PRICING OF ARITHMETIC BASKET AND ASIAN BASKET OPTIONS BY CONDITIONING

G. DEELSTRA[†],[*] *Université Libre de Bruxelles*

J. LIINEV AND M. VANMAELE,[**] *Ghent University*

Abstract

Determining the price of a basket option is not a trivial task, because there is no explicit analytical expression available for the distribution of the weighted sum of the assets in the basket. However, by conditioning the price processes of the underlying assets, this price can be decomposed in two parts, one of which can be computed exactly. For the remaining part we first derive a lower and an upper bound based on comonotonic risks, and another upper bound equal to that lower bound plus an error term. Secondly, we derive an approximation by applying some moment matching method.

Keywords: basket option; comonotonicity; analytical bounds; moment matching; Asian basket option; Black & Scholes model

AMS 2000 Subject Classification: Primary 91B28

Secondary 60E15;60J65

JEL Classification: G13

† This research was carried out while the author was employed at the Ghent University

* Postal address: Department of Mathematics, ISRO and ECARES, Université Libre de Bruxelles, CP 210, 1050 Brussels, Belgium

* Email address: Griselda.Deelstra@ulb.ac.be, Tel: +32 2 650 50 46, Fax: +32 2 650 40 12

** Postal address: Department of Applied Mathematics and Computer Science, Ghent University, Krijgslaan 281, building S9, 9000 Gent, Belgium

** Email address: Jan.Liinev@UGent.be, Michele.Vanmaele@UGent.be, Tel: +32 9 264 4895, Fax: +32 9 264 4995

1. Introduction

One of the more extensively sold exotic options is the basket option, an option whose payoff depends on the value of a portfolio or basket of assets. At maturity it pays off the greater of zero and the difference between the average of the prices of the n different assets in the basket and the exercise price.

The typical underlying of a basket option is a basket consisting of several stocks, that represents a certain economy sector, industry or region.

The main advantage of a basket option is that it is cheaper to use a basket option for portfolio insurance than to use the corresponding portfolio of plain vanilla options. Indeed, a basket option takes the imperfect correlation between the assets in the basket into account and moreover the transaction costs are minimized because an investor has to buy just one option instead of several ones.

For pricing simple options on one underlying the financial world has generally adopted the celebrated Black & Scholes model, which leads to a closed form solution for simple options since the stock price at a fixed time follows a lognormal distribution. However, using the famous Black & Scholes model for a collection of underlying stocks, does not provide us with a closed form solution for the price of a basket option. The difficulty stems primarily from the lack of availability of the distribution of a weighted average of lognormals, a feature that has hampered closed-form basket option pricing characterization. Indeed, the value of a portfolio is the weighted average of the underlying stocks at the exercise date.

One can use Monte Carlo simulation techniques (by assuming that the assets follow correlated geometric Brownian motion processes) to obtain a numerical estimate of the price. Other techniques consist of approximating the real distribution of the payoffs by another more tractable one. For instance, in industry it is common to use the lognormal distribution as an approximation for the sum of lognormals, although it is known that this methodology leads

sometimes to poor results. An extensive discussion of different methods can be found in the theses of Arts (1999), Beißer (2001) and Van Diepen (2002).

Obviously, the payoff structure of a basket option resembles the payoff structure of an Asian option. But whereas the Asian option is a path-dependent option, that is, its payoff at maturity depends on the price process of the underlying asset, the basket option is a path-independent option whose terminal payoff is a function of several asset prices at the maturity date. Nevertheless, in literature, different authors tried out initial methods for Asian options to the case of basket options. In this respect, it seems natural to adapt the methods developed in Vanmaele et al. (2002) for valuing Asian options and indeed, we have transferred them in a promising way to basket options.

Combining both types of options one can consider an Asian option on a basket of assets instead of on one single asset. In this case we talk about an Asian basket option. Dahl and Benth (2001a,b) value such options by quasi-Monte Carlo techniques and singular value decomposition.

But as these approaches are rather time consuming, it would be vital to have accurate, analytical and easily computable bounds or approximations of this price. As the financial institutions dealing with baskets are perhaps even more concerned about the ability of controlling the risks involved, it is important to offer an interval of hedge parameters.

Confronted with such issues, the objective of this paper is to obtain accurate analytical lower and upper bounds as well as approximations. To this end, we use on one hand the method of conditioning as in Curran (1994) and in Rogers and Shi (1995), and on the other hand results on a general technique based on comonotonic risks for deriving upper and lower bounds for stop-loss premiums of sums of dependent random variables (see Kaas, Dhaene and Goovaerts (2000)).

All lower and upper bounds can be expressed as an average of Black & Scholes option prices, sometimes with a synthetic underlying asset. Therefore, hedging parameters can be obtained

in a straightforward way.

A basket option is an option whose payoff depends on the value of a portfolio (or basket) of assets (stocks). Thus, an arithmetic basket call option with exercise date T, n risky assets and exercise price K generates a payoff $(\sum_{i=1}^{n} a_i S_i(T) - K)_+$ at T, that is, if the sum $\mathbb{S} = \sum_{i=1}^{n} a_i S_i(T)$ of asset prices S_i weighted by positive constants a_i at date T is more than K, the payoff equals the difference; if not, the payoff is zero. The price of the basket option at current time $t = 0$ is given by

$$BC(n, K, T) = e^{-rT} E^Q \left[\left(\sum_{i=1}^{n} a_i S_i(T) - K \right)_+ \right] \tag{1}$$

under a martingale measure Q and with r the risk-neutral interest rate.

Assuming a Black & Scholes setting, the random variables $S_i(T)/S_i(0)$ are lognormally distributed under the unique risk-neutral measure Q with parameters $(r - \sigma_i^2/2)T$ and $\sigma_i^2 T$, when σ_i is the volatility of the underlying risky asset S_i. Therefore we do not have an explicit analytical expression for the distribution of the sum $\sum_{i=1}^{n} a_i S_i(T)$ and determining the price of the basket option is not a trivial task. Since the problem of pricing arithmetic basket options turns out to be equivalent to calculating stop-loss premiums of a sum of dependent risks, we can apply the results on comonotonic upper and lower bounds for stop-loss premiums, which have been summarized in Section 2.

The paper is organized as follows. Section 2 recalls from Dhaene et al. (2002) and Kaas et al. (2000) procedures for obtaining the lower and upper bounds for prices by using the notion of comonotonicity. In Section 3 the price of the basket option in the Black & Scholes setting is decomposed in two parts, one of which is computed exactly. For the remaining part we derive bounds in Section 4, first by concentrating on the comonotonicity and then by applying the Rogers and Shi approach to carefully chosen conditioning variables. We discuss different conditioning variables in order to determine some superiority. In Section 5 the remaining part in the decomposition of the basket option price is approximated using a moment matching

method. Section 6 contains some general remarks. In Section 7, several sets of numerical results are given and the different bounds and approximations are discussed. Section 8 discusses the pricing of Asian basket options, which can be done by the same reasoning. Section 9 concludes the paper.

2. Some theoretical results

In this section, we recall from Dhaene et al. (2002) and the references therein the procedures for obtaining the lower and upper bounds for stop-loss premiums of sums $\mathbb{S}$ of dependent random variables by using the notion of comonotonicity. A random vector $(X_1^c, \ldots, X_n^c)$ is *comonotonic* if each two possible outcomes $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ of $(X_1^c, \ldots, X_n^c)$ are ordered componentwise.

In both financial and actuarial context one encounters quite often random variables of the type $\mathbb{S} = \sum_{i=1}^{n} X_i$ where the terms X_i are not mutually independent, but the multivariate distribution function of the random vector $\underline{X} = (X_1, X_2, \ldots, X_n)$ is not completely specified because one only knows the marginal distribution functions of the random variables X_i. In such cases, to be able to make decisions it may be helpful to find the dependence structure for the random vector $(X_1, \ldots, X_n)$ producing the least favourable aggregate claims $\mathbb{S}$ with given marginals. Therefore, given the marginal distributions of the terms in a random variable $\mathbb{S} = \sum_{i=1}^{n} X_i$, we shall look for the joint distribution with a smaller resp. larger sum, in the convex order sense. In short, the sum $\mathbb{S}$ is bounded below and above in convex order ($\preceq_{\text{cx}}$) by sums of comonotonic variables:

$$\mathbb{S}^{\ell} \preceq_{\text{cx}} \mathbb{S} \preceq_{\text{cx}} \mathbb{S}^c,$$

which implies by definition of convex order that

$$E[(\mathbb{S}^{\ell} - d)_+] \leq E[(\mathbb{S} - d)_+] \leq E[(\mathbb{S}^c - d)_+]$$

for all d in $\mathbb{R}^+$, while $E[\mathbb{S}^{\ell}] = E[\mathbb{S}] = E[\mathbb{S}^c]$ and $\text{var}[\mathbb{S}^{\ell}] \leq \text{var}[\mathbb{S}] \leq \text{var}[\mathbb{S}^c]$.

2.1. Comonotonic upper bound

As proven in Dhaene et al. (2002), the convex-largest sum of the components of a random vector with given marginals is obtained by the comonotonic sum $\mathbb{S}^c = X_1^c + X_2^c + \cdots + X_n^c$ with

$$\mathbb{S}^c \stackrel{d}{=} \sum_{i=1}^{n} F_{X_i}^{-1}(U), \tag{2}$$

where the usual inverse of a distribution function, which is the non-decreasing and left-continuous function $F_X^{-1}(p)$, is defined by

$$F_X^{-1}(p) = \inf \{ x \in \mathbb{R} \mid F_X(x) \geq p \}, \qquad p \in [0, 1],$$

with $\inf \emptyset = +\infty$ by convention.

Kaas et al. (2000) have proved that the inverse distribution function of a sum of comonotonic random variables is simply the sum of the inverse distribution functions of the marginal distributions. Moreover, in case of strictly increasing and continuous marginals, the cumulative distribution function (cdf) $F_{\mathbb{S}^c}(x)$ is uniquely determined by

$$F_{\mathbb{S}^c}^{-1} \left(F_{\mathbb{S}^c}(x) \right) = \sum_{i=1}^{n} F_{X_i}^{-1} \left(F_{\mathbb{S}^c}(x) \right) = x, \qquad F_{\mathbb{S}^c}^{-1}(0) < x < F_{\mathbb{S}^c}^{-1}(1). \tag{3}$$

Hereafter we restrict ourselves to this case of strictly increasing and continuous marginals.

In the following theorem Dhaene et al. (2002) have proved that the stop-loss premiums of a sum of comonotonic random variables can easily be obtained from the stop-loss premiums of the terms.

Theorem 1. *The stop-loss premiums of the sum $\mathbb{S}^c$ of the components of the comonotonic random vector $(X_1^c, X_2^c, \ldots, X_n^c)$ are given by*

$$E \left[(\mathbb{S}^c - d)_+ \right] = \sum_{i=1}^{n} E \left[\left(X_i - F_{X_i}^{-1} \left(F_{\mathbb{S}^c}(d) \right) \right)_+ \right], \qquad (F_{\mathbb{S}^c}^{-1}(0) < d < F_{\mathbb{S}^c}^{-1}(1)). \tag{4}$$

If the only information available concerning the multivariate distribution function of the random vector $(X_1, \ldots, X_n)$ are the marginal distribution functions of the X_i, then the distribution

function of $\mathbb{S}^c = F_{X_1}^{-1}(U) + F_{X_2}^{-1}(U) + \cdots + F_{X_n}^{-1}(U)$ is a prudent choice for approximating the unknown distribution function of $\mathbb{S} = X_1 + \cdots + X_n$. It is a supremum in terms of convex order. It is the best upper bound that can be derived under the given conditions.

2.2. Lower bound

Let $\underline{X} = (X_1, \ldots, X_n)$ be a random vector with given marginal cdfs $F_{X_1}, F_{X_2}, \ldots, F_{X_n}$. Let us now assume that we have some additional information available concerning the stochastic nature of $(X_1, \ldots, X_n)$. More precisely, we assume that there exists some random variable Λ with a given distribution function, such that we know the conditional distribution, given $\Lambda = \lambda$, of the random variables X_i, for all possible values of λ. We recall from Kaas et al. (2000) that a lower bound, in the sense of convex order, for $\mathbb{S} = X_1 + X_2 + \cdots + X_n$ is

$$\mathbb{S}^\ell = E\left[\mathbb{S} \mid \Lambda\right]. \tag{5}$$

This idea can also be found in Rogers and Shi (1995) for the continuous case.

Let us further assume that the random variable Λ is such that all $E\left[X_i \mid \Lambda\right]$ are non-decreasing and continuous functions of Λ and in addition assume that the cdfs of the random variables $E\left[X_i \mid \Lambda\right]$ are strictly increasing and continuous, then the cdf of $\mathbb{S}^\ell$ is also strictly increasing and continuous, and we get for all $x \in \left(F_{\mathbb{S}^\ell}^{-1}(0), F_{\mathbb{S}^\ell}^{-1}(1)\right)$,

$$\sum_{i=1}^{n} F_{E[X_i|\Lambda]}^{-1}\left(F_{\mathbb{S}^\ell}(x)\right) = x \quad \Leftrightarrow \quad \sum_{i=1}^{n} E\left[X_i \mid \Lambda = F_{\Lambda}^{-1}\left(F_{\mathbb{S}^\ell}(x)\right)\right] = x, \tag{6}$$

which unambiguously determines the cdf of the convex order lower bound $\mathbb{S}^\ell$ for $\mathbb{S}$. Using Theorem 1, the stop-loss premiums of $\mathbb{S}^\ell$ can be computed as:

$$E\left[\left(\mathbb{S}^\ell - d\right)_+\right] = \sum_{i=1}^{n} E\left[\left(E\left[X_i \mid \Lambda\right] - E\left[X_i \mid \Lambda = F_{\Lambda}^{-1}\left(F_{\mathbb{S}^\ell}(d)\right)\right]\right)_+\right], \tag{7}$$

which holds for all retentions $d \in \left(F_{\mathbb{S}^\ell}^{-1}(0), F_{\mathbb{S}^\ell}^{-1}(1)\right)$.

So far, we considered the case that all $E\left[X_i \mid \Lambda\right]$ are non-decreasing functions of Λ. The case

where all $E\left[X_i \mid \Lambda\right]$ are non-increasing and continuous functions of Λ also leads to a comonotonic vector $(E\left[X_1 \mid \Lambda\right], E\left[X_2 \mid \Lambda\right], \ldots, E\left[X_n \mid \Lambda\right])$, and can be treated in a similar way.

3. Basket options in a Black & Scholes setting: An exact part

We now shall concentrate on the pricing of basket options in the famous Black & Scholes model by noticing that there is a part which can be calculated in an exact way. The remaining part will be treated in subsequent paragraphs by lower and upper bounds and by approximations. Numerical computations show that the exact part contributes more than 90% to the price of basket options.

Denoting by $S_i(t)$ the price of the i-th asset in the basket at time t, the basket is given by

$$\mathbb{S}(t) = \sum_{i=1}^{n} a_i S_i(t),$$

where a_i are deterministic, positive and constant weights specified by the option contract. We assume that under the risk neutral measure

$$dS_i(t) = rS_i dt + \sigma_i S_i dW_i(t),$$

where $\{W_i(t), t \geq 0\}$ is a standard Brownian motion associated with the price process of asset i. Further, we assume the different asset prices to be instantaneously correlated according to

$$\text{corr}(dW_i, dW_j) = \rho_{ij} dt. \tag{8}$$

Given the above dynamics, the i-th asset price at time t equals

$$S_i(t) = S_i(0)e^{(r-\frac{1}{2}\sigma_i^2)t+\sigma_i W_i(t)}.$$

We can rewrite the basket as a sum of lognormal variables

$$\mathbb{S}(t) = \sum_{i=1}^{n} X_i(t) = \sum_{i=1}^{n} \alpha_i(t)e^{Y_i(t)}, \tag{9}$$

where $\alpha_i(t) = a_i S_i(0) e^{(r-\frac{1}{2}\sigma_i^2)t} (\geq 0)$ and $Y_i(t) = \sigma_i W_i(t) \sim N(0, \sigma_i^2 t)$ and thus $X_i(t)$ is lognormally distributed: $X_i(t) \sim LN(\ln(a_i S_i(0)) + (r - \frac{1}{2}\sigma_i^2)t, \sigma_i^2 t)$.

In this case the stop-loss premium with some retention d_i, namely $E^Q\left[(X_i - d_i)_+\right]$, is well-known since $\ln(X_i(t)) \sim N\left(\mu_{i(t)}, \sigma_{Y_i(t)}^2\right)$ with $\mu_i(t) = \ln(\alpha_i(t))$ and $\sigma_{Y_i(t)} = \sigma_i \sqrt{t}$, and equals for $d_i > 0$

$$E^Q[(X_i(t) - d_i)_+] = e^{\mu_i(t) + \frac{\sigma_{Y_i(t)}^2}{2}} \Phi(d_{i,1}(t)) - d_i \Phi(d_{i,2}(t)), \tag{10}$$

where $d_{i,1}$ and $d_{i,2}$ are determined by

$$d_{i,1}(t) = \frac{\mu_i(t) + \sigma_{Y_i(t)}^2 - \ln(d_i)}{\sigma_{Y_i(t)}}, \quad d_{i,2}(t) = d_{i,1}(t) - \sigma_{Y_i(t)}, \tag{11}$$

and where Φ is the cdf of the $N(0,1)$ distribution.

The case $d_i < 0$ is trivial.

In what follows we only consider the basket at maturity date T and for the sake of notational simplicity, we shall drop the explicit dependence on T in X_i, α_i and Y_i.

For any normally distributed random variable Λ, with cdf $F_\Lambda(\cdot)$, for which there exists a $d_\Lambda \in \mathbb{R}$, such that $\Lambda \geq d_\Lambda$ implies $\mathbb{S} \geq K$, it follows that

$$E^Q[(\mathbb{S} - K)_+ \mid \Lambda] = E^Q[\mathbb{S} - K \mid \Lambda] \overset{(5)}{=} \left(\mathbb{S}^\ell - K\right)_+. \tag{12}$$

For such Λ, we can decompose the option price (1) in the following way

$$e^{-rT} E^Q[(\mathbb{S} - K)_+] = e^{-rT} E^Q[E^Q[(\mathbb{S} - K)_+ \mid \Lambda]] \tag{13}$$
$$= e^{-rT} \left\{ \int_{-\infty}^{d_\Lambda} E^Q[(\mathbb{S} - K)_+ \mid \Lambda = \lambda] dF_\Lambda(\lambda) + \int_{d_\Lambda}^{+\infty} E^Q[\mathbb{S} - K \mid \Lambda = \lambda] dF_\Lambda(\lambda) \right\}.$$

We shall motivate the choice of Λ and, correspondingly, of d_Λ later.

The second term in (13) can be written out explicitly if for all i, (Y_i, Λ) is bivariate normally distributed. Then, $Y_i \mid \Lambda = \lambda$ is also normally distributed for all i with parameters $\mu(i) = r_i \frac{\sigma_{Y_i}}{\sigma_\Lambda}\left(\lambda - E^Q[\Lambda]\right)$ and $\sigma^2(i) = \left(1 - r_i^2\right)\sigma_{Y_i}^2$, where $r_i = r(Y_i, \Lambda) = \frac{\text{cov}(Y_i, \Lambda)}{\sigma_{Y_i}\sigma_\Lambda}$ is a correlation

between Y_i and Λ, and therefore we easily arrive at

$$E^Q[\mathbb{S} \mid \Lambda] = \sum_{i=1}^{n} a_i E^Q\left[S_i(T) \mid \Lambda\right] \stackrel{d}{=} \sum_{i=1}^{n} a_i S_i(0) e^{(r-\frac{\sigma_i^2}{2}r_i^2)T + \sigma_i r_i \sqrt{T}\Phi^{-1}(V)}, \tag{14}$$

where the random variable $V = \Phi\left(\frac{\Lambda - E^Q[\Lambda]}{\sigma_\Lambda}\right)$ is uniformly distributed on the unit interval and thus, $\Phi^{-1}(V) = \frac{\Lambda - E^Q[\Lambda]}{\sigma_\Lambda}$ is a standard normal variate. Next we apply the equality

$$\int_{-\infty}^{d_\Lambda} e^{b\Phi^{-1}(v)} f_\Lambda(\lambda) d\lambda = e^{\frac{b^2}{2}} \Phi\left(d_\Lambda^* - b\right), \quad d_\Lambda^* = \frac{d_\Lambda - E^Q[\Lambda]}{\sigma_\Lambda}, \tag{15}$$

where $f_\Lambda(\cdot)$ is the normal density function for Λ, and with $b = \sigma_i r_i \sqrt{T}$, we can express the second term in (13) in closed-form:

$$
\begin{aligned}
&e^{-rT} \int_{d_\Lambda}^{+\infty} E^Q[\mathbb{S} - K \mid \Lambda = \lambda] dF_\Lambda(\lambda) \\
&= e^{-rT} \int_{d_\Lambda}^{+\infty} E^Q[\mathbb{S} \mid \Lambda = \lambda] f_\Lambda(\lambda) d\lambda - e^{-rT} K(1 - F_\Lambda(d_\Lambda)) \\
&= e^{-rT} \sum_{i=1}^{n} a_i S_i(0) e^{(r-\frac{1}{2}\sigma_i^2 r_i^2)T} \int_{d_\Lambda}^{+\infty} e^{r_i \sigma_i \sqrt{T}\,\Phi^{-1}(v)} f_\Lambda(\lambda) d\lambda - e^{-rT} K(1 - \Phi(d_\Lambda^*)) \\
&= \sum_{i=1}^{n} a_i S_i(0) \Phi(r_i \sigma_i \sqrt{T} - d_\Lambda^*) - e^{-rT} K \Phi(-d_\Lambda^*). \tag{16}
\end{aligned}
$$

Next we discuss the choice of the conditioning variable Λ which should not only be normally distributed but also be chosen such that (Y_i, Λ) for all i are bivariate normally distributed. Hence, we define Λ by

$$\Lambda = \sum_{i=1}^{n} \beta_i \sigma_i W_i(T) \tag{17}$$

with β_i some real numbers. Since the conditioning variable only enters the exact part (16) via its correlation with the Y_i's, we report these correlations explicitly:

$$r_i = \frac{\operatorname{cov}\left(\sigma_i W_i(T), \Lambda\right)}{\sqrt{T}\sigma_i \sigma_\Lambda} = \frac{\sum_{j=1}^{n} \beta_j \rho_{ij} \sigma_j}{\sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} \beta_i \beta_j \rho_{ij} \sigma_i \sigma_j}}. \tag{18}$$

In this paper we consider the following types of conditioning variable Λ, motivated by the idea that one should put as much available information of $\mathbb{S}$ as possible in the conditioning variable.

Further on, we shall compare these choices to an optimal choice obtained by an optimization procedure.

- As a first conditioning variable we take a linear transformation of a first order approximation of $\mathbb{S}$ (denoted by *FA1*):

$$FA1 = \sum_{i=1}^{n} e^{(r-\frac{\sigma_i^2}{2})T} a_i S_i(0)\sigma_i W_i(T), \qquad (19)$$

and the correlation coefficients then read

$$r_i = \frac{\sum_{j=1}^{n} a_j S_j(0)e^{(r-\frac{\sigma_j^2}{2})T}\rho_{ij}\sigma_j}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} a_i S_i(0)e^{(r-\frac{\sigma_i^2}{2})T}a_j S_j(0)e^{(r-\frac{\sigma_j^2}{2})T}\rho_{ij}\sigma_i\sigma_j}}. \qquad (20)$$

- As a second conditioning variable (denoted by *FA2*), we consider

$$FA2 = \sum_{i=1}^{n} a_i S_i(0)\sigma_i W_i(T). \qquad (21)$$

In this case, the correlation between Y_i and Λ is easily found to be

$$r_i = \frac{\sum_{j=1}^{n} a_j S_j(0)\rho_{ij}\sigma_j}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} a_i S_i(0)a_j S_j(0)\rho_{ij}\sigma_i\sigma_j}}. \qquad (22)$$

Note that *FA2* is also a first order approximation of $\mathbb{S}$ and in fact of *FA1*.

- As a third conditioning variable (denoted by *GA*), we look at the standardized logarithm of the geometric average $\mathbb{G}$ which is defined by

$$\mathbb{G} = \prod_{i=1}^{n} S_i(T)^{a_i} = \prod_{i=1}^{n}\left(S_i(0)e^{(r-\frac{\sigma_i^2}{2})T+\sigma_i W_i(T)}\right)^{a_i}. \qquad (23)$$

Indeed, we can consider

$$GA = \frac{\ln\mathbb{G} - E^Q[\ln\mathbb{G}]}{\sqrt{\text{var}[\ln\mathbb{G}]}} = \frac{\sum_{i=1}^{n} a_i\sigma_i W_i(T)}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j\sigma_i\sigma_j\rho_{ij}T}}, \qquad (24)$$

since

$$E^Q[\ln\mathbb{G}] = \sum_{i=1}^{n} a_i\left(\ln(S_i(0)) + (r-\frac{\sigma_i^2}{2})T\right) \qquad (25)$$

$$\text{var}[\ln\mathbb{G}] = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j\sigma_i\sigma_j\rho_{ij}T. \qquad (26)$$

The correlation coefficients in this case are given by

$$r_i = \frac{\sum_{j=1}^n a_j \sigma_j \rho_{ij}}{\sqrt{\sum_{i=1}^n \sum_{j=1}^n a_i a_j \rho_{ij} \sigma_i \sigma_j}}. \tag{27}$$

We can determine the integration bound d_Λ for each of the three different Λ's (19), (21) and (23), such that $\Lambda \geq d_\Lambda$ implies that $\mathbb{S} \geq K$. Bounding the exponential function e^x below by its first order approximation $1 + x$ with $x = \sigma_i W_i(T)$, respectively $x = \left(r - \frac{\sigma_i^2}{2}\right) T + \sigma_i W_i(T)$, the integration bound corresponding to $\Lambda = FA1$ given by (19), respectively to $\Lambda = FA2$ given by (21), is found to be

$$d_{FA1} = K - \sum_{i=1}^n a_i S_i(0) e^{(r - \frac{\sigma_i^2}{2})T}, \tag{28}$$

$$\text{respectively,} \quad d_{FA2} = K - \sum_{i=1}^n a_i S_i(0)(1 + (r - \frac{\sigma_i^2}{2})T). \tag{29}$$

When Λ is the standardized logarithm of the geometric average (GA), see (24), we use the relationship $\mathbb{S} \geq \mathbb{G} \geq K$ and (25)–(26) in order to arrive at

$$d_{GA} = \frac{\ln(K) - \sum_{i=1}^n a_i \left(\ln(S_i(0)) + (r - \frac{\sigma_i^2}{2})T \right)}{\sqrt{\sum_{i=1}^n \sum_{j=1}^n a_i a_j \sigma_i \sigma_j \rho_{ij} T}}. \tag{30}$$

4. Bounds

In Section 3 we decomposed the basket option price into the exact part (16) and the remaining part. Deriving bounds for the latter according to Section 2, and adding up to the exact part we obtain the bounds for the basket option price (1).

4.1. Lower bound.

By means of Jensen's inequality, the first term in (13) can be bounded below as follows:

$$\int_{-\infty}^{d_\Lambda} E^Q[(\mathbb{S} - K)_+ \mid \Lambda = \lambda] dF_\Lambda(\lambda) \geq \int_{-\infty}^{d_\Lambda} (E^Q[\mathbb{S} \mid \Lambda = \lambda] - K)_+ dF_\Lambda(\lambda).$$

By adding the exact part (16) and introducing notation (5), we end up with the inequality

$$E^Q[(\mathbb{S} - K)_+] \geq E^Q[(\mathbb{S}^\ell - K)_+], \tag{31}$$

where we recall that $\mathbb{S}^\ell$ is in convex order a lower bound of $\mathbb{S}$. When $\mathbb{S}^\ell$ is a sum of n comonotonic risks we can apply the results of Section 2.2. Hereto the correlations r_i (22) should all have the same sign. For the time being we assume that the correlations are positive for all i, we shall come back to this issue later on. Invoking (7) and (10)-(11), and substituting α_i and the standard deviation of Y_i, we find the following lower bound for the price of the basket call option:

$$BC(n, K, T) \geq \sum_{i=1}^{n} a_i S_i(0) \; \Phi\left[\sigma_i \sqrt{T} r_i - \Phi^{-1}\left(F_{\mathbb{S}^\ell}(K)\right)\right] - e^{-rT} \; K\left(1 - F_{\mathbb{S}^\ell}(K)\right) \tag{32}$$

which holds for any $K > 0$ and where $F_{\mathbb{S}^\ell}(K)$, according to (6), solves

$$\sum_{i=1}^{n} a_i S_i(0) e^{(r - \frac{1}{2} r_i^2 \sigma_i^2)T + r_i \sigma_i \sqrt{T} \Phi^{-1}(F_{\mathbb{S}^\ell}(K))} = K. \tag{33}$$

The lower bound (32)-(33) can be formulated as an average of Black & Scholes formulae with new underlying assets and new exercise prices. The new assets $\tilde{S}_i$ are with $\tilde{S}_i(0) = S_i(0)$ and with new volatilities $\tilde{\sigma}_i = \sigma_i r_i$ for $i = 1, \ldots, n$. The new exercise prices $\tilde{K}_i$, $i = 1, \ldots, n$, are given by

$$\tilde{K}_i = \tilde{S}_i(0) e^{(r - \frac{\tilde{\sigma}_i^2}{2})T + \tilde{\sigma}_i \sqrt{T} \Phi^{-1}(F_{\mathbb{S}^\ell}(K))}. \tag{34}$$

Indeed,

$$BC(n, K, T) \geq \sum_{i=1}^{n} a_i \left[\tilde{S}_i(0)\Phi\left(d_{1i}\right) - e^{-rT}\tilde{K}_i \, \Phi\left(d_{2i}\right)\right] \tag{35}$$

with

$$d_{1i} = \frac{\ln\left(\frac{\tilde{S}_i(0)}{\tilde{K}_i}\right) + (r + \frac{\tilde{\sigma}_i^2}{2})T}{\tilde{\sigma}_i \sqrt{T}} \quad \text{and} \quad d_{2i} = d_{1i} - \tilde{\sigma}_i \sqrt{T}, \quad \text{for } i = 1, \ldots, n.$$

We stress that (7) is an equality and thus the decomposition in a sum of n terms is an optimal one. Hence also in this sense, given a conditioning variable Λ, (32) or (35) is the best lower

bound that can be obtained if one wants the lower bound in the form of a linear combination of Vanilla European call options. However, the conditioning variable is still a free parameter and its choice can be optimized. Hereby note that the integration bound d_Λ is just a technical tool for making the link between the exact part and the lower bound, but does not appear in the final expression (32) or (35) of this lower bound because this is based on the inequality (31). Further remark that the lower bound only depends on the conditioning variable Λ through the correlations r_i. In case r_i equals one we obtain the exact price. In practice we did not find up to now a conditioning variable Λ such that $r_i = 1$ for all i. But we do have that for the conditioning variables (19), (21) and (24) the lower bound is quite good. Beißer (2001), who has obtained the same lower bound by using other arguments, chooses along intuitive arguments the numerator of the standardized logarithm of the geometric average (24). This is indeed a good choice since the geometric average and arithmetic average are based on the same information. In this case, the correlation coefficients in the formulae for the lower bound are given by (27). Note however that these correlation coefficients are independent of the initial value of the assets in the basket which can lead to a lower quality of the lower bound when the assets in the basket have different initial values. Beißer (2001) therefore considers also the conditioning variable Λ given by (21) with correlations r_i (22), which depend on the weights, the initial values and the volatilities of the assets in the basket.

It is easily seen that the lower bound will coincide for the three different choices of Λ when the initial values as well as the volatilities are equal for the different assets. When only the volatilities σ_i are equal for all i then the correlation coefficients (20) and (22) coincide and hence also the corresponding lower bounds. Similarly, when only the initial values are equal the correlation coefficients (22) and (27) lead to the same lower bound.

Next we go deeper into the assumption of positiveness for the correlation coefficients r_i (18). This condition is needed for $\mathbb{S}^\ell$ (14) to be a comonotonic sum and thus for (7) to hold and to be applied.

When the correlations ρ_{ij} (8) are positive for all i and j then it suffices to take all coefficients β_i also with a positive sign in order to satisfy the assumption. However when a ρ_{ij} is negative a general discussion is much more involved. Therefore, we first look at the special case when $n = 2$ and $\rho_{12} = \rho_{21} \overset{not}{=} \rho \leq 0$. The conditions $r_1, r_2 \geq 0$ are equivalent to

$$
\begin{cases}
\beta_1 \sigma_1 - \beta_2 \sigma_2 |\rho| \geq 0 \\
\beta_2 \sigma_2 - \beta_1 \sigma_1 |\rho| \geq 0
\end{cases}
\quad \Leftrightarrow \quad \beta_2 \sigma_2 |\rho| \leq \beta_1 \sigma_1 \leq \beta_2 \sigma_2 \frac{1}{|\rho|},
\tag{36}
$$

and imply that β_1 and β_2 should have the same sign and differ from zero. For simplicity assume that β_1 and β_2 are both strictly positive, then the condition (36) can be rewritten as

$$
|\rho| \leq \frac{\beta_1 \sigma_1}{\beta_2 \sigma_2} \leq \frac{1}{|\rho|}.
\tag{37}
$$

Note that since $|\rho| \leq 1$, the second inequality is trivially fulfilled when $\beta_1 \sigma_1 \leq \beta_2 \sigma_2$ while in the case $\beta_1 \sigma_1 \geq \beta_2 \sigma_2$ the first inequality is trivial. Hence only one of these inequalities has to be checked. Beißer (2001) made a similar reasoning but only for the particular correlation coefficients (22).

When ρ is negative, it can happen that for none of the three choices for Λ, namely (19), (21) and (24), relation (37) is satisfied. However, since we derived a lower bound for any Λ given by (17) we are not restricted to the three choices. Indeed, it is always possible to find a β_1 and β_2 since the interval $[|\rho| \frac{\sigma_2}{\sigma_1}, \frac{1}{|\rho|} \frac{\sigma_2}{\sigma_1}]$ is non-empty.

In fact, one might search for the β_1 and β_2 which leads to an optimal lower bound. When we write r_i, $i = 1, 2$ in function of $x = \frac{\beta_2 \sigma_2}{\beta_1 \sigma_1}$, we find that $r_1 = \frac{1+x\rho}{\sqrt{1+x^2+2x\rho}}$ and $r_2 = \frac{\rho+x}{\sqrt{1+x^2+2x\rho}}$. If one assumes that $r_2 \neq 1$ and if we rewrite the equation defining r_2, we find the relation $r_1 - r_2 \rho = \sqrt{(1-\rho^2)(1-r_2^2)}$. As a consequence, the optimal lower bound becomes the solution

to the optimization program

$$\max_{r_1, r_2} LB(r_1, r_2) = \sum_{i=1}^{2} a_i [\tilde{S}_i(0)\Phi(d_{1i}) - e^{-rT}\tilde{K}_i\Phi(d_{1i})] \tag{38}$$

such that $0 \leq r_1 \leq 1, 0 \leq r_2 < 1$

$$\sum_{i=1}^{2} a_i \tilde{K}_i = K$$

$$r_1 - r_2\rho = \sqrt{1 - \rho^2}\sqrt{1 - r_2^2},$$

where we used the notation introduced in (34)-(35). Solving this problem by Lagrange optimization leads to a conclusion of three cases:

1. If $r_2 = 0$ and $r_1 = \sqrt{1 - \rho^2}$, which is only possible if $\rho < 0$, the lower bound is maximized under the above conditions if

$$a_2 \tilde{K}_2 \sigma_2 + \rho\sigma_1(K - a_2\tilde{K}_2) \leq 0 \quad \text{with} \quad \tilde{K}_2 = S_2(0)e^{rT}.$$

2. If r_1 and r_2 are strictly between 0 and 1, r_1 and r_2 are solutions to the equations:

$$\begin{cases} e^{-rT}\varphi(\Phi^{-1}(F_{\mathbb{S}^\ell}(K)))a_1\tilde{K}_1\sigma_1\sqrt{T} - \lambda_2 = 0 \\ e^{-rT}\varphi(\Phi^{-1}(F_{\mathbb{S}^\ell}(K)))a_2\tilde{K}_2\sigma_2\sqrt{T} - \lambda_2\left(-\rho + r_2\sqrt{\frac{1-\rho^2}{1-r_2^2}}\right) = 0 \\ a_1\tilde{K}_1 + a_2\tilde{K}_2 = K \\ r_1 - r_2\rho = \sqrt{1 - \rho^2}\sqrt{1 - r_2^2}, \end{cases}$$

which are four non-linear equations in four unknowns $r_1, r_2, \lambda_2, \Phi^{-1}(F_{\mathbb{S}^\ell}(K))$ and where φ is the density function of the $N(0,1)$ distribution.

3. If $r_1 = 0$ and $r_2 = \sqrt{1 - \rho^2}$, which is only possible if $\rho < 0$, the lower bound is maximized under the above conditions if

$$a_1 \tilde{K}_1 \sigma_1 + \rho\sigma_2(K - a_1\tilde{K}_1) \leq 0 \quad \text{with} \quad \tilde{K}_1 = S_1(0)e^{rT}.$$

We now turn to the general case for $n \geq 3$ where at least one correlation ρ_{ij}, defined in (8), is supposed to be strictly negative. As a conclusion of the following statement we see that a lower bound can be computed also in a general case. However, the optimization program will be much more involved when there are more than two assets in the basket.

Theorem 2. *There always exist coefficients $\beta_i \in \mathbb{R}$, $i = 1, \ldots, n$, in (17) such that all correlations r_i, $i = 1, \ldots, n$, (18) are positive.*

Proof. Denoting $\mathbf{A}$ for the correlation matrix $(\rho_{ij})_{1 \leq i,j \leq n}$ and putting $\tilde{\boldsymbol{\beta}}^T = (\beta_1 \sigma_1, \ldots, \beta_n \sigma_n)$, the conditions $r_i \geq 0$, $i = 1, \ldots, n$ are equivalent to $\mathbf{A}\tilde{\boldsymbol{\beta}} \geq \mathbf{0}$.

As all variance-covariance matrices, this matrix $\mathbf{A}$ is symmetric and positive semi-definite. Moreover it is non-singular and positive definite since we assume that the market is complete. By a reasoning ex absurdo we show that at least one of the coefficients β_i is strictly positive: Assume that all β_i are negative then from $\mathbf{A}\tilde{\boldsymbol{\beta}} \geq \mathbf{0}$ it follows that $\tilde{\boldsymbol{\beta}}^T \mathbf{A}\tilde{\boldsymbol{\beta}} \leq \mathbf{0}$ which is a contradiction to the positive definiteness of $\mathbf{A}$.

Finally, using the link between the primal and the dual of a linear programming problem, and again by the positiveness of matrix $\mathbf{A}$ the assertion can then be proved. $\qquad\square$

For a more detailed discussion we refer to Section 7.

4.2. Upper bounds.

4.2.1. *Partially exact/comonotonic upper bound.* Recalling that $Y_i \mid \Lambda = \lambda$ is normally distributed for all i with parameters $\mu(i)$ and $\sigma^2(i)$, see Section 3, we bound the first term of (13) above by replacing $\mathbb{S} \mid \Lambda = \lambda$ by its comonotonic upper bound $\mathbb{S}^c_{\Lambda=\lambda}$ (in convex order sense), given by

$$\mathbb{S}^c_{\Lambda=\lambda} = (\mathbb{S} \mid \Lambda = \lambda)^c = \sum_{i=1}^{n} \alpha_i e^{r_i \sigma_{Y_i} \Phi^{-1}(v) + \sqrt{1-r_i^2} \sigma_{Y_i} \Phi^{-1}(U)}$$

where U and $V = \Phi\left(\frac{\Lambda - E^Q[\Lambda]}{\sigma_\Lambda}\right)$ are mutually independent uniform(0,1) random variables. Combining (3)-(4) with (10)-(11) and substituting α_i and the standard deviation of Y_i, the

comonotonic upper bound for the first term of (13) reads:

$$
\begin{aligned}
& e^{-rT} \int_{-\infty}^{d_\Lambda} E^Q[(\mathbb{S} - K)_+ \mid \Lambda = \lambda] f_\Lambda(\lambda) d\lambda \\
& \leq e^{-rT} \int_{-\infty}^{d_\Lambda} E^Q[(\mathbb{S}^c_{\Lambda=\lambda} - K)_+] f_\Lambda(\lambda) d\lambda \\
& = e^{-rT} \int_0^{\Phi(d_\Lambda^*)} E^Q[(\mathbb{S}^c_{\Lambda=\lambda} - K)_+] \, dv \\
& = \sum_{i=1}^n a_i S_i(0) e^{-\frac{1}{2}\sigma_i^2 r_i^2 T} \int_0^{\Phi(d_\Lambda^*)} e^{r_i \sigma_i \sqrt{T} \Phi^{-1}(v)} \Phi\left(\sqrt{1 - r_i^2}\, \sigma_i \sqrt{T} - \Phi^{-1}\left(F_{\mathbb{S}^c_{\Lambda=\lambda}}(K)\right)\right) dv \\
& \quad - e^{-rT} K \left(\Phi(d_\Lambda^*) - \int_0^{\Phi(d_\Lambda^*)} F_{\mathbb{S}^c_{\Lambda=\lambda}}(K) \, dv\right),
\end{aligned}
\tag{39}
$$

where we recall that d_Λ^* is defined as in (15), and the cumulative distribution $F_{\mathbb{S}^c_{\Lambda=\lambda}}(K)$ is, according to (3), determined by

$$
\sum_{i=1}^n a_i S_i(0) \exp\left[(r - \frac{\sigma_i^2}{2})T + r_i \sigma_i \sqrt{T} \Phi^{-1}(v) + \sqrt{1 - r_i^2} \sigma_i \sqrt{T} \Phi^{-1}\left(F_{\mathbb{S}^c_{\Lambda=\lambda}}(K)\right)\right] = K.
$$

Adding (39) to the exact part (16) of the decomposition (13) results in the so-called partially exact/comonotonic upper bound for $BC(n, K, T)$.

For the random variables Λ given by (19), (21) and (24) we derived a d_Λ, see (28), (29) and (30), and thus we can compute this upper bound.

Note that we can rewrite the comonotonic upper bound, applied to the first term in (13), as a linear combination of Black & Scholes prices for European call options. For this purpose, given $\Lambda = \lambda$ or equivalently given $V = v$, we introduce some artificial underlying assets $\tilde{S}_{i,v}$ having volatilities $\tilde{\sigma}_{i,v} = \sigma_i \sqrt{1 - r_i^2}$ and with initial value

$$
\tilde{S}_{i,v}(0) = S_i(0) e^{-\frac{1}{2}\sigma_i^2 r_i^2 T + r_i \sigma_i \sqrt{T} \Phi^{-1}(v)}.
$$

We also consider new exercise prices:

$$
\tilde{K}_{i,v} = S_i(0) e^{(r - \frac{\sigma_i^2}{2})T + r_i \sigma_i \sqrt{T} \Phi^{-1}(v) + \sqrt{1 - r_i^2} \sigma_i \sqrt{T} \Phi^{-1}(F_{\mathbb{S}^c_{\Lambda=\lambda}}(K))}.
$$

We also stress here that in view of the equality in (4) and given $\Lambda = \lambda$ or equivalently, given $V = v$, this linear combination of Black & Scholes prices is the best, i.e. the smallest, upper bound constructed in this way. As for the lower bound, the choice of the conditioning variable is an additional parameter to further optimize this upper bound but is limited by the knowledge of an integration bound d_Λ.

4.2.2. *Upper bound based on lower bound.* In this section we follow the ideas of Rogers and Shi (1995) in order to derive an upper bound based on the lower bound. By Jensen's inequality and according to (12) we can find an error bound to the first term in (13):

$$
\begin{aligned}
0 &\leq E^Q\left[E^Q[(\mathbb{S}-K)_+ \mid \Lambda] - \left(\mathbb{S}^\ell - K\right)_+\right] \\
&= \int_{-\infty}^{d_\Lambda} \left(E^Q[(\mathbb{S}-K)_+ \mid \Lambda = \lambda] - \left(E^Q\left[\mathbb{S} \mid \Lambda = \lambda\right] - K\right)_+\right) f_\Lambda(\lambda)d\lambda \\
&\leq \frac{1}{2} \int_{-\infty}^{d_\Lambda} \left(\operatorname{var}\left(\mathbb{S} \mid \Lambda = \lambda\right)\right)^{\frac{1}{2}} f_\Lambda(\lambda)d\lambda \\
&\leq \frac{1}{2} \left(E^Q\left[\operatorname{var}\left(\mathbb{S} \mid \Lambda\right) 1_{\{\Lambda < d_\Lambda\}}\right]\right)^{\frac{1}{2}} \left(E^Q\left[1_{\{\Lambda < d_\Lambda\}}\right]\right)^{\frac{1}{2}},
\end{aligned} \tag{40}
$$

where Hölder's inequality has been applied in the last inequality and where $1_{\{\Lambda < d_\Lambda\}}$ is the indicator function.

Now we shall derive an easily computable expression for (40).

The second expectation factor in the product (40) equals $\Phi(d_\Lambda^*)$, where d_Λ^* is defined in (15).

The first expectation factor in the product (40) can be expressed as

$$
E^Q\left[\operatorname{var}\left(\mathbb{S}|\Lambda\right) 1_{\{\Lambda < d_\Lambda\}}\right] = E^Q\left[E^Q[\mathbb{S}^2|\Lambda]1_{\{\Lambda < d_\Lambda\}}\right] - E^Q\left[(E^Q[\mathbb{S}|\Lambda])^2 1_{\{\Lambda < d_\Lambda\}}\right]. \tag{41}
$$

The second term of the right-hand side of (41) can according to (14) be rewritten as

$$
\begin{aligned}
E^Q\left[(E^Q[\mathbb{S}|\Lambda])^2 1_{\{\Lambda < d_\Lambda\}}\right] &= \int_{-\infty}^{d_\Lambda} (E^Q[\mathbb{S}|\Lambda = \lambda])^2 f_\Lambda(\lambda)d\lambda \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j S_i(0) S_j(0) e^{(2r - \frac{\sigma_i^2 r_i^2 + \sigma_j^2 r_j^2}{2})T} \int_{-\infty}^{d_\Lambda} e^{(\sigma_i r_i + \sigma_j r_j)\sqrt{T}\Phi^{-1}(v)} f_\Lambda(\lambda)d\lambda,
\end{aligned} \tag{42}
$$

where we recall that $\Phi^{-1}(v) = \frac{\lambda - E^Q[\Lambda]}{\sigma_\Lambda}$ and $\Phi(\cdot)$ is the cumulative distribution function of a standard normal variable. Applying the equality (15) with $b = (\sigma_i r_i + \sigma_j r_j)\sqrt{T}$ we can express $E^Q\left[(E^Q[\mathbb{S}|\Lambda])^2 1_{\{\Lambda < d_\Lambda\}}\right]$ as

$$\sum_{i=1}^n \sum_{j=1}^n a_i a_j S_i(0) S_j(0) e^{(2r + \sigma_i \sigma_j r_i r_j)T} \Phi\left(d_\Lambda^* - (r_i \sigma_i + \sigma_j r_j)\sqrt{T}\right). \tag{43}$$

To transform the first term of the right-hand side of (41) we can rely on the fact that the product of two lognormal random variables is again lognormal:

$$E^Q\left[\mathbb{S}^2|\Lambda\right] = \sum_{i=1}^n \sum_{j=1}^n E^Q\left[a_i a_j S_i(T) S_j(T) \mid \Lambda\right]$$
$$= \sum_{i=1}^n \sum_{j=1}^n a_i a_j S_i(0) S_j(0) e^{(2r - \frac{\sigma_i^2 + \sigma_j^2}{2})T + r_{ij}\sigma_{ij}\sqrt{T}\Phi^{-1}(V) + \frac{1}{2}(1 - r_{ij}^2)T\sigma_{ij}^2}, \tag{44}$$

with $\sigma_{ij}^2 = \sigma_i^2 + \sigma_j^2 + 2\sigma_i \sigma_j \rho_{ij}$ and $r_{ij} = \frac{\sigma_i}{\sigma_{ij}} r_i + \frac{\sigma_j}{\sigma_{ij}} r_j$, and where $V = \Phi(\frac{\Lambda - E^Q[\Lambda]}{\sigma_\Lambda})$ is uniformly distributed on the interval $(0,1)$. Next we apply (15) again with $b = r_{ij}\sigma_{ij}\sqrt{T} = (\sigma_i r_i + \sigma_j r_j)\sqrt{T}$:

$$E^Q\left[E^Q[\mathbb{S}^2 \mid \Lambda] 1_{\{\Lambda < d_\Lambda\}}\right]$$
$$= \sum_{i=1}^n \sum_{j=1}^n a_i a_j S_i(0) S_j(0) e^{(2r - \frac{\sigma_i^2 + \sigma_j^2}{2})T + \frac{1}{2}(1 - r_{ij}^2)\sigma_{ij}^2 T} \int_{-\infty}^{d_\Lambda} e^{r_{ij}\sigma_{ij}\sqrt{T}\Phi^{-1}(v)} f_\Lambda(\lambda) d\lambda$$
$$= \sum_{i=1}^n \sum_{j=1}^n a_i a_j S_i(0) S_j(0) e^{(2r + \sigma_i \sigma_j \rho_{ij})T} \Phi\left(d_\Lambda^* - (r_i \sigma_i + \sigma_j r_j)\sqrt{T}\right). \tag{45}$$

Combining (43) and (45) into (41), and then substituting $\Phi(d_\Lambda^*)$ and (41) into (40) we get the following expression for the error bound, shortly denoted by $\varepsilon(d_\Lambda)$

$$\varepsilon(d_\Lambda) = \frac{1}{2} \left\{\Phi(d_\Lambda^*)\right\}^{\frac{1}{2}} \left\{ \sum_{i=1}^n \sum_{j=1}^n a_i a_j S_i(0) S_j(0) e^{(2r + \sigma_i \sigma_j r_i r_j)T} \Phi\left(d_\Lambda^* - (r_i \sigma_i + \sigma_j r_j)\sqrt{T}\right) \times \right.$$
$$\left. \times \left(e^{\sigma_i \sigma_j (\rho_{ij} - r_i r_j)T} - 1\right)\right\}^{1/2}. \tag{46}$$

The upper bound for the basket option price $BC(n, K, T)$ is then given by adding the lower bound (32) (which contains the exact part (16)) and the error term (46) multiplied with the discount factor e^{-rT}.

5. Approximations

In this section we investigate approximations to the option price by using a moment matching method.

From Section 3 we know that $\mathbb{S} \mid \Lambda = \lambda$ is a sum of n lognormal random variables with parameters $\mu(i) + \ln(\alpha_i)$ and $\sigma^2(i)$, $i = 1, \ldots, n$. We approximate this sum by assuming that it is also lognormally distributed with parameters μ and σ (depending on $\Lambda = \lambda$ or equivalently on $V = v = \Phi(\frac{\lambda - E^Q[\Lambda]}{\sigma_\Lambda})$) and having the same first two moments as the sum itself:

$$e^{\mu + \frac{1}{2}\sigma^2} = E^Q[\mathbb{S} \mid \Lambda = \lambda] \tag{47}$$

$$e^{2(\mu + \sigma^2)} = E^Q[\mathbb{S}^2 \mid \Lambda = \lambda]. \tag{48}$$

Explicit expressions for the right-hand sides are given by (14) and (44) for $V = v$.

Taking the logarithm of both sides of (47)-(48) and solving for σ^2 yields

$$\sigma^2 = 2(\mu + \sigma^2) - 2(\mu + \frac{1}{2}\sigma^2) = \ln E^Q[\mathbb{S}^2 \mid \Lambda = \lambda] - 2 \ln E^Q[\mathbb{S} \mid \Lambda = \lambda].$$

Then, the integrand of the first integral in (13) can be approximated by a European call option price for which the standard Black & Scholes formula (10)-(11) applies, leading to:

$$\int_{-\infty}^{d_\Lambda} E^Q[(\mathbb{S} - K)_+ \mid \Lambda = \lambda] dF_\Lambda(\lambda) \approx \int_{-\infty}^{d_\Lambda} \left[E^Q[\mathbb{S} \mid \Lambda = \lambda] \Phi(d_1(\lambda)) - K\Phi(d_2(\lambda)) \right] dF_\Lambda(\lambda),$$

with

$$d_1(\lambda) = \frac{\frac{1}{2} \ln E^Q[\mathbb{S}^2 \mid \Lambda = \lambda] - \ln(K)}{\sigma}, \qquad d_2(\lambda) = d_1(\lambda) - \sigma.$$

Instead of approximating $\mathbb{S} \mid \Lambda = \lambda$ when Λ is given by (19) or (21), we consider the difference $\mathbb{H}$ between the sum $\mathbb{S}$ and its approximation $\Lambda + K - d_\Lambda$ (as in the derivation of d_Λ):

$$\mathbb{S} - K = (\mathbb{S} - (\Lambda + K - d_\Lambda)) - (K - (\Lambda + K - d_\Lambda)) \overset{not}{=} \mathbb{H} - \tilde{K},$$

and approximate $\mathbb{H} \mid \Lambda = \lambda$ by a lognormal with parameters $\mu_{\mathbb{H}}$ and $\sigma_{\mathbb{H}}^2$ (depending on $\Lambda = \lambda$, or equivalently on $V = v$) using a moment matching method:

$$
\begin{aligned}
e^{\mu_{\mathbb{H}} + \frac{1}{2}\sigma_{\mathbb{H}}^2} &= E^Q[\mathbb{H} \mid \Lambda = \lambda] = E^Q[\mathbb{S} \mid \Lambda = \lambda] - (\lambda + K - d_\Lambda) \\
e^{2(\mu_{\mathbb{H}} + \sigma_{\mathbb{H}}^2)} &= E^Q[\mathbb{H}^2 \mid \Lambda = \lambda] \\
&= E^Q[\mathbb{S}^2 \mid \Lambda = \lambda] - 2(\lambda + K - d_\Lambda)E^Q[\mathbb{S} \mid \Lambda = \lambda] + (\lambda + K - d_\Lambda)^2,
\end{aligned}
$$

where the right-hand sides are expressed in terms of the expectations (14) and (44) for $V = v$. Note that $\tilde{K} = d_\Lambda - \Lambda$ is known and positive for a given $\Lambda = \lambda$ with $\lambda \leq d_\Lambda$. An approximation to the first integral in (13) then reads

$$
\begin{aligned}
&\int_{-\infty}^{d_\Lambda} E^Q[(\mathbb{H} - \tilde{K})_+ \mid \Lambda = \lambda]dF_\Lambda(\lambda) \\
&\approx \int_{-\infty}^{d_\Lambda} \left[(E^Q[\mathbb{S} \mid \Lambda = \lambda] - (\lambda + K - d))\Phi(d_1(\lambda)) - (d_\Lambda - \lambda)\Phi(d_2(\lambda)) \right] dF_\Lambda(\lambda) \\
&= \int_0^{\Phi(d_\Lambda^*)} \left[(E^Q[\mathbb{S} \mid V = v] - (\sigma_\Lambda \Phi^{-1}(v) + E^Q[\Lambda] + K - d_\Lambda))\Phi(d_1(v)) \right. \\
&\qquad\qquad \left. -(d_\Lambda - \sigma_\Lambda \Phi^{-1}(v) - E^Q[\Lambda])\Phi(d_2(v)) \right] dv
\end{aligned}
$$

with d_Λ^* defined in (15) and with, recalling that $\lambda = \sigma_\Lambda \Phi^{-1}(v) + E^Q[\Lambda]$,

$$
d_1(\lambda) = d_1(v) = \frac{\mu_{\mathbb{H}} + \sigma_{\mathbb{H}}^2 - \ln(\sigma_\Lambda \Phi^{-1}(v) + E^Q[\Lambda] + K - d_\Lambda)}{\sigma_{\mathbb{H}}}, \quad d_2(v) = d_1(v) - \sigma_{\mathbb{H}}.
$$

We can further approximate this result by assuming that the parameters $\mu_{\mathbb{H}}$ and $\sigma_{\mathbb{H}}^2$ are independent of λ and thus are constant. Since the main contribution to this integral comes from values λ close to d_Λ (for $\lambda \ll d_\Lambda$, $(\mathbb{S} - K)_+ \mid \Lambda = \lambda$ equals zero by definition of d_Λ), we now put

$$
\begin{aligned}
\mu_{\mathbb{H}} + \sigma_{\mathbb{H}}^2 &= \frac{1}{2} \ln E^Q[\mathbb{H}^2 \mid \Lambda = d_\Lambda] \\
\sigma_{\mathbb{H}}^2 &= 2(\mu_{\mathbb{H}} + \sigma_{\mathbb{H}}^2) - 2 \ln E^Q[\mathbb{H} \mid \Lambda = d_\Lambda].
\end{aligned}
$$

Note that d_1 and d_2 however remain dependent on λ through the term $-\ln(\lambda + K - d_\Lambda)$ in de numerator.

In case $\Lambda = GA = \frac{\ln \mathbb{G} - E^Q[\ln \mathbb{G}]}{\sqrt{\text{var}[\ln \mathbb{G}]}}$ (24), we follow the approach of Curran (1994) and consider the difference between the sum $\mathbb{S}$ and the geometric average $\mathbb{G}$:

$$\mathbb{S} - K = (\mathbb{S} - \mathbb{G}) - (K - \mathbb{G}) \stackrel{not}{=} \mathbb{H} - \tilde{K},$$

where $\mathbb{H} = \mathbb{S} - \mathbb{G}$ and $\tilde{K} = K - \mathbb{G}$, and approximate $\mathbb{H} \mid \mathbb{G}$ by a lognormal variable using a moment matching method. Note that we have a new strike price $\tilde{K} = (K - \mathbb{G})$ which is known once $\mathbb{G}$ (or equivalently Λ) is known, and which is positive for $\mathbb{G} = g$ and $g \leq K$. The latter is also equivalent with $\Lambda = \lambda$ and $\lambda \leq d_{GA}$. Hence the first integral in (13) can be transformed to:

$$\int_{-\infty}^{d_{GA}} E^Q\left[(\mathbb{S} - K)_+ \mid \Lambda = \lambda\right] dF_\Lambda(\lambda) = \int_0^K E^Q\left[(\mathbb{S} - K)_+ \mid \mathbb{G} = g\right] dF_{\mathbb{G}}(g).$$

Thus we assume that $\mathbb{H} \mid \mathbb{G} = g$ is lognormal with parameters $\mu_{\mathbb{H}}$ and $\sigma_{\mathbb{H}}^2$ (depending on $\mathbb{G} = g$) which are such that

$$e^{\mu_{\mathbb{H}} + \frac{1}{2}\sigma_{\mathbb{H}}^2} = E^Q[\mathbb{H} \mid \mathbb{G} = g] = E^Q[\mathbb{S} \mid \mathbb{G} = g] - g, \tag{49}$$

$$\begin{aligned} e^{2(\mu_{\mathbb{H}} + \sigma_{\mathbb{H}}^2)} &= E^Q[\mathbb{H}^2 \mid \mathbb{G} = g] \\ &= E^Q[\mathbb{S}^2 \mid \mathbb{G} = g] - 2g E^Q[\mathbb{S} \mid \mathbb{G} = g] - g^2, \end{aligned} \tag{50}$$

where the first two moments of $\mathbb{S} \mid \mathbb{G} = g$ are given by (14) and (44) with $\Phi^{-1}(v) = \frac{\ln g - E^Q[\ln \mathbb{G}]}{\text{var}[\ln \mathbb{G}]}$ ((25)-(26)). Taking the logarithm of both sides of (49)–(50) and solving for $\sigma_{\mathbb{H}}^2$, we find

$$\begin{aligned} \sigma_{\mathbb{H}}^2 &= 2(\mu_{\mathbb{H}} + \sigma_{\mathbb{H}}^2) - 2(\mu_{\mathbb{H}} + \frac{1}{2}\sigma_{\mathbb{H}}^2) \\ &= \ln E^Q[\mathbb{H}^2 \mid \mathbb{G} = g] - 2\ln E^Q[\mathbb{H} \mid \mathbb{G} = g]. \end{aligned}$$

By assuming that the distribution of $\mathbb{H} \mid \mathbb{G} = g$ is lognormal, the integrand of the first integral in (13) can be approximated by a European call option price for which the standard Black & Scholes formula (10)-(11) applies, leading to:

$$\begin{aligned} &\int_0^K E^Q\left[(\mathbb{S} - K)_+ \mid \mathbb{G} = g\right] dF_{\mathbb{G}}(g) \\ &\approx \int_0^K \left[(E^Q[\mathbb{S} \mid \mathbb{G} = g] - g)\Phi(d_1(g)) - (K - g)\Phi(d_2(g))\right] dF_{\mathbb{G}}(g) \end{aligned}$$

with

$$d_1(g) = \frac{\mu_\mathbb{H} + \sigma_\mathbb{H}^2 - \ln \tilde{K}}{\sigma_\mathbb{H}} = \frac{\frac{1}{2} \ln E^Q[\mathbb{H}^2 \mid \mathbb{G} = g] - \ln(K - g)}{\sigma_\mathbb{H}}$$

and

$$d_2(g) = d_1(g) - \sigma_\mathbb{H}.$$

Note that the moments depend in fact on g and have to be recomputed for each value that g takes. One could avoid this by assuming that the moments are constant and equal to the moments for $\mathbb{G} = K$. Thus the first integral in (13) is in that case approximated by

$$\int_0^K \left[(E^Q[\mathbb{S} \mid \mathbb{G} = K] - g)\Phi(d_1(g)) - (K - g)\Phi(d_2(g)) \right] dF_\mathbb{G}(g) \tag{51}$$

with

$$d_1(g) = \frac{\frac{1}{2} \ln E^Q[\mathbb{H}^2 \mid \mathbb{G} = K] - \ln(K - g)}{\sigma_\mathbb{H}}$$

and

$$d_2(g) = d_1(g) - \sigma_\mathbb{H},$$

where $\sigma_\mathbb{H}^2 = \ln E[\mathbb{H}^2 \mid \mathbb{G} = K] - 2 \ln E[\mathbb{H} \mid \mathbb{G} = K]$.

In the formulas (14) and (44) we then have that $\Phi^{-1}(v) = \frac{\ln K - E^Q[\ln \mathbb{G}]}{\text{var}[\ln \mathbb{G}]} = d_{GA}$. Fixing the moments on $\mathbb{G} = K$ is motivated by the fact that the main contribution to the integral comes from values of g close to K since for $g \ll K$ it holds that $(\mathbb{S} - K)_+ \mid \mathbb{G} = g$ equals zero.

The appearing integrals have to be computed by means of numerical quadrature. Numerical experiments are carried out for the different conditioning variables and we consider both cases of variable moments and of constant moments.

6. General remarks

In this section we summarize some general remarks:

- The price of the basket *put* option with exercise date T, n underlying assets and fixed exercise price K, given by $BP(n, K, T) = e^{-rT} E^Q\left[(K - \mathbb{S}(T))_+ \right]$ satisfies the put-call

parity at the present: $BC(n, K, T) - BP(n, K, T) = \mathbb{S}(0) - e^{-rT}K$. Hence, we can derive bounds for the basket put option from the bounds for the call. These bounds for the put option coincide with the bounds that are obtained by applying the theory of comonotonic bounds or the Rogers and Shi approach directly to basket put options. This stems from the fact that the put-call parity also holds for these bounds.

- The case of a continuous dividend yield q_i can easily be dealt with by replacing the interest rate r by $r - q_i$.

- For $n = 1$ there is only one asset in the basket and hence the comonotonic sums $\mathbb{S}^c_{\Lambda=\lambda}$ and $\mathbb{S}^\ell$ coincide with the sum $\mathbb{S}$ which consists of only one term: this asset. In this case, the comonotonic upper and lower bound reduce to the well-known Black & Scholes price for an option on a single asset. This is also true for the bound based on the Rogers & Shi approach since the error bound is zero.

- As for the Asian options (see Vanmaele et al. (2002)), we can easily derive the hedging Greeks for the upper and lower bounds as well as for the approximations of a basket option since we found analytical expressions for them. Moreover the expressions are in terms of Black & Scholes prices.

7. Numerical illustration

In this section we give a number of numerical examples on basket options in the Black & Scholes setting. We first concentrate on the bounds proposed in Section 4 and afterwards on the approximations in Section 5. Therefore, we first discuss Tables 1, 3 and 5 and secondly, Tables 2, 4 and 6.

We introduce the following notations where Λ can be *FA*1, *FA*2, *GA*: LBΛ for lower bound, PECUBΛ for partially exact/comonotonic upper bound, and UBΛ for upper bound based on

lower bound.

The first set of input data was taken from Arts (1999). Note that we consider here the forward-moneyness, which is defined as the ratio of the forward price of the basket and the exercise price K. The input parameters correspond to a two-dimensional basket. We first consider equal weights and afterwards, unequal weights. The spot prices are first assumed to be equal to 100 units, and then allowed to vary. The risk-free interest rate is fixed at 5% and we assume no dividends. Moneyness ranges from 10% in-the-money to 10% out-of-the-money. For the time to maturity T two cases are considered ($T = 1, 3$ years). For the correlation (8), two values are considered, representing low and high correlation respectively. We consider equal volatilities (high and low) for both individual assets in the basket.

Concerning the upper bounds, we present only the results that lead to the best upper bound together with the corresponding type of the bound. That is, the upper bound given in the Tables 1, 3 and 5 is the bound which satisfies min(UBΛ, PECUBΛ), where the bounds were computed for all three choices $FA1$, $FA2$ and GA of the conditioning variable Λ. The detailed numerical results for all bounds are available upon request. Notice that, in general, the Monte Carlo (MC) price is closer to the best lower bound than to the best upper bound. One can also note that the relative difference between the best lower and upper bound is smaller for higher correlation. We start by discussing Table 1 which corresponds to the case of equal weights, spot prices and volatilities for both assets. In this case the lower bound (32)-(33) applied with Λ given by (19), (21) and (24), which are denoted by LB$FA1$, LB$FA2$ and LBGA, are equal. The optimized lower bound LBopt, which is obtained by solving the optimization program (38), gave practically the same values, therefore it is not reported in the table. From all the upper bounds considered, UBGA uniformly performs the best.

Table 3 refers to the case of unequal weights and spot prices with equal volatilities. From Table 3, we notice that LB$FA1$ = LB$FA2$ gives sharper results than LBGA. The lower bound LBopt only slightly improves the lower bound LB$FA1$. However, for high volatilities,

small correlation ρ and long maturity, the improvement is significant. As for the upper bounds, we could observe some pattern, namely for out- and at-the-money options, and in-the-money options with the maturity of three years, the upper bound UB$FA1$ performs the best for smaller volatility (0.2), whereas UB$FA2$ is the best for larger volatility (0.4) with the exception of three years to maturity out-of-the-money option. In the latter case the partially exact/comonotonic upper bound PECUBGA based on the standardized logarithm of the geometric average outperforms the other bounds for larger volatility. For in-the-money options with the maturity of one year, the pattern is reversed compared to that of in-the-money options with three years to maturity. The second set of input data was taken from Brigo et al. (2002). Here we consider two assets with weights 0.5956 and 0.4044, and spot prices of 26.3 and 42.03, respectively. Maturity is equal to 5 years. The discount factor at payoff is 0.783895779. This example refers to a realistic basket, for which we allow the volatilities and correlations of individual assets to vary in order to facilitate the comparative price analysis. From Table 5 we see that the optimized lower bound gives the best value, and that the lower bound LBGA leads to the closest values to LBopt. The lower bound LB$FA2$ led to the worst results and is therefore not reported. For this example the partially exact/comonotonic upper bound PECUB$FA2$, i.e. with Λ given by $FA2$ (21), turns out to be the sharpest upper bound, except for very high correlation when PECUBGA is to be preferred, and for $\sigma_1 = 0.1$, $\sigma_2 = 0.3$ (for both $\rho = 0.2$ and $\rho = 0.6$) when UBGA is the best.

As mentioned before for a negative correlation between the assets in the basket the lower bound (32) is not applicable if any of the correlations r_1 and r_2 is negative. If this happens, one should turn to the optimization procedure which enables to choose the coefficients β_1 and β_2 such that r_1 and r_2 would be positive. Consider a case where $\sigma_1 = 0.3$, $\sigma_2 = 0.6$, and $\rho = -0.6$. In this instance we have that the correlations r_1 and r_2 are positive for the conditioning variables $FA1$ and GA and therefore we can find the lower bounds based on those variables: LB$FA1$ = 29.39746493, LBGA = 29.77084284. The optimization procedure (38)

gives LBopt = 29.773172314, which shows again that the geometric average is a fairly good choice for a conditioning variable when $a_1 = a_2$, and $S_1(0) = S_2(0)$.

We now turn to the discussion of the approximations. In Tables 2 and 4, we used as in Tables 1 and 3 the dataset of Arts respectively with equal weights and spot prices and with different weights and spot prices, but in both cases with equal volatility. The numerical results clearly show that the approximations improve for shorter maturities and this for in-the-money, at-the-money as well as for out-of-the-money, and always lie between the corresponding bounds. For long maturities (3 year) and high volatility (0.4), the approximation by using $\mathbb{H}$ and constant moments turns out to be disappointing. However, the other approximation methods seem to deliver approximating prices with an equivalent level of high quality.

In Table 6, we discuss the approximations for the dataset of Brigo et al. (2002) used in Table 5 with different weights, different spot prices and different correlations. The numerical results for a very high correlation (0.99) are excellent since in that case, the proportion of the exact part in the price is more than 99%. We have attempted to prove this fact analytically by using the derivative of the exact part with respect to the correlation, but the expression of the derivative is too long and too complicated to obtain straightforward general conclusions. For this dataset, the moment matching method with the first order approximation $FA1$ and with fixed moments turns out to be somewhat poor, especially for high volatility and low correlation. For the first order approximation $FA2$ and with fixed moments, nearly all approximation values are close to the MC value but are smaller than the lower bound LBopt. In general, all moment matching methods with variable moments superperform the methods with fixed moments and deliver approximating prices which are close to the Monte Carlo estimates. Since nowadays even quite simple PC's can treat numerical quadratures with high speed, we therefore recommend to use moment matching approximations with variable moments.

8. Asian basket options

An Asian basket option is an option whose payoff depends on an average of values at different dates of a portfolio (or basket) of assets, or which is equivalent on the portfolio value of an average of asset prices taken at different dates. The price of a discrete arithmetic Asian basket call option at current time $t = 0$ is given by

$$ABC(n, K, T) = e^{-rT} E^Q \left[\left(\sum_{\ell=1}^{n} a_\ell \sum_{j=0}^{m-1} b_j S_\ell(T - j) - K \right)_+ \right]$$

with a_ℓ and b_j positive coefficients. For $T \leq m - 1$ we call this Asian basket call option in progress and for $T > m - 1$, we call it forward starting.

Remark that the double sum $\mathbb{S} = \sum_{\ell=1}^{n} a_\ell \sum_{j=0}^{m-1} b_j S_\ell(T - j)$ is a sum of lognormal distributed variables:

$$\mathbb{S} = \sum_{i=1}^{mn} X_i = \sum_{i=1}^{mn} \alpha_i e^{Y_i}$$

with

$$\alpha_i = a_{\lceil \frac{i}{m} \rceil} b_{(i-1) \bmod m} S_{\lceil \frac{i}{m} \rceil}(0) e^{(r - \frac{1}{2} \sigma^2_{\lceil \frac{i}{m} \rceil})(T - (i-1) \bmod m)}$$

and

$$Y_i = \sigma_{\lceil \frac{i}{m} \rceil} W_{\lceil \frac{i}{m} \rceil}(T - (i - 1) \bmod m) \sim N \left(0, \sigma^2_{Y_i} = \sigma^2_{\lceil \frac{i}{m} \rceil}(T - (i - 1) \bmod m) \right)$$

for all $i = 1, \ldots, mn$, where $\lceil x \rceil = \inf\{k \in \mathbb{Z} \mid x \leq k\}$.

Hence, we can apply the general formulae for lognormals from Section 3 (see also Vanmaele et al. (2002)).

9. Conclusion

We derived lower and upper bounds and approximations for the price of the arithmetic basket call options by decomposing the option price into an exact and an approximating part, and

applying different techniques to the latter, such as firstly results based on comonotonic risks and bounds for stop-loss premiums of sums of dependent random variables as in Kaas, Dhaene and Goovaerts (2000), and secondly, conditioning on some random variable as in Rogers and Shi (1995), and finally using a moment matching method. Notice that all bounds and approximations have analytical and easily computable expressions. For the numerical illustration it was important to find and motivate a good choice of the conditioning variables appearing in the formulae. We also managed to find the best lower bound through an optimization procedure.

Acknowledgements

References

ARTS P. (1999). *Pricing the basket option and exploring its volatility structure.* MSc Thesis, Utrecht University.

BEISSER J. (2001). *Topics in Finance - A conditional expectation approach to value Asian, Basket and Spread Options.* Ph.D. Thesis, Johannes Gutenberg University Mainz.

BRIGO D., MERCURIO F., RAPISPARDA F., AND SCOTTI R. (2002). *Approximated moment-matching dynamics for basket-options simulation.* Working paper, Banca IMI, Milano.

CURRAN M. (1994). Valuing Asian and portfolio options by conditioning on the geometric mean price. *Management Science*, **40**(12), 1705–1711.

DAHL L.O., AND BENTH F.E. (2001a). Valuation of Asian basket options with quasi-Monte Carlo techniques and singular value decomposition. *Pure Mathematics*, No. 5, February, 1–21.

DAHL L.O., AND BENTH F.E. (2001b). Fast evaluation of the Asian basket option by singular value decomposition. *Pure Mathematics*, No. 8, March.

DHAENE J., DENUIT, M., GOOVAERTS, M.J., KAAS, R. AND VYNCKE, D. (2002). The concept of comonotonicity

in actuarial science and finance: theory. *Insurance: Mathematics & Economics*, **31**, 3–33.

KAAS R., DHAENE J., AND GOOVAERTS M.J. (2000). Upper and lower bounds for sums of random variables. *Insurance: Mathematics & Economics*, **27**, 151–168.

ROGERS L.C.G., AND SHI Z. (1995). The value of an Asian option. *Journal of Applied Probability*, **32**, 1077–1088.

VAN DIEPEN E. (2002). *Baskets with a smile. An alternative model for pricing basket options.* MSc Thesis, Free University Amsterdam.

VANMAELE M., DEELSTRA G., LIINEV J., DHAENE J., AND GOOVAERTS M.J. (2002). Bounds for the price of discretely sampled arithmetic Asian options. Submitted.

TABLE 1: Comparing bounds, equal weights and spot prices.

Data		T	corr	vol	MC	LB$FA1$=LB$FA2$=LBGA	UB	Type
$n = 2$	10%OTM	1	0.3	0.2	2.90	2.8810	3.2428	UBGA
$r = 0.05$	$K = 115.64$			0.4	9.12	9.0280	10.2168	UBGA
$a_1 = a_2 = 0.5$			0.7	0.2	3.72	3.7172	3.8605	UBGA
$S_1(0) = S_2(0) = 100$				0.4	10.88	10.8647	11.3373	UBGA
	10%OTM	3	0.3	0.2	7.39	7.3290	8.2487	UBGA
	$K = 127.80$			0.4	18.85	18.4242	21.6818	UBGA
			0.7	0.2	8.92	8.9054	9.2733	UBGA
				0.4	21.66	21.5913	22.8329	UBGA
	ATM	1	0.3	0.2	6.44	6.4245	6.6658	UBGA
	$K = 105.13$			0.4	12.90	12.8088	13.7572	UBGA
			0.7	0.2	7.35	7.3445	7.4447	UBGA
				0.4	14.64	14.6281	15.0098	UBGA
	ATM	3	0.3	0.2	11.17	11.1071	11.8210	UBGA
	$K = 116.18$			0.4	22.41	21.9985	24.8118	UBGA
			0.7	0.2	12.69	12.6885	12.9784	UBGA
				0.4	25.12	25.0568	26.1324	UBGA
	10%ITM	1	0.3	0.2	12.37	12.3620	12.4836	UBGA
	$K = 94.61$			0.4	17.88	17.8093	18.5009	UBGA
			0.7	0.2	13.08	13.0861	13.1412	UBGA
				0.4	19.47	19.4565	19.7426	UBGA
	10%ITM	3	0.3	0.2	16.34	16.2843	16.7788	UBGA
	$K = 104.57$			0.4	26.62	26.2563	28.5970	UBGA
			0.7	0.2	17.71	17.6942	17.9022	UBGA
				0.4	29.19	29.1130	30.0151	UBGA

TABLE 2: Comparing approximations, equal weights and spot prices.

					$\mathbb{S}\vert_{FA1=\lambda}$	$\mathbb{H}\vert_{FA1=\lambda}$		$\mathbb{S}\vert_{FA2=\lambda}$	$\mathbb{H}\vert_{FA2=\lambda}$		$\mathbb{S}\vert_{\mathbb{G}=g}$	$\mathbb{H}\vert_{\mathbb{G}=g}$	
					λ	λ	λ	λ	λ	λ	g	g	g
K	T	corr	vol	MC	variable	variable	$= d_{FA1}$	variable	variable	$= d_{FA2}$	variable	variable	$= K$
115.64	1	0.3	0.2	2.90	2.8993	2.9010	2.9045	2.8995	2.9010	2.9082	2.8982	2.900	2.9008
			0.4	9.12	9.1186	9.1215	9.1853	9.1168	9.1178	9.1697	9.1110	9.1146	9.1426
		0.7	0.2	3.72	3.7191	3.7191	3.7205	3.7190	3.7190	3.7226	3.7187	3.7190	3.7191
			0.4	10.88	10.8800	10.8818	10.8939	10.8797	10.8815	10.8889	10.8780	10.8796	10.8820
127.80	3	0.3	0.2	7.39	7.3874	7.3915	7.4248	7.3900	7.3968	7.4865	7.3824	7.3865	7.4008
			0.4	18.85	18.8980	18.8366	19.7958	18.8819	18.8072	19.4074	18.8526	18.8025	19.1520
		0.7	0.2	8.92	8.9153	8.9165	8.9232	8.9154	8.9160	8.9534	8.9140	8.9152	8.9163
			0.4	21.66	21.6703	21.6755	21.8864	21.6683	21.6698	21.7629	21.6599	21.6609	21.6956
105.13	1	0.3	0.2	6.44	6.4350	6.4355	6.4369	6.4353	6.4361	6.4377	6.4349	6.4359	6.4368
			0.4	12.90	12.8929	12.8880	12.9207	12.8917	12.8859	12.9139	12.8908	12.8907	12.9143
		0.7	0.2	7.35	7.3461	7.3463	7.3465	7.3462	7.3465	7.3466	7.3461	7.3464	7.3464
			0.4	14.64	14.6420	14.6429	14.6465	14.6416	14.6421	14.6453	14.6413	14.6425	14.6447
116.18	3	0.3	0.2	11.17	11.1616	11.1599	11.1766	11.1656	11.1670	11.1977	11.1606	11.1620	11.1741
			0.4	22.41	22.4383	22.3577	22.8886	22.4224	22.3375	22.6925	22.4102	22.3521	22.6520
		0.7	0.2	12.69	12.6975	12.6981	12.6999	12.6984	12.6994	12.7055	12.6971	12.6980	12.6991
			0.4	25.12	25.1308	25.1312	25.2160	25.1270	25.1236	25.1702	25.1229	25.1229	25.1538
94.61	1	0.3	0.2	12.37	12.3707	12.3702	12.3696	12.3703	12.3700	12.3701	12.3700	12.3702	12.3708
			0.4	17.88	17.8828	17.8748	17.8884	17.8835	17.8755	17.8858	17.8826	17.8793	17.8973
		0.7	0.2	13.08	13.0876	13.0876	13.0872	13.0875	13.0875	13.0875	13.0874	13.0876	13.0876
			0.4	19.47	19.4685	19.4682	19.4699	19.4688	19.4683	19.4694	19.4684	19.4693	19.4711
104.57	3	0.3	0.2	16.34	16.3318	16.3276	16.3333	16.3318	16.3283	16.3401	16.3312	16.3302	16.3392
			0.4	26.62	26.6469	26.5613	26.8472	26.6383	26.5553	26.7548	26.6372	26.5741	26.8217
		0.7	0.2	17.71	17.7021	17.7020	17.7026	17.7021	17.7024	17.7039	17.7019	17.7026	17.7034
			0.4	29.19	29.1779	29.1735	29.2125	29.1735	29.1689	29.1950	29.1746	29.1737	29.2005

TABLE 3: Comparing bounds, different weights and spot prices.

Data		T	corr	vol	MC	LBGA	LBFA1 =LBFA2	LBopt	UB	Type
$n = 2$	10%OTM	1	0.3	0.2	2.57	2.4677	2.5611	2.5611	2.8737	UBFA1
$r = 0.05$	$K = 101.76$			0.4	8.07	7.7665	7.9855	7.9855	9.0400	UBFA2
$a_1 = 0.3$			0.7	0.2	3.28	3.2381	3.2788	3.2788	3.4057	UBFA1
$a_2 = 0.7$				0.4	9.61	9.4864	9.5767	9.5767	9.9963	UBFA2
$S_1(0) = 130$	10%OTM	3	0.3	0.2	6.53	6.2970	6.4823	6.4823	7.3026	UBFA1
$S_2(0) = 70$	$K = 112.47$			0.4	16.65	15.8604	16.2771	16.2772	18.9776	PECUBGA
			0.7	0.2	7.84	7.7706	7.8478	7.8478	8.1762	UBFA1
				0.4	19.07	18.8624	19.0234	19.0234	20.0905	PECUBGA
	ATM	1	0.3	0.2	5.69	5.5582	5.6750	5.6750	5.8848	UBFA1
	$K = 92.51$			0.4	11.39	11.0722	11.3112	11.3113	12.1387	UBFA2
			0.7	0.2	6.47	6.4267	6.4724	6.4724	6.5595	UBFA1
				0.4	12.90	12.7972	12.8889	12.8889	13.2216	UBFA2
	ATM	3	0.3	0.2	9.89	9.6011	9.8066	9.8066	10.4308	UBFA1
	$K = 102.24$			0.4	19.76	18.9795	19.4182	19.4186	21.9157	UBFA2
			0.7	0.2	11.18	11.0985	11.1778	11.1778	11.4310	UBFA1
				0.4	22.12	21.9132	22.0729	22.0729	23.0263	UBFA2
	10%ITM	1	0.3	0.2	10.90	10.7924	10.8905	10.8906	10.9984	UBFA2
	$K = 83.26$			0.4	15.78	15.4667	15.7025	15.7027	16.3073	UBFA1
			0.7	0.2	11.52	11.4815	11.5195	11.5195	11.5680	UBFA2
				0.4	17.13	17.0467	17.1329	17.1329	17.3822	UBFA1
	10%ITM	3	0.3	0.2	14.41	14.1593	14.3585	14.3586	14.7923	UBFA1
	$K = 92.02$			0.4	23.46	22.7133	23.1587	23.1598	25.2074	UBFA2
			0.7	0.2	15.58	15.5092	15.5827	15.5827	15.7644	UBFA1
				0.4	25.69	25.4874	25.6415	25.6416	26.4286	UBFA2

TABLE 4: Comparing approximations, different weights and spot prices.

K	T	corr	vol	MC	$\mathbb{S}|_{FA1=\lambda}$ λ variable	$\mathbb{H}|_{FA1=\lambda}$ λ variable	$\lambda = d_{FA1}$	$\mathbb{S}|_{FA2=\lambda}$ λ variable	$\mathbb{H}|_{FA2=\lambda}$ λ variable	$\lambda = d_{FA2}$	$\mathbb{S}|_{\mathbb{G}=g}$ g variable	$\mathbb{H}|_{\mathbb{G}=g}$ g variable	$g = K$
101.76	1	0.3	0.2	2.57	2.5699	2.5712	2.5743	2.5700	2.5712	2.5774	2.5639	2.5718	2.5762
			0.4	8.07	8.0619	8.0648	8.1182	8.0604	8.0616	8.1049	8.0381	8.0463	8.1024
		0.7	0.2	3.28	3.2803	3.2804	3.2816	3.2803	3.2803	3.2835	3.2797	3.2805	3.2904
			0.4	9.61	9.5897	9.5913	9.6017	9.5895	9.5910	9.5974	9.5854	9.5870	9.6194
112.47	3	0.3	0.2	6.53	6.5315	6.5352	6.5631	6.5337	6.5396	6.6163	6.5128	6.5239	6.5586
			0.4	16.65	16.6792	16.6303	17.4395	16.6655	16.6048	17.1084	16.6410	16.5471	16.9490
		0.7	0.2	7.84	7.8563	7.8573	7.8631	7.8564	7.8568	7.8895	7.8534	7.8549	7.8785
			0.4	19.07	19.0910	19.0956	19.2774	19.0892	19.0907	19.1703	19.0777	19.0689	19.2126
92.51	1	0.3	0.2	5.69	5.6838	5.6842	5.6854	5.6841	5.6848	5.6861	5.6829	5.6815	5.6857
			0.4	11.39	11.3825	11.3787	11.4058	11.3814	11.3768	11.4001	11.3729	11.3605	11.4132
		0.7	0.2	6.47	6.4738	6.4740	6.4741	6.4739	6.4741	6.4742	6.4741	6.4736	6.4849
			0.4	12.90	12.9007	12.9015	12.9045	12.9004	12.9008	12.9035	12.8988	12.8974	12.9286
102.24	3	0.3	0.2	9.89	9.8527	9.8515	9.8654	9.8561	9.8577	9.8833	9.8454	9.8388	9.8717
			0.4	19.76	19.7930	19.7267	20.1726	19.7793	19.7090	20.0060	19.7904	19.6592	20.0299
		0.7	0.2	11.18	11.1854	11.1860	11.1875	11.1862	11.1871	11.1923	11.1845	11.1835	11.2067
			0.4	22.12	22.1363	22.1368	22.2092	22.1330	22.1302	22.1698	22.1275	22.1141	22.2466
83.26	1	0.3	0.2	10.90	10.8979	10.8975	10.8969	10.8976	10.8973	10.8973	10.8956	10.8914	10.8944
			0.4	15.78	15.7650	15.7585	15.7697	15.7656	15.7591	15.7674	15.7680	15.7366	15.7825
		0.7	0.2	11.52	11.5208	11.5207	11.5204	11.5207	11.5207	11.5206	11.5215	11.5198	11.5296
			0.4	17.13	17.1432	17.1429	17.1444	17.1434	17.1430	17.1439	17.1433	17.1392	17.1671
92.02	3	0.3	0.2	14.41	14.3989	14.3955	14.4002	14.3989	14.3962	14.4060	14.4012	14.3784	14.4066
			0.4	23.46	23.4933	23.4217	23.6611	23.4859	23.4162	23.5825	23.5271	23.3639	23.6957
		0.7	0.2	15.58	15.5895	15.5893	15.5899	15.5895	15.5898	15.5910	15.5900	15.5866	15.6074
			0.4	25.69	25.6972	25.6936	25.7267	25.6947	25.6895	25.7116	25.6950	25.6774	25.7963

TABLE 5: Comparing bounds, different weights and spot prices, different correlations.

Data	σ_1	σ_2	corr	MC	LB$FA1$	LBGA	LBopt	UB	Type
$n = 2$	0.1	0.3	0.2	26.2824	26.2206	26.2346	26.2380	26.7612	UBGA
$T = 5$ years			0.6	27.4454	27.4304	27.4321	27.4356	27.7353	UBGA
$K = 32.661$			0.99	28.5055	28.5083	28.5083	28.5084	28.5213	PECUBGA
$r = 4.8696\%$	0.1	0.6	0.2	34.0869	33.9755	34.0185	34.0233	34.7658	PECUB$FA2$
$a_1 = 0.5956$			0.6	35.5348	35.4995	35.5172	35.5206	35.9272	PECUB$FA2$
$a_2 = 0.4044$			0.99	36.8755	36.8709	36.8713	36.8714	36.8829	PECUBGA
$S_1(0) = 26.3$	0.3	0.6	0.2	39.8587	38.8396	38.9627	38.9640	42.6078	PECUB$FA2$
$S_2(0) = 42.03$			0.6	42.5568	42.2886	42.3316	42.3318	43.9641	PECUB$FA2$
			0.99	45.1953	45.1919	45.1926	45.1926	45.2273	PECUBGA

TABLE 6: Comparing approximations, different weights and spot prices, different correlations.

σ_1	σ_2	corr	MC	$\mathbb{S}|_{FA1=\lambda}$ λ variable	$\mathbb{H}|_{FA1=\lambda}$ λ variable	$\lambda = d_{FA1}$	$\mathbb{S}|_{FA2=\lambda}$ λ variable	$\mathbb{H}|_{FA2=\lambda}$ λ variable	$\lambda = d_{FA2}$	$\mathbb{S}|_{\mathbb{G}=g}$ g variable	$\mathbb{H}|_{\mathbb{G}=g}$ g variable	$g = K$
0.1	0.3	0.2	26.2824	26.2747	26.2776	26.2542	26.2699	26.2742	26.2350	26.2828	26.2812	26.2979
		0.6	27.4454	27.4451	27.4457	27.4400	27.4427	27.4449	27.4355	27.4465	27.4460	27.4520
		0.99	28.5055	28.5084	28.5084	28.5080	28.5084	28.5084	28.5080	28.5084	28.5084	28.5085
0.1	0.6	0.2	34.0869	34.1239	34.0882	34.3969	34.0687	34.0865	33.9912	34.0729	34.1170	34.0761
		0.6	35.5348	35.5478	35.5380	35.6267	35.5315	35.5380	35.4984	35.5342	35.5464	35.5348
		0.99	36.8755	36.8713	36.8714	36.8753	36.8715	36.8715	36.8652	36.8714	36.8714	36.8714
0.3	0.6	0.2	39.8587	40.3488	39.7310	42.7168	39.9498	39.6478	39.6556	40.0362	39.7428	41.0706
		0.6	42.5568	42.6382	42.5399	43.2703	42.5406	42.4922	42.5142	42.5692	42.5429	42.7702
		0.99	45.1953	45.1931	45.1932	45.2111	45.1927	45.1927	45.1917	45.1927	45.1927	45.1928