INTERACTION BETWEEN ASSET LIABILITY MANAGEMENT AND RISK THEORY.

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SUMMARY

We start from the model of Janssen[1] (1992) and the papers of Ars & Janssen[2,3](1994, 1995), in which they developed some applications of the Janssen model of Asset Liability Management (ALM) to real life situations. We study an extension of the Janssen model in which the asset fund A takes into account fixed-income securities. Therefore, we take into account the rates of return of the asset portfolio, which we model by a Vasicek[4] process. The liability process B is defined by a geometric Brownian motion with drift which may be correlated with the asset process.

In this generalized Janssen model, we study the relations between the asset process A and the liability process B in order to point out some management principles. More exactly, we study the probability that the assets and liabilities of a company have no good matching and we propose a degree of the mismatching. Therefore, we look at the process $a = (a_t, t \geq 0)$ defined by $a_t = \ln\left(\frac{A_t}{B_t}\right)$ and at the first mismatching time $\tau = \inf\{t : 0 \leq t \leq T, a(t) \leq 0\}$. The determination of the probability of mismatching leads to the calculation of crossing probabilities $P[\tau < T]$. Only in special cases, explicit results are obtained and we turn to the approximations proposed by Durbin[5,6] (1985, 1992) and Sacerdote & Tomasetti[7] (1992). The degree of mismatching follows from option theory. These results are important as they are useful to determine ALM-strategies for insurance companies.

Keywords: stochastic differential equation, Ornstein-Uhlenbeck process, probability of mismatching, ALM, option theory.

1. INTRODUCTION

The last few years, the increasing importance of various risks associated with their financial activities has led many insurance companies to pay more and more importance to modern techniques of asset liability management largely introduced in banks as it is well known that the bad management of interest rates risks can lead to heavy financial losses and possibly requires a significant increase in the free reserve of the enterprise.

Our goal is to measure the riskiness of the insurance company by using a stochastic model of both the asset and the liability side of the balance and we consider the possibilities of perfect matching and partially matching. We propose an indicator of riskiness which we call the mismatching probability or mismatching degree. This information is interesting for the management of the company who can check whether they stay within the risk limits and can approve their strategies with respect to investment, reinsurance, pricing and acceptation of policies. This kind of information would also be useful for the determination of a contingency reserve or the solvency of a portfolio of insurance policies.

We do not propose these measures as an alternative of other ALM approaches but rather as a complement. Starting from a good database, we advise to use different ALM-tools like duration analysis, gap management, simulation and our mismatching probability and/or mismatching degree in order to obtain more useful information and a more complete idea of the situation of the company. The proposed measures of risk are also useful from the point of view of regulating authorities. In fact the goals of an insurance company and regulatory bodies are the same to a certain degree.

We start from the model of Janssen[1] (1992) which is symmetric in A and B and assumes geometric Brownian motions both for $A(t)$ and $B(t)$. We study an extension of the Janssen model in which the asset fund A takes into account fixed-income securities and this introduces asymmetry for A and B. This is particularly useful for insurance companies whose investments are more in bonds than in shares.

We suppose that the asset portfolio can be modeled by a fund containing only pure-discount bonds which reflect the rates of return of the asset portfolio in the past and with maturity the time horizon of the period that we are interested in. In this paper, we assume that the rates of return follow an Ornstein-Uhlenbeck process. Then the stochastic differential equation of the assets follow from the paper of Vasicek[4] (1979).

We further assume that the liability process B is defined by a geometric Brownian motion with drift, which is correlated with the asset process in a constant way.

In this generalized Janssen's model, we study the perfect matching and final matching of assets and liabilities by determining the probability of mismatching and the degree of mismatching.

To begin with, we present the generalized Janssen model in section 2. Section 3 is devoted to the study of probabilities of mismatching. In section 4, we concentrate on a degree of mismatching between the assets and the liabilities. Section 5 concludes the paper.

2. THE GENERALIZED JANSSEN MODEL

The most realistic model is to look at a portfolio of asset pools $A_1, A_2, \ldots, A_n$ with some segments containing only interest rate sensitive securities and some only shares. This model will be called the multidimensional model and will be treated in another paper[8].

First, we concentrate on a less realistic but more treatable model in order to obtain an increased understanding of different influences. Instead of dividing the assets up in different classes, we suppose that we can model the assets as one group of interest rate sensitive securities, reflecting the rates of returns of the asset portfolio in the past. Since insurance companies invest particularly in bonds, we model the asset portfolio by assuming that it contains N zero-coupon bonds which are modeled by the rates of returns which have been obtained by the portfolio over the last years.

The maturity $\tilde{T}$ of the bonds representing the asset portfolio certainly should be larger than or equal to the time horizon T if [0,T] is the period that we are interested in. In order to simplify the notations, we choose $\tilde{T}$=T. The results about the mismatching probabilities can easily be generalized to longer maturities. In case of the proposed risk measure of final mismatching for stochastic rates of return, however, it makes a difference whether $\tilde{T}$ = T or whether $\tilde{T}$ > T.

The rates of return are assumed to follow an Ornstein-Uhlenbeck process of the form

$$dr_t = \kappa(\theta - r_t)dt + \eta dZ_t,$$

where $(Z_t)_{t\geq 1}$ is a Brownian motion and where $\kappa, \theta, \eta \in \Re^+$. This model has the realistic property of being mean reverting towards the long term value θ where the speed of adjustment is determined by the parameter κ. The rates in the Vasicek model can be negative but in our opinion, negative rates of return are possible since assets can be invested in many different financial instruments.

We assume that financial markets are complete and frictionless and that trading takes place continuously. In this setting, Harrison and Kreps[9] (1979) have shown that there exists a unique risk-neutral probability.

Under these assumprions, the assets A_t, modeled by the investment in N pure-discount bonds with maturity T, are modeled by (see e.g. Vasicek's paper):

$$dA_t = A_t\left(r_t + \frac{\eta\lambda}{\kappa}\left(1 - e^{-\kappa(T-t)}\right)\right)dt - A_t\frac{\eta}{\kappa}\left(1 - e^{-\kappa(T-t)}\right)dZ_t$$

with λ the parameter of market risk and with A_T=N.

We model the liability process B by a lognormal process with positive constants μ_B, σ_B which is correlated with $(Z_t)_{t\geq 1}$. Cummins and Ney[10] (1980) argue that the lognormal distribution is a reasonable model for insurer liabilities if there is a good reinsurance program to hedge catastrophic jumps in the liabilities.

We now consider the process $a = (a_t, t \geq 0)$, which has been defined in Janssen[1] (1992), namely $a_t = \ln\left(\frac{A_t}{B_t}\right)$ and $a_0 = \ln\left(\frac{A_0}{B_0}\right)$. This process has the same meaning as the surplus process in risk theory. The stochastic differential equation of $a = (a_t, t \geq 0)$ follows from Ito's lemma:

Theorem 1

The stochastic process $a = (a_t, t \geq 0)$ *is a solution of the stochastic differential equation*

$$da_t = \mu(r_t, t, T)dt + \tilde{\sigma}(t, T)d\overline{W}_t$$

where

$$\mu(r_t, t, T) = r_t + \frac{\eta\lambda}{\kappa}\left(1 - e^{-\kappa(T-t)}\right) - \mu_B - \frac{\eta^2}{2\kappa^2}\left(1 - e^{-\kappa(T-t)}\right)^2 + \frac{\sigma_B^2}{2}$$

$$\tilde{\sigma}^2(t, T) = \frac{\eta^2}{\kappa^2}\left(1 - e^{-\kappa(T-t)}\right)^2 + \sigma_B^2 + \frac{2\eta}{\kappa}\left(1 - e^{-\kappa(T-t)}\right)\rho\sigma_B$$

and where $\overline{W} = (\overline{W}_t, t \geq 0)$ *denotes a standard Brownian motion.*

In the next section, we use this theorem to derive the probabilities of mismatching.

3. PROBABILITIES OF MISMATCHING

3.1 Perfect matching

Using the generalized Janssen model presented in the previous section, we study the relations between the assets process A and the liabilities process B in order to point out some management principles. We say that the assets and liabilities have no perfect match if for some $t \geq 0$ the asset value $A(t)$ becomes lower than the liability value $B(t)$

or equivalently if $a(t)$ becomes negative (see Janssen[1] (1992) and Ars & Janssen[2,3] (1994,1995)). Therefore, we define the first mismatching time in the period [0,T] as

$$\tau = \inf\{t : 0 \le t \le T, a(t) \le 0\}$$

or in case of the Ornstein-Uhlenbeck process

$$\tau = \inf\left\{ \begin{array}{l} t : 0 \le t \le T,\, a_0 + \int_0^t r_s ds + \int_0^t \tilde{\sigma}_s d\overline{W}_s + \left(\frac{\eta\lambda}{\kappa} - \frac{\eta^2}{2\kappa^2} - \mu_B + \frac{\sigma_B^2}{2} \right) t \\ \qquad + e^{-\kappa T}\left(\frac{\eta\lambda}{\kappa^2} - \frac{\eta^2}{\kappa^3} \right) + \frac{\eta^2}{4\kappa^3} e^{-2\kappa T} + e^{-\kappa(T-t)}\left(\frac{\eta^2}{\kappa^3} - \frac{\eta\lambda}{\kappa^2} \right) - \frac{\eta^2}{4\kappa^3} e^{-2\kappa(T-t)} \le 0 \end{array} \right\}$$

where we restrict ourselves to times smaller than T. We now concentrate on the crossing probabilities $P[\tau < T]$, which cannot be obtained explicitely in the general model. To obtain more insight, we first treat deterministic rates of return.

3.1.1 Special case: Non-stochastic rates of return

First, let us assume that the volatility coefficient η equals zero so that the rates of return are deterministic

$$r_t = e^{-\kappa t}(r_0 - \theta) + \theta.$$

In this case, we can rewrite the first mismatching time τ as

$$\tau = \inf\left\{ t : 0 \le t \le T,\, W_t \ge \frac{1}{\sigma_B}\left(a_0 + \frac{r_0 - \theta}{\kappa} - \left(\mu_B - \theta - \frac{\sigma_B^2}{2} \right) t + \frac{e^{-\kappa t}}{\kappa}(\theta - r_0) \right) \right\}.$$

a/ Constant rates of return

In order to be able to use the nice and well-known results in case of a Brownian motion, we concentrate first on the special case of constant rates of return r. Clearly, if $r_0 = \theta$, then $a = (a_t, t \ge 0)$ is a Brownian motion with drift and denoting $\mu = \theta - \mu_B + \frac{\sigma_B^2}{2}$ and $\sigma = -\sigma_B$, the results of Ars-Janssen[2,3] (1994, 1995) hold. Indeed,

$$da_t = \mu dt + \sigma dZ_t$$

and the first mismatching time equals

$$\begin{aligned} \tau &= \inf\{t : 0 \le t \le T, a(t) \le 0\} \\ &= \inf\left\{ t : 0 \le t \le T, \frac{a_0}{\sigma} \le W_t - \frac{\mu t}{\sigma} \right\}. \end{aligned}$$

The probability of no perfect match in the period [0, T] turns out to be (see for an overview e.g. Deelstra[11] (1994)):

$$P[\tau < T] = P\left[\sup_{0 \le t \le T} \left(W_t - \frac{\mu}{\sigma} t \right) \ge \frac{a_0}{\sigma} \right].$$

$$=1 \qquad \text{if } \frac{a_0}{\sigma} \le 0$$

$$= 1 - \int_{-\infty}^{\frac{a_0}{\sigma\sqrt{T}}+\frac{\mu\sqrt{T}}{\sigma}} \frac{e^{-u^2/2}}{\sqrt{2\pi}}du + e^{-2a_0\mu/\sigma^2} \int_{-\infty}^{\frac{-a_0}{\sigma\sqrt{T}}+\frac{\mu\sqrt{T}}{\sigma}} \frac{e^{-u^2/2}}{\sqrt{2\pi}}du \qquad \text{if } \frac{a_0}{\sigma} > 0.$$

Notice that the formula in case of $\frac{a_0}{\sigma} > 0$ can be expressed in terms of the cumulative Normal distribution function $\Phi(\ )$:

$$P[\tau < T] = 1 - \Phi\left(\frac{a_0}{\sigma\sqrt{T}} + \frac{\mu\sqrt{T}}{\sigma}\right) + e^{-2a_0\mu/\sigma^2}\Phi\left(\frac{-a_0}{\sigma\sqrt{T}} + \frac{\mu\sqrt{T}}{\sigma}\right).$$

An interesting measure of the mismatching risk is to calculate the probability of mismatching in the period [0, ∞[. For T tending to infinity, we consequently find that:

$$P[\tau < \infty] = e^{-2\mu a_0/\sigma^2} \qquad \mu, a_0 > 0$$
$$= 1 \qquad \mu \le 0 \ or\ a_0 \le 0.$$

Therefore, if μ is negative or a_0 is negative, then there will be no perfect match with probability 1. Otherwise, the probability of having at least once a mismatch equals $e^{-2\mu a_0/\sigma^2}$. This probability decreases if $a_0 = \ln\left(\frac{A_0}{B_0}\right)$ or $\mu = r - \mu_B + \frac{\sigma_B^2}{2}$ increases and/or $\sigma = \sigma_B$ decreases. So, the initial assets should be as large as possible in comparison with the liabilities. The instantaneous inflation and the volatility of the liabilities should be as low as possible. Indeed, if one starts with very low assets and high liabilities or with liabilities which are increasing very quickly, one can expect a mismatch. As motivated before, this information number, this indicator of mismatching can be interesting for the managers of the company, the regulators as well as the clients and everyone who has to deal with the insurance company because it is a measure of the risk position of the company.

Notice that even if μ is negative, then the probability of mismatching over the period [0,T] does not equal 1. But of course, in order to lower the probability of mismatching, the company should increase μ and try to keep μ positive.

b/ Time-dependent rates of return

If $r_0 \neq \theta$, the determination of the crossing probability $P[\tau < T]$ is not so easy since the drift term of $a = (a_t, t \ge 0)$ is time-dependent and therefore, we cannot rely on results about Brownian motions crossing (piecewise) linear boundaries. The time of first mismatching is the crossing time of a standard Brownian motion to a boundary $l(t)$ which is wholly convex for $r_0 < \theta$, and wholly concave for $r_0 > \theta$. Therefore, we can apply the results of Durbin[5] (with an appendix by Williams) (1992). It was shown in Durbin[6] (1985) that the first-passage density $p(t)$ of $W(u)$ to a boundary $l(u)$ at time $u = t$ is

$$p(t) = b(t).\, f(t) \qquad 0 < t < T$$

where f(t) is the density of W(t) on the boundary, i.e.

$$f(t)=\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{l(t)^2}{2t}\right)$$

and where

$$b(t)=\lim_{s\uparrow t}\frac{1}{t-s}E\left[I_{(s,W)}\left(l(s)-W(s)\right)\middle|W(t)=l(t)\right]$$

with $I_{(s,W)}$ an indicator function which is equal to 1 if the sample path does not cross the boundary prior to s and equal to 0 otherwise. As b(t) usually is not computable in a direct way, Durbin (1992) expands the first-passage density $p(t)$ of $W(u)$ to $l(u)$ at $u=t$ as a series of multiple integrals, namely

$$p(t)=\sum_{j=1}^{k}(-1)^{j-1}q_j(t)+(-1)^k\tilde{r}_k(t)\quad k=1,2,\ldots$$

where

$$q_1(t)=\left[\frac{l(t)}{t}-l'(t)\right]f(t),$$

$$q_2(t)=\int_0^t\left[\frac{l(t)}{t}-l'(t)\right]\left[\frac{l(t)-l(t_1)}{t-t_1}-l'(t)\right]f(t_1,t)dt_1,$$

$$q_j(t)=\int_0^t\int_0^{t_1}\ldots\int_0^{t_{j-2}}\left[\frac{l(t_{j-1})}{t_{j-1}}-l'(t_{j-1})\right]$$

$$\times\prod_{i=1}^{j-1}\left[\frac{l(t_{i-1})-l(t_i)}{t_{i-1}-t_i}-l'(t_{i-1})\right]f(t_{j-1},\ldots,t_1,t)dt_{j-1}\ldots dt_1\qquad j>2$$

with $f(t_{j-1},...,t_1,t)$ is the joint density of $W(t_{j-1}),\ldots,W(t_1),W(t)$ on the boundary, i.e. at values $l(t_{j-1}),\ldots,l(t_1),l(t)$ and where

$$\tilde{r}_k(t)=\int_0^t\int_0^{t_1}\ldots\int_0^{t_{k-1}}b(t_k)$$

$$\times\prod_{i=1}^{k}\left[\frac{l(t_{i-1})-l(t_i)}{t_{i-1}-t_i}-l'(t_{i-1})\right]f(t_k,\ldots,t_1,t)dt_k\ldots dt_1.$$

By truncating the series, one obtains the successive approximations:

$$p_k(t)=\sum_{j=1}^{k}(-1)^{j-1}q_j(t)\qquad k=1,2,\ldots.$$

If $l(t)$ is concave everywhere, thus $r_0>\theta$, the error $\tilde{r}_k(t)$ in the k-the approximation $p_k(t)$ is less than the last computed term $q_k(t)$ and less than the next term $q_{k+1}(t)$.

If the boundary is wholly convex, then the error is bounded from above:

$$\left|\tilde{r}_k(t)\right|\le\left|u_k(t)\right|\qquad k=1,2,\ldots$$

where

$$u_k(t)=\int_0^t\int_0^{t_1}\ldots\int_0^{t_{k-1}}\frac{l(0)}{t_k}\times\prod_{i=1}^{k}\left[\frac{l(t_{i-1})-l(t_i)}{t_{i-1}-t_i}-l'(t_{i-1})\right]f(t_k,\ldots,t_1,t)dt_k\ldots dt_1.$$

The probability that a sample path of W(t) crosses the boundary at least once in the interval [0,T], namely $P = \int_0^T p(t)dt$, can be approximated by

$$P_k = \int_0^T p_k(t)\,dt \qquad k = 1,2,\ldots.$$

As for the first-passage density, in the wholly concave case the error R_k is bounded by

$$Q_j = \int_0^T q_j(t)dt$$

for both j=k and j=k+1, while for the wholly convex case $|R_k|$ is bounded by

$$\left|\int_0^T u_k(t)dt\right|.$$

These formulae easily can be programmed.

Sacerdote & Tomassetti (1996) propose also approximations for the first passage probabilities and indicate error bounds by using a series expansion for the solution of the integral equation for the first-passage time probability density function. However, in order to apply these results, a hypothesis has to be fulfilled which clearly depends on the parameters.

3.1.2 General case: Ornstein-Uhlenbeck process

Let us now concentrate on the first mismatching time τ in the case of stochastic rates of return, modeled by an Ornstein-Uhlenbeck process with $\eta \neq 0$. Some simple calculations show that τ can be rewritten as

$$\tau = \inf\left\{\begin{array}{l} t : 0 \le t \le T, y(t) \equiv \int_0^t \left(E\left[r_s\right] - r_s\right)ds - \int_0^t \tilde{\sigma}_s d\overline{W}_s \ge \\ \qquad a_0 + \frac{r_0 - \vartheta}{\kappa} + e^{-\kappa T}\left(\frac{\eta\lambda}{\kappa^2} - \frac{\eta^2}{\kappa^3}\right) + \frac{\eta^2}{4\kappa^3}e^{-2\kappa T} \\ \qquad + \left(\frac{\eta\lambda}{\kappa} - \frac{\eta^2}{2\kappa^2} - \mu_B + \vartheta + \frac{\sigma_B^2}{2}\right)t + e^{-\kappa t}\frac{\vartheta - r_0}{\kappa} \\ \qquad + e^{-\kappa(T-t)}\left(\frac{\eta^2}{\kappa^3} - \frac{\eta\lambda}{\kappa^2}\right) - \frac{\eta^2}{4\kappa^3}e^{-2\kappa(T-t)} \equiv l(t) \end{array}\right\}.$$

with $\tilde{\sigma}^2(t,T) = \frac{\eta^2}{\kappa^2}\left(1 - e^{-\kappa(T-t)}\right)^2 + \sigma_B^2 + \frac{2\eta}{\kappa}\left(1 - e^{-\kappa(T-t)}\right)\rho\sigma_B$.

Expressed this way, we see that we have to compute the first-passage density from below of the continuous Gaussian process $y = (y(t); 0 \le t \le T)$ with $y(t) = -\int_0^t e^{-\kappa s}\int_0^s \eta e^{\kappa l} dZ_l ds - \int_0^t \tilde{\sigma}_s d\overline{W}_s$ to the boundary $l(t)$. Neither the process $y = (y(t); 0 \le t \le T)$ nor $l(t)$ satisfies the assumptions of Durbin[6] (1992) or Sacerdote &

Tomassetti[7] (1996). Therefore, we turn to the approximations of Durbin[5] (1985), although in this paper no error-bounds are given.

Under mild restrictions on $l(t)$ and on the covariance function cov(y(u),y(v)), Durbin (1985) derives approximations for the crossing probabilities and the first-passage density $p(t)$ of a continuous Gaussian process y(t) at a boundary $l(t)$ at $u = t$. Long calculations show that the covariance function cov(y(u),y(v)), which we denote by $\rho(u,v)$, equals:

$$\rho(u,v) = E\left[\left(-\int_0^u e^{-\kappa s}\int_0^s \eta e^{\kappa l} dZ_l ds - \int_0^u \tilde{\sigma}_s d\overline{W}_s\right).\left(-\int_0^v e^{-\kappa t}\int_0^t \eta e^{\kappa j} dZ_j dt - \int_0^v \tilde{\sigma}_t d\overline{W}_t\right)\right]$$

$$= -\frac{\eta^2}{\kappa^3} + \frac{2\eta\sigma_B\varphi}{\kappa^2} e^{-\kappa T} - \frac{\eta^2}{2\kappa^3} e^{-2\kappa T} + \min(u,v)\sigma_B^2$$

$$+ e^{-\kappa\min(u,v)}\frac{\eta^2}{\kappa^3} + e^{2\kappa\min(u,v)}\left[\frac{\eta^2}{2\kappa^3} e^{-2\kappa T} - \frac{\eta^2}{2\kappa^3}\left(e^{-\kappa(u+T)} + e^{-\kappa(v+T)}\right)\right]$$

$$- \frac{\eta^2}{2\kappa^3} e^{-\kappa\max(u,v)}\left[e^{\kappa\min(u,v)} + e^{-\kappa\min(u,v)} - 2\right] - \frac{2\eta\sigma_B\varphi}{\kappa^2} e^{\kappa(\min(u,v)-T)}$$

$$+ (e^{-\kappa v} + e^{-\kappa u})\left[\frac{\eta^2}{\kappa^3}\left(e^{\kappa\min(u,v)} - 1 + \frac{e^{-\kappa T}}{2}\right) + \frac{\sigma_B\eta\varphi}{\kappa^2}\left(e^{\kappa\min(u,v)} - 1\right)\right].$$

It is easy to verify that the assumptions of Durbin's paper (1985) are fulfilled and therefore, we may apply the approximations proposed in his paper. A first approximation P_g for the crossing probability is:

$$P_g = \left(\frac{\left.\frac{\partial\rho(u,t)}{\partial u}\right|_{u=t}}{\rho(t,t)} - \frac{l'(t)}{l(t)}\right).\left(-\frac{2\rho(t,t)/l^2(t)}{\left.\frac{d^2}{du^2}\left(\frac{\rho(u,u)}{l^2(u)}\right)\right|_{u=t}}\right)^{1/2}.\exp\left(\frac{-l^2(t,t)}{2\rho(t,t)}\right)$$

where t is such that $\frac{\rho(u,u)}{l^2(u)}$ is maximized.

Another approximation P_1 of the no-perfect-match probability $P[\tau < T]$ proposed by Durbin (1985) is:

$$P_1(T) = \int_0^T p_1(t)dt$$

where

$$p_1(t) = \frac{l(t)}{\rho(t,t)}\left.\frac{\partial\rho(u,t)}{\partial u}\right|_{u=t} - l'(t) \qquad u \le t.$$

A last approximation uses this expression, namely

$$P_2(T) = \int_0^T p_2(t)dt$$

where

$$p_2(t) = p_1(t) + \int_0^t \left[l'(t) - \beta_1(r,t)l(r) - \beta_2(r,t)l(r)\right] f(t|r) p_1(r)dr$$

with

$$\begin{pmatrix} \beta_1(r,t) \\ \beta_2(r,t) \end{pmatrix} = \begin{pmatrix} \rho(r,r) & \rho(r,t) \\ \rho(r,t) & \rho(t,t) \end{pmatrix}^{-1} \begin{pmatrix} \left.\dfrac{\partial\rho(r,u)}{\partial u}\right|_{u=t} \\ \left.\dfrac{\partial\rho(s,t)}{\partial s}\right|_{s=t} \end{pmatrix}$$

and where f(t|r) is the conditional density of y(t) at l(t) given that y(r) = l(r).

The first approximation P_g is the least accurate and there may arise some problems with finding the maximizing value t. The last approximation P_2 appears to be the most accurate but involves more calculations. Therefore, we suggest to use the approximation P_1.

3.2 Final mismatching

In practice, perfect matching of insurance liabilities might be too demanding since low-risk investment strategies associated with the highest degree of matching possible usually produce lower expected returns. Therefore, we also observe final matching which means that we only check whether the assets cover the liabilities at the end of the period [0,T]: A(T)>B(T). Therefore, the probability of no final matching is the probability

$$P\left[A_T < B_T\right] = P\left[a_T < 0\right]$$

where $(a_t)_{t\geq 0}$ is the process of mismatching defined above and this probability follows from the distribution of a_T.

From theorem 1, it is easy to see that $(a_t)_{t\geq 0}$ is a Gaussian process since

$$a_t = a_0 + \int_0^t r_s ds + \int_0^t \frac{\eta\lambda}{\kappa}\left(1 - e^{-\kappa(T-s)}\right)ds - \mu_B t - \frac{\eta^2}{2\kappa^2}\int_0^t \left(1 - e^{-\kappa(T-s)}\right)^2 ds + \frac{\sigma_B^2}{2}t + \int_0^t \tilde{\sigma}(s,t)d\overline{W}_s$$

with the rates of return following an Ornstein-Uhlenbeck process, thus

$$\int_0^t r_s ds \sim \mathrm{N}\left(\theta t + \frac{r_0 - \theta}{\kappa}\left(1 - e^{-\kappa t}\right), \frac{\eta^2}{\kappa^2}t + \frac{2\eta^2}{\kappa^3}e^{-\kappa t} - \frac{\eta^2}{2\kappa^3}e^{-2\kappa t} - \frac{3\eta^2}{2\kappa^3}\right).$$

Therefore, a_T has a Normal distribution with mean

$$m(T) = a_0 + \frac{r_0 - \vartheta}{\kappa} + e^{-\kappa T}\left(\frac{\eta\lambda}{\kappa^2} - \frac{\eta^2}{\kappa^3}\right) + \frac{\eta^2}{4\kappa^3} e^{-2\kappa T}$$
$$+\left(\frac{\eta\lambda}{\kappa} - \frac{\eta^2}{2\kappa^2} - \mu_B + \vartheta + \frac{\sigma_B^2}{2}\right)T + e^{-\kappa T}\frac{\vartheta - r_0}{\kappa}$$
$$+\left(\frac{\eta^2}{\kappa^3} - \frac{\eta\lambda}{\kappa^2}\right) - \frac{\eta^2}{4\kappa^3}$$

and with variance

$$\sigma^2(T) = -\frac{\eta^2}{\kappa^3} + \frac{2\eta\sigma_B\varphi}{\kappa^2} e^{-\kappa T} - \frac{\eta^2}{2\kappa^3} e^{-2\kappa T} + \sigma_B^2 T$$
$$+ e^{-\kappa T}\frac{\eta^2}{\kappa^3} + -\frac{\eta^2}{2\kappa^3} - \frac{\eta^2}{2\kappa^3} e^{-\kappa T}\left[e^{\kappa T} + e^{-\kappa T} - 2\right] - \frac{2\eta\sigma_B\varphi}{\kappa^2}$$
$$+ 2e^{-\kappa T}\left[\frac{\eta^2}{\kappa^3}\left(e^{\kappa T} - 1 + \frac{e^{-\kappa T}}{2}\right) + \frac{\sigma_B\eta\varphi}{\kappa^2}\left(e^{\kappa T} - 1\right)\right].$$

Remark that the mean is l(T) of the previous section and that the expression of the variance follows from a substitution of u=v=T in the covariance function $\rho(u,v)$ also presented in the previous section.

We conclude that the probability of no final matching equals

$$P\left[A_T < B_T\right] = P\left[a_T < 0\right] = 1 - \Phi\left(\frac{m(T)}{\sigma(T)}\right)$$

with m(T) and $\sigma(T)$ as above and where $\Phi(z)$ denotes the cumulative standard normal distribution function in z.

In case of deterministic rates of return, the expressions for the mean and variance simplify and the probability of no final matching equals

$$P\left[A_T < B_T\right] = P\left[a_T < 0\right] = 1 - \Phi\left(\frac{a_0 + \frac{1 - e^{-\kappa T}}{\kappa}(r_0 - \theta) + \left(\theta - \mu_B + \frac{\sigma_B^2}{2}\right)T}{\sigma_B\sqrt{T}}\right).$$

Remark that perfect matching implies final matching since final matching puts only a restriction on the portfolio at time T and therefore the probability of no final matching is always lower than the probability of no perfect match.

4. MISMATCHING DEGREE

In the case of no final matching, we propose a risk measure of final matching which gives an idea of the difference between liabilities and assets at the time horizon T. We use the approach of Cummins[12] (1988) in his calculation of risk-based premiums and of Kusakabe[13] (1995) in his discrete ALM model; and we propose as a measure of risk at time t:

$$M_t(B_T - A_T) = E\left[\left(B_T - A_T\right)^+ e^{-\int_t^T i_u du} \mid F_t\right]$$

with $(i_t)_{t\geq 0}$ modeling the short-term interest rates, with F_t the sigma-field of information until time t and where the conditional expectation is taken with respect to the risk-neutral probability. In the case that the assets are higher than the liabilities, the risk measure thus equals zero.

At time T itself, we know that the measure M_T equals

$$(B_T - A_T)^+ = \max(B_T - A_T, 0).$$

The value at time t can be obtained by using techniques from option theory and in particular from the formulae of Black & Scholes[14] (1973), Merton[15] (1973) and/or Rabinovitch[16] (1989).

Indeed, it is well-known that the value of a call option at time t which gives the right (but not the obligation) "to buy" at time T the liabilities B_T, modeled by the geometric Brownian motion

$$dB_t = \mu_B B_t dt + \sigma_B B_t dW_t,$$

at the exercise value K=A_T of the assets at time T, equals

$$E\left[\left(B_T - A_T\right)^+ e^{-\int_t^T i_u du} \mid F_t\right]$$

where the conditional expectation is taken with respect to the risk-neutral probability.

If we assume that the interest rates are constant and $\tilde{T}$ = T, we can use the well-known Black & Scholes (1973) formula:

$$M_t(B_T - A_T) = e^{-i(T-t)} E\left[\left(B_T - A_T\right)^+ \mid F_t\right] = B_t \Phi(z) - e^{-i(T-t)} K \Phi(z - \sigma_B \sqrt{T-t})$$

with

$$z = \frac{\log\left(B_t/K\right) + \left(i + \sigma_B^2/2\right)(T-t)}{\sigma_B \sqrt{T-t}}$$

with K=A_T=N and where $\Phi(z)$ denotes the cumulative standard normal distribution function in z. If we are interested at time 0 in the risk measure of no-final-match, we just have to plug in t=0.

Remark that the assumption of constant interest rates is not necessary. The interest rates may be stochastic. Then the value of the risk measure follows from generalizations of the Black & Scholes formula obtained by e.g. Merton[15] (1973) and Rabinovitch[16] (1989).

Merton[15] (1973) extended the Black & Scholes formulae to the case of stochastic interest rates which are such that the zero-coupon bonds are determined by a stochastic differential equation of the form

$$dP_t = P_t \nu(t)dt + P_t \delta(t)dZ_t$$

with the bond and stock prices correlated by $E[dZ_t\, dW_t] = \rho dt$. Using this notation and with $\tilde{T}$ = T, the Merton (1973) formula implies that the risk measure at time t equals

$$M_t(B_T - A_T) = B_t\Phi(z) - A_T P(T-t)\Phi(z - \overline{V(T-t)})$$

with

$$z = \frac{\log\left(B_t/K\right) - \log(P(T-t))}{\overline{V(T-t)}} + \frac{1}{2}\overline{V(T-t)}$$

and where

$$\overline{V(T-t)} = \sqrt{\sigma_B^2(T-t) + \int_0^{T-t} \delta^2(s)ds - 2\sigma_B\rho\int_0^{T-t}\delta(s)ds}\,.$$

In case the short-term interest rates are modeled by a mean-reverting Ornstein-Uhlenbeck process of the form

$$di_t = q(m - i_t)dt + \omega dZ_t$$

with $E[dZ_t\, dW_t] = \rho dt$ describing the correlation between the short-term interest rates and the return on the liabilities, this formula leads to an explicit expression (see Rabinovitch (1989)). Indeed, a default-free discount bond P that matures at the time horizon T is priced in this model by the formula (see e.g. Vasicek (1977)):

$$P(T-t) = G \cdot \exp[-i_t H]$$

where

$$H \equiv H(T-t) = \frac{1 - \exp[-q(T-t)]}{q}$$

and

$$G \equiv G(T-t) = \exp\left[\left(m + \frac{\omega\lambda}{q} - \frac{\omega^2}{2q^2}\right)(H - T + t) - \frac{\omega^2 H^2}{4q}\right]$$

with constant market price of risk λ. Using Itô's lemma, it is known that the instantaneous return variance of the bond δ is a function of time, namely $\delta(t) = \omega H(t)$.

Using this expression for δ, Rabinovitch rewrites Merton's (1973) formula for the call value with given exercise price K=A_T=N for interest rates *i* modeled by a Vasicek process:

$$M_t(B_T - A_T) = E\left[(B_T - A_T)^+ e^{-\int_t^T i_u du} \mid F_t\right] = B_t\Phi(z) - A_T P(T-t)\Phi(z - \overline{V(T-t)})$$

with

$$z = \frac{\log\left(B_t/K\right) - \log(P(T-t))}{\overline{V(T-t)}} + \frac{1}{2}\overline{V(T-t)}$$

and where

$$\overline{V(T-t)}^2 = \sigma_B^2(T-t) + \frac{\omega^2}{q^2}\left(T - t - 2H + \frac{1-\exp(-2q(T-t))}{2q}\right) - \frac{2\rho\sigma_B\omega}{q}(T-t-H).$$

Substituting t=0, delivers us the risk measure at time t=0 of the expected deficit at time T, i.e. the expected value of the difference between the liabilities and the assets when there is no final match.

If $\tilde{T}$ > T, then the results remain the same in the case with deterministic rate of return with

$$K = A_T = A_0 \exp\left(\theta T + \frac{r-\theta}{\kappa}\left(1 - e^{-\kappa T}\right)\right).$$

In the general case, however, the risk measure of no final match has to be determined numerically since now not only the liabilities B_T but also the assets A_T at time T are random.

4. CONCLUSIONS

We have sucessfully extended the Janssen model in such a way that the asset fund A takes into account fixed-income securities. This is important for insurance companies whose investments are more in bonds than in shares, especially for life-insurance companies.

We have considered a treatable model in which we assume that the assets can be represented by only zero-coupon bonds which reflect the historical rates of return. Those rates of return of the portfolio in the past are supposed to be presented by an Ornstein-Uhlenbeck process. In this generalized Janssen model, we have studied the probability of mismatching of the assets and liabilities of the company in a period [0,T] by introducing the first mismatching time $\tau = \inf\{t : 0 \le t \le T, a(t) \le 0\}$ where $a = (a_t, t \ge 0)$ is defined by $a_t = \ln\left(\frac{A_t}{B_t}\right)$ and where T can be assumed to be infinity. Further, we have proposed a risk measure of no final matching which indicates the difference between the assets and the liabilities at time T.

These results are important as they are useful to determine ALM-objectives to be achieved by the company. In a forthcoming paper, we will study a more realistic multi-dimensional model and develop some tools needed to encounter these objectives.

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