paper, Garcia et al. (2009) compare this model to the standard Gaussian model. They find that their model yields superior results in terms of calibration performances and provides more reliable correlation estimates. Finally using Proposition 1 we can easily evaluate the characteristic function of the $Y(t)$ vector

$$\varphi_{Y(t)}(u) = \left[\varphi_{X(1)}(\sum_{j=1}^{d} u_j) \right]^{\rho t} \prod_{j=1}^{d} \left[\varphi_{X(1)}(u_j) \right]^{(1-\rho)t}.$$

3.3 Multivariate processes via Lévy copula

The third and last method that can be used to build multivariate Lévy processes is based on the so called Lévy copula. Since Lévy copulas are involved mathematical objects we will not enter a complete discussion of their properties here. Rather we will briefly sketch the intuition behind their construction and give some references. In the appendix of the paper we derive the bivariate counter-monotonic Lévy copula.

Lévy copulas were originally introduced for processes with only positive jumps by Tankov (2004), they were extended to arbitrary Lévy processes in Kallsen and Tankov (2006). Tankov pointed out that the standard theory of copulas is unsuited to model the dependence in a time series context. Three problems arise when one tries to use standard copulas in a dynamic context; first the copulas may need to be time dependent; second the multivariate process may lose some of the desirable properties of its univariate components (for example the univariate components are all Lévy processes while the multivariate is not); third standard copulas are unsuited to model processes with discontinuous paths (see Tankov (2004) for examples). It is to avoid those problems that Tankov introduced the concept of Lévy copula.

The major difference with respect to standard copulas is that Lévy copulas are not defined on the cumulative distribution function but on the characteristic function of the process. Lévy copulas are defined on the Lévy measure $\nu(dx)$ itself. They can be seen as functions glutting the marginal Lévy measures together to form the Lévy measure of the joint process. The advantages of this method are twofold: first, it avoids the problems posed by standard copulas in a dynamic context. Second, since the behavior of the process is directly related to the form of the Lévy measure we have a straightforward interpretation of the behavior of our multivariate process.

The current literature on Lévy copulas can be resumed as follows: Tankov (2004) introduced the concept for spectrally positive processes and proposed a simulation algorithm. Spectrally positive Lévy copulas are

further studied in Barndorff-Nielsen and Linder (2005). Kallsen and Tankov (2006) generalized the concept to arbitrary multivariate processes and developed the class of Archimedian Lévy copulas. Finally Tankov (2006) proposed a new simulation algorithm for general copulas. There are still a lot of unscratched issues on Lévy copulas and no estimation procedure has been proposed yet.

4 Lévy processes with stochastic volatility

Lévy processes have been successful at fitting the implied volatility surface across strikes at a given maturity. Unfortunately it has been found that their performances are considerably reduced if one tries to fit the volatility surface for different maturities. Two properties of the Lévy processes have been held responsible for this failure. Namely the homogeneity and independence properties. In our context the homogeneity property implies that the distribution of the returns should only depend on the frequency of the observations. The independence property implies that the increments of the process on disjoint time intervals should be independent.

These assumptions have two major drawbacks. First, they exclude phenomenas like volatility clustering. This is an undesirable property since it has been recognized that the higher moments of financial time series are dependent (see for example, Barndorff-Nielsen and Shepard (1999)). Second, the homogeneity and independence properties have some rather strong implications for the scaling of the variance, kurtosis and skewness (see Konikov and Madan (2002)). Recently some authors have tried to remedy to those problems in a univariate set-up. These approaches can be grouped in five categories: two state Markov chain Lévy processes, stochastic clock Lévy processes, stochastic volatility Lévy processes, local Lévy models and Sato processes. We start this section by reviewing these five approaches. Afterwards we will propose new ways of modeling stochastic volatility in a multivariate set-up.

4.1 Some existing stochastic volatility Lévy processes

4.1.1 Two States Variance Gamma Process

Konikov and Madan (2002) proposed to model the asset returns using a discrete state hidden Markov chain. In this model there are two states of the world and the process can switch from one state to the other with a strictly positive probability. Konikov and Madan (2002) specify the price process as

$$S(t) = S(0)e^{rt+w(t)+X(t)},$$

where

$$E\left[e^{X(t)}\right] = e^{-w(t)},$$

and with

$$X(t) = \int_0^t (1 - U(s))dX_0(s) + \int_0^t U(s)dX_1(s),$$

$U(s)$ is an independent Bernoulli distributed random variable indicating the state of the world, and X_0, X_1 are two independent Lévy processes. Using daily returns of the Dow Jones Industrial Average the authors find support for the presence of only two underlying states. The authors argue that by modifying the probability of the initial state, their model can capture declining term structure volatility in the option maturity

direction. By adjusting the probabilities of transition from one state to another one can capture volatility clustering. Even if the probability density function of the model is not immediately available, the characteristic function of the model remains tractable and can be used to do option pricing or maximum likelihood estimation (see Proposition 2 in Konikov and Madan (2002)). The model was originally developed in the case where X_0, X_1 are two independent univariate variance gamma processes. However it can easily be extended to the case of a more general multivariate Lévy process.

4.1.2 Lévy processes with stochastic clock

Another way to get rid of the homogeneity and independence property is to time change the clock of a Lévy process by a process whose increments are non-homogeneous and dependent. This method was originally introduced by Carr et al. (2003). Let $\overline{\sigma}(t)$ be an increasing right continuous process with left limits such that $\overline{\sigma}(t)$ is a stopping time $\forall t$ and $\overline{\sigma}(t) \to +\infty$ as $t \to +\infty$. Then define the time changed Lévy process as

$$Z(t) = X(\overline{\sigma}(t)),$$

where $X(t)$ is a standard Lévy process. Carr et al. (2003) considered the case where the processes $X(t)$ and $\overline{\sigma}(t)$ are independent. In this case the characteristic function of the process can easily be derived since

$$E(e^{iuZ(t)}) = E(e^{\overline{\sigma}(t)\Psi_x(u)}) = \varpi(-i\Psi_x(u)),$$

where $\Psi_x(u)$ is the characteristic exponent of $X(1)$ and ϖ the characteristic function of $\overline{\sigma}(t)$. To derive this expression we used the independence between $X(t)$ and $\overline{\sigma}(t)$ and the homogeneity property of $X(t)$. Carr et al. (2003) provide many examples for this new class of models.

These results where further extended by Carr and Wu (2004) in order to allow correlation between the time change process $\overline{\sigma}(t)$ and the process $X(t)$. They justify the introduction of correlation between $\overline{\sigma}(t)$ and $X(t)$ as a mean to capture the leverage effect. In order to derive the characteristic function of their process, Carr and Wu introduced a new concept in the option pricing literature: the complex valued measure change.

4.1.3 Lévy processes with stochastic volatility

Recently Madan (2009) pointed out a problem with the original approach of Carr et al. (2003). He notices that the introduction of the stochastic volatility as an independent time change has strong implications for the relationship between volatility and skewness. More precisely he shows that for the stochastic clock Lévy processes there is a negative relation between the variance of the returns and the skewness. However using option prices on 44 stocks, Madan finds that most of the time the observed relation between the variance and the skewness is positive. To fix this problem Madan proposes to introduce stochastic volatility by immediately modifying the scale of the process itself. The return process is given by

$$Z(t) = \int_0^t \sigma(s) dX(s),$$

where $\sigma(s)$ is a stochastic volatility and $X(s)$ a Lévy process. In his paper, Madan studies the case when $\sigma(s)$ is a Cox-Ross-Ingersoll process and $X(s)$ a variance gamma process. The principal drawback of this approach is that the characteristic function of the process $Z(t)$ does not have a closed form expression.

4.1.4 Local Lévy processes

Local Lévy processes were introduced by Carr et al. (2004). They can be seen as a generalization of the local volatility model introduced by Dupire (1993). In its seminal paper Dupire modeled the stock return process as

$$dS(t) = rS(t)dt + \sigma(S(t), t)S(t)dB(t).$$

$\sigma(x, t)$ is called the local volatility, and is a deterministic function of x and t. Dupire showed how, under some suitable conditions, observed put and call prices could be used to recover $\sigma(x, t)$.

Following Dupire's approach, Carr and al. (2004) had the idea to modify the Lévy measure of the process via a local speed function. Starting from a Lévy process whose Lévy measure is given by

$$k(x)dx,$$

Carr and al. define the local Lévy measure as

$$\nu(dx, dt) = a(S(t-), t)k(x)dxdt,$$

where the local speed $a(x, t)$ is a deterministic function of x and t. Using this, they model the stock price with a Doleans-Dade's stochastic exponential namely

$$dS(t) = rS(t-)dt + \sigma(S(t-), t)S(t-)dB(t) + \int_{-\infty}^{+\infty} S(t-)(e^x - 1)(m(dx, dt) - \nu(dx, dt)),$$

where $\sigma(x, t)$ is a local volatility function as defined by Dupire (1993) and where $m(dx, dt)$ is the counting measure associated with the jumps in the logarithm of the stock price. In their paper Carr et al. (2004) show how observed option prices can be used to estimate the local speed function.

4.1.5 Modeling returns with Sato processes

These processes were introduced in the option pricing literature by Carr et al. (2007). We start this section by stating some useful definitions:

- **Definition 2:** Let Z_k, $k = 1, 2, ...$ be a sequence of independent random variables. Denote their sum by $S_n = \sum_{k=1}^{n} Z_k$. A random variable X is said to have the class L property if there exists a sequence of real numbers b_n and c_n such that $b_n S_n + c_n$ converges in distribution to X.

- **Definition 3:** (see Sato (1999)) The distribution of a random variable X is self-decomposable if for every constant $c \in (0, 1)$ we have

$$X =^d cX + X^{(c)},$$

  where $X^{(c)}$ is a random variable independent of X.

- **Definition 4:** A $\mathbb{R}^d$ valued process $\{X(t)\}_{t \geq 0}$ is self-similar (s.s) if for every $c > 0$, $\exists a_c > 0$ such that $\{X(ct)\} =^d \{a_c X(t)\}$. If $a_c = c^H$ then $X(t)$ is said to be self-similar with exponent H (Sato (1999)).

- **Definition 5:** Additive processes are processes satisfying conditions 1, 2, 4 and 5 of Definition 1.

- **Definition 6:** $\{X(t)\}_{t \geq 0}$ is called a Sato process if it is an additive s.s. with exponent H and has independent increments[4].

- **Property:** A probability distribution is in the class L if and only if it is self-decomposable.

From definition 2 we see that every law in the class L can be seen as a limit distribution of a normalized sum of random variables. For example it can be shown that random variables with stable and normal distributions belong to this class. From definition 3 we see that self-decomposable random variables are random variables that can be rewritten as a sum of a scaled down version of themselves and another random variable.

In their paper, Carr et al. (2007) start by justifying the use of a law from the class L to model the distribution of the returns at a given frequency (e.g. the weekly returns). Their argumentation is based on the limit law interpretation of the distributions from the class L: if the returns are generated by adding uncertainty of different order then their distribution should be in the class L. After this, they start discussing the relation between a self-decomposable distribution holding at a fixed time and the path property of the process. Put it differently: if the distribution of the weekly returns is self-decomposable, what are the implications for the properties of their paths?

An answer to this question can be found in the paper of Sato (1991). Sato showed that if a random variable X has a self-decomposable distribution, then for each $H > 0$ one can find a Sato process $\{Y^{(H)}(t)\}_{t \geq 0}$ with exponent H, such that $Y^{(H)}(1)$ has the same law as X. In term of characteristic functions this means

$$E(e^{iuX}) = E(e^{iuY^{(H)}(1)}). \tag{6}$$

The characteristic function of this Sato process at any time t can now be computed by using the self-similarity with the exponent H property

$$E(e^{iuY^{(H)}(t)}) = E(e^{iut^H Y^{(H)}(1)}). \tag{7}$$

For example assume that the weekly returns follow a variance gamma process (proof of the self-decomposability of the variance gamma distribution can be found in the paper of Carr et al. (2004)). Then using (6) we can compute the characteristic function of the corresponding Sato process at $t = 1$

$$E(e^{iuY^{(H)}(1)}) = \left(\frac{1}{1 - iu\theta\nu + \frac{\sigma^2\nu}{2}u^2}\right)^{\frac{1}{\nu}}.$$

Now using (7) we can compute the characteristic function of our process at any time t.

$$E(e^{iuY^{(H)}(t)}) = \left(\frac{1}{1 - iu\theta t^H\nu + \frac{\sigma^2\nu}{2}t^{2H}u^2}\right)^{\frac{1}{\nu}}.$$

Note that from (7) we see that in general

$$E(e^{iuY^{(H)}(t)}) \neq \left[E(e^{iuY^{(H)}(1)})\right]^t,$$

implying non-homogeneous paths. Remark that we only needed to introduce one additional parameter to produce non-homogeneous paths, thus Sato processes have the advantage of being parsimonious. However the increments of Sato processes are independent while those of a stochastic volatility models are allowed to have dependent increments. One consequence of this is that we should not expect Sato processes to capture volatility clustering phenomenas.

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[4]In the original paper of Sato (1991) these processes were called: processes of the class L with exponent H. The name Sato processes was introduced by Carr et al. (2007).

4.2 Multivariate Stochastic Volatility Lévy Processes

In the previous subsection we reviewed the existing literature on non-homogeneous Lévy processes. In this subsection we study multivariate stochastic volatility Lévy processes. We show how the model we saw above can be generalized to a multivariate set up in a natural way.

4.2.1 Stochastic Volatility Through Stochastic Time Change

The first method to creat multivariate stochastic volatility Lévy processes is the one proposed by Carr et al. (2003) and already described in subsection 4.1.2 in an univariate framework. More specificaly let $X(t)$ be a d-dimensional Lévy processes and $\overline{\sigma}(t)$ an independent increasing right continuous process with left limit such that $\overline{\sigma}(t)$ is a stopping time and $\overline{\sigma}(t) \to \infty$ as $t \to \infty$. Then

$$Y(t) = X(\overline{\sigma}(t)),$$

is a d-dimensional stochastic volatility Lévy Processes whose characteristic function was derived in Carr et al. (2003).

4.2.2 Stochastic Volatility for Multivariate Affine Lévy Processes

We can also create multivariate processes with stochastic volatility by combining the approach of sections 3.2 and 4.1.2. Keeping our previous notations we could build the process

$$Y(t) = \mathbf{A}X(\overline{\sigma}(t)),$$

where $X(t)$ is a $n \times 1$ vector of independent Lévy processes, $\mathbf{A}$ a $d \times n$ matrix and

$$\overline{\sigma}(t) = (\overline{\sigma}_1(t), \ldots, \overline{\sigma}_n(t))',$$

where the $\overline{\sigma}_i(t)$ are independent time changes. It is understood that

$$X(\overline{\sigma}(t)) = (X_1(\overline{\sigma}_1(t)), \ldots, X_n(\overline{\sigma}_n(t)))'.$$

First, assume that the processes $X(t)$ and $\overline{\sigma}(t)$ are independent. Such a construction has the advantage that the characteristic function of $Y(t)$ can be easily evaluated.

Theorem 2. *The characteristic function of a multivariate affine Lévy process with stochastic volatility is given by*

$$\varphi_{Y(t)}(u) \quad = \quad \prod_{s=1}^{n} \varpi_s(-i \ln(\varphi_{X_s}(\sum_{j=1}^{d} u_j(\mathbf{A})_{js}))).$$

Proof
As in Carr et al. (2003) note that by the tower property

$$\varphi_{Y(t)}(u) = E\left(e^{iuY(t)}\right) = E\left[E\left(e^{iuY(t)} \mid \mathcal{F}\right)\right],$$

where

$$\mathcal{F} \quad = \quad \{\overline{\sigma}_i(t) \mid i = 1, \ldots, n, \},$$

the result follows from standard computations. $\qquad\square$

One advantage of the multivariate affine Lévy process with stochastic volatility is that the characteristic function of those models can be derived even if the processes X_j and $\overline{\sigma}_j(t)$ are dependent for $j = 1, \ldots, n$. Indeed we have from Carr and Wu (2004):

Theorem 3. *The characteristic function of a multivariate affine Lévy process with stochastic volatility and leverage effect is given by*

$$\varphi_{Y(t)}(u) \;=\; \prod_{s=1}^{n} E(e^{i(\sum_{j=1}^{d} u_j(\mathbf{A})_{js})X_s(\overline{\sigma}_s(t))}) = \prod_{s=1}^{n} E^{u'\mathbf{A}}(e^{-\overline{\sigma}_s(t)\ln(\varphi_{X_s}(\sum_{j=1}^{d} u_j(\mathbf{A})_{js}))}),$$

where $E(\cdot)$ and $E^{u'\mathbf{A}}(\cdot)$ denote the expectation with respect to the measures P and $Q(u'\mathbf{A})$. The measure $Q(u'\mathbf{A})$ is defined by

$$\left.\frac{dQ(u'\mathbf{A})}{dP}\right|_t := exp\left(i(u'\mathbf{A})X(\overline{\sigma}(t)) + \overline{\sigma}(t)'\bar{\Psi}_x(u'\mathbf{A})\right),$$

where

$$\bar{\Psi}_x(u'\mathbf{A}) = (ln(E\left[e^{i\sum_{j=1}^{d} u_j(\mathbf{A})_{j1}X_1(1)}\right]),\ldots, ln(E\left[e^{i\sum_{j=1}^{d} u_j(\mathbf{A})_{jn}X_n(1)}\right]))'.$$

Proof
To prove this Theorem, we follow the same steps as in Theorem 2 and use Theorem 1 from Carr and Wu (2004). □

4.2.3 An Alternative to the Stochastic Time Change

In the preceding subsections we followed Carr et al. (2003) using stochastic time change to introduce heteroskedasticty in our model. We could also have followed Madan (2009) and build a multivariate affine Lévy process using

$$Y(t) = \mathbf{A}X(t),$$

where $X(t)$ is of the form

$$X(t) = (\int_0^t \sigma_1(s)dL_1(s),\ldots, \int_0^t \sigma_n(s)dL_n(s))',$$

where $L_i(s)$ are n independent Lévy processes and $\sigma_i(s)$ are n independent stochastic volatilities. In this set up we allow $L_i(s)$ and $\sigma_i(s)$ to be dependent for $i = 1,\ldots,n$.

Using the results of Madan (2009) we can derive the characteristic function of $Y(t)$:

$$\varphi_{Y(t)}(u) \;=\; \prod_{v=1}^{n} E\left[\exp\left\{\int_0^t \Psi_v((\sum_{j=1}^{d} u_j(\mathbf{A})_{jv})\sigma_v(s))ds\right\}\right],$$

where $\Psi_v(u)$ is the characteristic exponent of $X_v(1)$. The characteristic function has no closed form expression. The expectation with respect to the path of $\sigma_v(s)$ has to be computed numerically. For this reason we choose not to pursue this approach further.

4.2.4 Multivariate Models via Sato processes

The most straightforward approach to build a multivariate Sato process would be to follow the approach of section 4.1.5 but using a multivariate self-decomposable distribution. For example let us assume that the distribution at $t = 1$ of our Sato process is given by the CCVG model of section 3.1.1. The characteristic function of the multivariate Sato process can be evaluated as

$$(1 - iu't^H\nu\theta + \frac{1}{2}t^{2H}\nu u'\Sigma u)^{-\frac{1}{\nu}}.$$

This process will have non-homogeneous and independent increments. However the main drawback of this specification is that the non homogeneity of the sample paths will be similar for each component since each component is affected by the same H.

Another possibility is to combine independent univariate Sato processes via an affine transformation. Let $X^{\tilde{H}}(t)$ be a d-dimensional random vector whose components are independent Sato processes with exponent $H_i > 0, i = 1, \ldots, d$. Define the process $Y(t)$ as

$$Y(t) = \mathbf{A}X^{\tilde{H}}(t), \tag{8}$$

where A is an invertible matrix, we call this process the affine Sato process. Remark that we allowed the different components of the vector $X^{\tilde{H}}(t)$ to have different exponents.

Before going further let us state the following definition from Sato (1991)

Definition 7: a process $\{X(t)\}_{t\geq 0}$ on $\mathbb{R}^d$ is operator self-similar if, for every $c > 0$ there is a matrix $\mathbf{D} \in \mathbb{R}^{d\times d}$ such that

$$\{X(ct)\} =^d \{\mathbf{D}X(t)\}.$$

Strictly speaking an affine Sato process is not a Sato process as defined in section 4.1.5. Indeed we have the following Proposition

Proposition 2. *Let Y(t) be an affine Sato process as defined in (8). Then Y(t) is an operator self-similar process with a self-decomposable distribution.*

Proof
To prove the operator self-similar property simply note that

$$\{Y(ct)\} = \left\{\mathbf{A}X^{\tilde{H}}(ct)\right\} =^d \left\{\mathbf{A}\mathbf{C}^{\tilde{H}}X^{\tilde{H}}(t)\right\} = \left\{\underbrace{\mathbf{A}\mathbf{C}^{\tilde{H}}A^{-1}}_{=M_{\tilde{H}}}AX^{\tilde{H}}(t)\right\} = \left\{\mathbf{M}_{\tilde{H}}\mathbf{A}X^{\tilde{H}}(t)\right\},$$

where $\tilde{H}$ is a diagonal matrix such that $(\mathbf{C}^{\tilde{H}})_{ii} = c^{H_i}$.

To prove the self-decomposability of the distribution of $Y(t)$ use Proposition 1 to express the characteristic function of the process $Y(t)$ in terms of the characteristic function of the marginal components of $X^{\tilde{H}}(t)$

$$\varphi_{Y(t)}(u) = \prod_{s=1}^{d} \varphi_{X_s(t)}(\sum_{j=1}^{d} u_j(\mathbf{A})_{js}). \tag{9}$$

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Now remember that a distribution μ is self-decomposable if and only if for any $a \in (0,1)$ its characteristic function $\widehat{\mu}(z)$ can be rewritten as $\widehat{\mu}(z) = \widehat{\mu}(az)\rho_a(z)$ (Sato (1999)), where $\rho_a(z)$ is the characteristic function of a certain random variable. Now using (9) and the self-decomposability of each component of the vector $X^{\tilde{H}}(t)$ we have

$$
\begin{aligned}
\varphi_{Y(t)}(u) &= \prod_{s=1}^{d} \varphi_{X_s(t)}\Big(a \sum_{j=1}^{d} u_j(\mathbf{A})_{js}\Big)\rho_{as}\Big(\sum_{j=1}^{d} u_j(\mathbf{A})_{js}\Big) \\
&= \left[\prod_{s=1}^{d} \varphi_{X_s(t)}\Big(a \sum_{j=1}^{d} u_j(\mathbf{A})_{js}\Big)\right]\left[\prod_{s=1}^{d} \rho_{as}\Big(\sum_{j=1}^{d} u_j(\mathbf{A})_{js}\Big)\right].
\end{aligned}
$$

$\square$

Proposition 2 teaches us that affine Sato processes fail to be Sato processes because they are not self-similar with exponent $\tilde{H}$ anymore. However we can see that affine Sato processes still have self-decomposable distributions, this implies that we can still justify their use on the basis of the central limit Theorem.

Example 9: Let $X^{\tilde{H}}(t)$ be a $\mathbb{R}^2$ valued stochastic process with independent components. Assume that the characteristic function of its sth component is given by

$$
\varphi_{X^{\tilde{H}_s}(t)}(u) = \left(\frac{1}{1 - iu\theta_s t^{H_s}\nu_s + \frac{\sigma_s^2\nu_s}{2}t^{2H_s}u^2}\right)^{\frac{1}{\nu_s}}.
$$

Then the characteristic function of the affine Sato process $Y(t) = \mathbf{A}X^{\tilde{H}}(t)$, where

$$
\mathbf{A} = \left(\begin{array}{cc} A_{11} & 0 \\ A_{21} & A_{22} \end{array}\right),
$$

is given by

$$
\begin{aligned}
\varphi_{Y(t)}(u) &= \left(\frac{1}{1 - iu_2 A_{22}\theta_2 t^{H_2}\nu_2 + \frac{\sigma_2^2\nu_2}{2}t^{2H_2}u_2^2 A_{22}^2}\right)^{\frac{1}{\nu_2}} \times \\
&\quad \left(\frac{1}{1 - i(u_1 A_{11} + u_2 A_{21})\theta_1 t^{H_1}\nu_1 + \frac{\sigma_1^2\nu_1}{2}t^{2H_1}((u_1 A_{11} + u_2 A_{21})^2)}\right)^{\frac{1}{\nu_1}}.
\end{aligned}
$$

5   Conclusion

In this paper we expose the existing literature on multivariate Lévy processes. We resume the current literature on univariate non-homogeneous Lévy processes and propose some extensions to a multivariate set-up. We study the main consequences of each modeling choice on the properties of the multivariate processes and derive the characteristic function of each model. Further work should focus on the calibration of the multivariate non homogenous Lévy processes.

Appendix

In this appendix we define and derive the bivariate counter-monotonic and group comonotonic Lévy copulas.

We start by recalling some definitions from Kallsen and Tankov (2006). For any $x \in \mathbb{R}^d$ define

$$I(x) := \begin{cases} (x, \infty) & x \geq 0 \\ (\infty, x] & x < 0 \end{cases}$$

Definition 1. *Let X be a $\mathbb{R}^d$ valued Lévy process with Lévy measure ν. The tail integral of X is the function $U{:}(\mathbb{R}\setminus\{0\})^d \to \mathbb{R}$ defined by*

$$U(x_1, \ldots, x_d) := \prod_{i=1}^{d} sign(x_i)\nu(\prod_{i=1}^{d} I(x_i)).$$

A subset $S \in \mathbb{R}^d$ is said to be ordered if, for any $u, v \in S$, either $u_k \geq v_k$ or $v_k \geq u_k$ $k = 1, \ldots, d$. It is said strictly ordered if the inequalities are strict. Define also

$$K := \left\{ x \in \mathbb{R}^d \mid \text{sign}(x_1) = \cdots = \text{sign}(x_d) \right\},$$

where $\text{sign}(x_i)$ is the sign of x_i. We will also need the following definition 4.5 from Kallsen and Tankov (2006):

Definition 2. *Let X be a $\mathbb{R}^d$ valued Lévy process with Lévy measure ν. Its jumps are said to be completely dependent or comonotonic if there exists a strictly ordered subset S of K such that $\nu(\mathbb{R}^d\setminus S) = 0$.*

We will now derive the bivariate counter-monotonic Lévy copula. We say that a subset $\widehat{S} \in \mathbb{R}^2$ is negatively ordered if, for any $u, v \in \widehat{S}$, if $u_i \geq v_i$ implies $v_j \geq u_j$ $i, j = 1, 2$. It is strictly negatively ordered if the inequalities are strict. Set also

$$\widehat{K} = \left\{ x \in \mathbb{R}^2 \mid \text{sign}(x_1) \neq \text{sign}(x_2) \right\}.$$

Finally we define the counter-monotonic Lévy copula as follows

Definition 3. *Let X be a $\mathbb{R}^2$ valued Lévy process with Lévy measure ν. Its jumps are said to be counter-monotonic or completely negatively dependent if there exists a strictly negatively ordered subset S of K such that $\nu(\mathbb{R}^2\setminus S) = 0$.*

The following Theorem characterizes the complete jump counter dependence in terms of Lévy copula.

Theorem 4. [5] *Let Y be a $\mathbb{R}^2$ valued Lévy process whose Lévy measure is supported by a strictly negatively ordered subset $\widehat{S} \subset \widehat{K}$. Then the counter-monotonic Lévy copula given by*

$$F_{cnt}(w_1, w_2) = -\min(|U_{y_1}(w_1)|, |U_{y_2}(w_2)|)1_{\widehat{K}}(U_{y_1}(w_1), U_{y_2}(w_2)).$$

is a Lévy copula of X.

Conversly, if F_{cnt} is a Lévy copula of Y then the Lévy measure of Y is supported by a negatively ordered subset of $\widehat{K}$. In addition if the tail integral U_i of Y^i are continuous and satisfy $\lim_{z \to 0} U_i(z) = \infty$ for $i = 1, 2$ then the jumps of Y are completely counter-monotonic.

---

[5]From here we write $U_{x_i}(w)$ for the marginal tail integral of the ith component of the random vector X.

To proof the Theorem we will need two lemmas.

Lemma 1. *Let X be a $\mathbb{R}$ valued Lévy process with Lévy measure $\nu_x(dx)$, then $Y = -X$ is a $\mathbb{R}$ valued Lévy process with Lévy measure $\nu_y(dx) = \nu_x(-dx)$. Furthermore the tail integrals of X and Y verify*

$$U_y(z) = -U_x(-z).$$

Proof
The first part of the lemma is a direct consequence of Proposition 11.10 from Sato (1999). The second part can be proved as follow

$$
\begin{aligned}
U_y(z) &= \int_z^{+\infty} \nu_y(dx) \quad z > 0 \\
&= \int_{-\infty}^{-z} \nu_x(dx) \\
&= -U_x(-z).
\end{aligned}
$$

The same method can be used for $z < 0$. □

Lemma 2. *Let X be a $\mathbb{R}^2$ valued Lévy process whose jumps are completely dependent or comonotonic. Then the Lévy process Y defined as*

$$Y = \mathbf{A}X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} X,$$

is a counter-monotonic Lévy process.

Proof
The proof follows from the definition. □

We are now ready to proof Theorem 4.

Proof
$\Rightarrow$ Let X be a comonotonic Lévy process. Thus from Theorem 4.6 in Kallsen and Tankov (2006) we know that the tail integral of X is given by

$$
\begin{aligned}
U_X(w_1, w_2) &= \prod_{j=1}^2 \text{sign}(w_j) \nu_X(\prod_{j=1}^2 I(w_j)) &\quad (10) \\
&= \min(|U_{x_1}(w_1)|, |U_{x_2}(w_2)|) 1_K(U_{x_1}(w_1), U_{x_2}(w_2)). &\quad (11)
\end{aligned}
$$

We know from lemma 2 that $Y = \mathbf{A}X$ is a counter-monotonic Lévy process. Furthermore we can use Proposition 11.10 from Sato (1999) to express the tail integrals of Y in terms of the tail integrals of X. By definition the tail integrals of Y are

$$U_Y(w_1, w_2) := \prod_{j=1}^2 \text{sign}(w_j) \nu_Y(\prod_{j=1}^2 I(w_j)).$$

Two cases are possible:

- Suppose that $\text{sign}(w_1) = \text{sign}(w_2)$, in this case since the process is counter-monotonic we have $U_Y(w_1, w_2) = 0$.

- Assume now that $\text{sign}(w_1) \neq \text{sign}(w_2)$. Set for example $w_1 > 0$ and $w_2 < 0$, then using Proposition 11.10 from Sato (1999) we have

$$
\begin{aligned}
\nu_Y([w_1, +\infty) \times (-\infty, w_2]) &= \nu_X([w_1, +\infty) \times [-w_2, +\infty)) \\
&= \min(|U_{x_1}(w_1)|, |U_{x_2}(-w_2)|),
\end{aligned}
$$

and for $w_1 < 0$ and $w_2 > 0$

$$
\begin{aligned}
\nu_Y((-\infty, w_1] \times [w_2, +\infty)) &= \nu_X((-\infty, w_1] \times (-\infty, -w_2]) \\
&= \min(|U_{x_1}(w_1)|, |U_{x_2}(-w_2)|).
\end{aligned}
$$

These remarks imply that the tail integral of Y can be rewritten in terms of the tail integrals of X as

$$
U_Y(w_1, w_2) = -\min(|U_{x_1}(w_1)|, |U_{x_2}(-w_2)|)1_K(U_{x_1}(w_1), U_{x_2}(-w_2)).
$$

Using Lemma 1 and the properties of the indicator function we can express the tail integral in terms of the tail integral of Y itself

$$
U_Y(w_1, w_2) = -\min(|U_{y_1}(w_1)|, |U_{y_2}(w_2)|)1_{\widehat{K}}(U_{y_1}(w_1), U_{y_2}(w_2)).
$$

$\Leftarrow$ Let X be a Lévy process whose Lévy copula is comonotonic. Then we know from Theorem 4.6 in Kallsen and Tankov (2006) that the support of X is an ordered subset $S \subset K$. Then, by lemma 2, $Y = \mathbf{A}X$ is a counter-monotonic Lévy process . Theorem 11.10 from Sato (1999) tells us that the support of $Y = \mathbf{A}X$ is given by $\{x \in \mathbb{R}^2 \mid y = \mathbf{A}x, x \in S\}$. Thus the support of Y is negatively ordered if S is ordered. The strictly negatively ordered property follows from Theorem 4.6 in Kallsen and Tankov. $\qquad\square$

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