The Role of the Dependence between Mortality and Interest Rates when pricing Guaranteed Annuity Options

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Abstract

In this paper we investigate the consequences on the pricing of insurance contingent claims when we relax the typical independence assumption made in the actuarial literature between mortality risk and interest rate risk. Starting from the Gaussian approach of Liu et al. (2014), we consider some multifactor models for the mortality and interest rates based on more general affine models which remain positive and we derive pricing formulas for insurance contracts like Guaranteed Annuity Options (GAOs). In a Wishart affine model, which allows for a non-trivial dependence between the mortality and the interest rates, we go far beyond the results found in the Gaussian case by Liu et al. (2014), where the value of these insurance contracts can be explained only in terms of the initial pairwise linear correlation.

Key words: Stochastic mortality; affine interest rate models; dependence; guaranteed annuity options; Wishart process.

JEL codes: G13; G22.

1 Introduction

A large number of life insurance products, such as annuities, include interest rate and mortality risks. Mortality risk is generally considered to be less difficult to be modelled than interest risk. Indeed, by virtue of the law of large numbers, actuarial practice considered for a long time that the mortality risk can be diversified away by holding a sufficiently large portfolio of similar contracts (see e.g. Milevsky and Young (2007) for a discussion). Therefore, the traditional approach of actuaries consisted in modeling mortality in a deterministic way as opposed to interest rates which were assumed stochastic for quite some time now. Later, when stochastic mortality models showed up during the 90's, the actuarial community made the assumption that mortality risk is independent of interest risk. This assumption may seem acceptable in the short term. Nevertheless, it seems also natural that catastrophic risks, like an earthquake or a severe pandemic, will affect the economy and the financial markets in the short run. Furthermore, in the long term, it seems intuitive that demographic changes can affect the economy. For example, in Favero et al. (2011), the authors investigate the possibility that the slowly evolving mean in the log dividend-price ratio is related to demographic trends. Maurer (2014) explores how demographic changes affect the value of financial assets. He considers a continuous time overlapping generations model where birth and mortality rates are stochastic. His model suggests that demographic transitions explain substantial parts of the time variation in the real interest rate, equity premium and conditional stock price volatility. Moreover, he gives sufficient conditions for the interest rate to be decreasing in the birth rate and increasing in the death rate. In Dacorogna and Cadena (2015), the authors are interested in providing some empirical evidence of a changing behavior of the economy and the financial markets during periods where the mortality is relatively high. Another motivation of dependence between the mortality and interest rates can be found in Nicolini (2004) where it is shown that the increase in adult life expectancy in the 17th and 18th century can be considered a key factor in explaining the increase in the accumulation of assets and the decline in the interest rate that took place in pre-industrial England. To conclude, we mention that Dhaene et al. (2013) investigate the conditions under which it is possible (or not) to transfer the independence assumption from the physical world to the pricing world. In particular, they show that this independence relation is not maintained in general. Therefore, as also suggested by Miltersen and Persson (2005) and Liu et al. (2014), it is more reasonable to have a pricing framework that allows for a dependence between mortality and interest rates.

Nowadays, it is widely admitted that mortality intensities behave in a stochastic way. There are important similarities between the force of mortality and interest rates (see e.g. Milevsky and Promislow (2001), Dahl (2004), and Biffis (2005)). Luciano and Vigna (2008) calibrate some affine models where the force of mortality follows either a Cox-Ingersoll-Ross (CIR) process or an Ornstein-Ulhenbeck process to different generations in the UK population and investigate their empirical relevance. They find that Ornstein-Ulhenbeck processes (with zero long-term mean) seem to be appropriate descriptors of human mortality. Dahl (2004), Dahl and Moller (2006) and Dahl et al. (2008) treat the classical affine situation for both stochastic mortality rates and interest rates, but in the independent situation under both the historical and the real world measure. Russo et al. (2011) focus upon the calibration of affine stochastic mortality models using term insurance premiums.

In this paper we assume that the interest rate and the mortality dynamics are not independent of each other. More precisely, we consider a general affine framework like in Keller-Ressel and Mayerhofer (2015); our goal is to study the influence in pricing of their dependence structure when focusing upon mortality and interest rates in different models constructed by a linear combination of positive idiosyncratic and systematic processes, inspired by factor models as in Duffie and Kan (1996) and Duffie and Garleanu (2001). These linear combinations can be chosen so that the model either avoids or allows interest rates and/or mortality rates to be negative. For mortality rates, it is a desirable property to remain positive. However, the financial markets show the recent years negative interest rates, see e.g. Borovkova (2016), Recchioni and Sun (2016) or Russo and Fabozzi (2016). Although the spreads between the LIBOR rates and the overnight indexed swap (OIS) rates of the same maturity have been far from negligible since 2007, and several multiple curve interest rate models have been introduced (see e.g. (Grbac and Runggaldier, 2015)), we choose in the present paper the interest rate models still from the traditional single-curve models in order to study the influence of the dependence between interest rates and mortality rates.

In particular, generalizing the investigations of Liu et al. (2014), we are interested in two multifactor specifications that are nested in the general affine framework, namely the multi-CIR and the Wishart model that have been successfully applied to many fields of quantitative finance. Wishart processes have been first defined by Bru (1991) and recently introduced in finance by Gouriéroux and Sufana (2003, 2011). They represent a matrix extension of the square-root model that allows for a non trivial correlation between the diagonal terms which are by definition positive (see e.g. Cuchiero et al. (2011) for a complete characterization). This property enables to overcome an intrinsic constraint of the standard affine Duffie and Kan (1996) model. In addition, the affine property of the Wishart process leads to a closed form expression for its moment-generating function, so that pricing within the Wishart framework can be efficiently performed via Fourier methods[1].

From the empirical side, the advantages of interest rate models based on the Wishart process have been underlined by Buraschi et al. (2008) and Chiarella et al. (2016). The latter extends the former by estimating, using Kalman filtering techniques, an extended version of the classical Wishart model, together with a multifactor CIR model. The

[1] The Wishart process has found applications to many fields of quantitative finance like multivariate option pricing (see e.g. Da Fonseca et al. (2007, 2008); Da Fonseca and Grasselli (2011)), yield curve modeling (see e.g. Buraschi et al. (2008); Gnoatto (2012); Chiarella et al. (2014); Da Fonseca et al. (2013)), credit risk (Gouriéroux and Sufana (2010)), portfolio management (Buraschi et al. (2010), Da Fonseca et al. (2011)), commodity derivative pricing (Chiu et al. (2015)) and foreign exchange models (Leung et al. (2013); Branger and Muck (2012); Gnoatto and Grasselli (2014)).

Wishart based model was found to outperform the multifactor CIR both in terms of goodness of fit and hedging performance.

To the best of our knowledge, the first attempt to introduce dependence between mortality and interest rates, was done by Miltersen and Persson (2005), see also Cairns et al. (2006). We follow the methodology of Jalen and Mamon (2009) who introduced a pricing framework in which the dependence between the mortality and the interest rates is explicitely modelled. We apply the same change of probability measure to the valuation of some insurance contracts such as indexed annuities and Guaranteed Annuity Options (GAOs). These options are available to holders of certain life insurance policies and give them the right to convert their accumulated funds to a life annuity at a fixed rate when the policy matures. These kind of guarantees became very popular in the 1970's and 1980's when long term rates in many countries were quite high. We mention that in Liu et al. (2014), a pricing formula for the GAO has been obtained where the interest rate and mortality processes follow bivariate Gaussian dynamics. In their setting, the dependence between mortality and interest rates is described just by one constant, namely the pairwise linear correlation coefficient. We will show that in (positive) structures such as the Wishart framework the aforementioned dependence shows richer features.

The remainder of this paper is organized as follows. In Section 2, we present our model in a general affine approach and we define the change of probability measure used in the sequel for the pricing purpose. In Section 3, we concentrate upon some insurance products and we present different ways for determining their fair values in our setting. In Section 4, we specify the dynamics of the affine processes in two important specifications: the multidimensional CIR process and the Wishart case. For both settings we derive formulas for the price of the insurance contracts of Section 3. In Section 5 we perform a sensitivity analysis with respect to parameters that rule the dependence structure between interest rates and mortality risks. We provide concluding remarks in Section 6. Finally, we gather in the Appendix some technical results and a brief overview on affine processes.

2 The general pricing model

We consider a filtered probability space $(\Omega,\ \mathcal{G},\ (\mathcal{G}_t)_t,\ \mathbb{Q})$ where $\mathbb{Q}$ is a risk-neutral martingale measure. By the presence of both mortality and interest rate risk, we are dealing with a pricing problem in an incomplete market. Following the standard practice, we assume that the right risk-neutral probability can be selected on the basis of e.g. available market data (for another approach about the non-diversifiable mortality risk in an incomplete market, see e.g. Bayraktar et al. (2009)).

We denote by $\tau_M(x)$ the positive random variable corresponding to the future lifetime of an individual aged x at time 0, admitting a random intensity $\mu(s, x+s)$. As in e.g. Biffis (2005), we regard $\tau_M(x)$ as the first jump-time of a nonexplosive $\mathcal{G}$-counting process N recording at each time $t \geq 0$ whether the individual died ($N_t \neq 0$) or not ($N_t = 0$). Let $\mathcal{R}_t$ be the filtration generated by the interest rate process and $\mathcal{M}_t$ the one associated to the mortality intensity. We denote by $\mathcal{F}_t := \mathcal{R}_t \vee \mathcal{M}_t$ the minimal σ-algebra containing $\mathcal{R}_t \cup \mathcal{M}_t$. The filtration $(\mathcal{G}_t)_t$ denotes the flow of information available as time goes by: this includes knowledge of the evolution of both state variables above up to each time t and of whether the policyholder has died by then. Therefore, $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t$, with

$$\mathcal{H}_t := \sigma(1_{\{\tau_M(x) \leq s; 0 \leq s \leq t\}}),$$

being the smallest filtration with respect to which $\tau_M(x)$ is a stopping time. We further assume[2] that N is a doubly stochastic process or Cox process driven by the subfiltration $\mathcal{F}$ of $\mathcal{G}$, which implies that for all $0 \leq t \leq T$ and $k \in \mathbb{N}$:

$$\mathbb{Q}\left(N_T - N_t = k \mid \mathcal{F}_T \vee \mathcal{H}_t\right) = \frac{\left(\int_t^T \mu(s, x+s)ds\right)^k \exp\left(-\int_t^T \mu(s, x+s)ds\right)}{k!}. \quad (1)$$

The idea behind the specification of $\mathcal{F}$ is that it provides enough information about the evolution of the interest rates and the intensity of mortality, i.e. about the likelihood of death happening, but not enough information about the actual occurrence of death. Such information is carried by the larger filtration $\mathcal{G}$, with respect to which $\tau_M(x)$ is a stopping time. The application of the law of iterated expectations and the use of (1) with $k = 0$ yield that the 'probability of survival' up to time $T \geq t$, on the set $\{\tau_M(x) > t\}$, is given by

$$\mathbb{Q}(\tau_M(x) > T|\mathcal{G}_t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \mu(s,x+s)ds}|\mathcal{G}_t\right].$$

We notice that this kind of reasonings is widely used in the credit risk literature, see e.g. Duffie and Singleton (1999).

Given a time-homogeneous affine Markov process X taking values in a non empty convex subset E of $\mathbb{R}^d$ ($d \geq 1$), endowed with the inner product $\langle .,. \rangle$, we assume that the dynamics of the force of mortality $\mu(t, x+t)$ are given by

$$\mu(t, x+t) = \bar{\mu}^x + \langle M^x, X_t \rangle, \quad (2)$$

and the dynamics of interest rate r_t by

$$r_t = \bar{r} + \langle R, X_t \rangle, \quad (3)$$

for some constants $\bar{r}$, $\bar{\mu^x} \in \mathbb{R}$ and M^x, R in $\mathbb{R}^d$, that is mortality and interest rate are linear projections of the common stochastic factor X along constant directions given by

[2]Notice that Biffis (2005) uses an analogous hypothesis under the assumption of independence of mortality and interest rates.

the parameters M^x, R. We will be interested in the cases where X is a classical affine process with state space $\mathbb{R}_+^m \times \mathbb{R}^n$ or an (affine) Wishart process on the state space S_d^+ (the set of $d \times d$ symmetric positive definite matrices). When R and M^x are vectors in $\mathbb{R}^d$ (resp. matrices in M_d), the corresponding inner product between R and M^x is the scalar product (resp. the trace of the matrix product RM^x).
In the following, if there is no confusion about the age x, we will denote $\mu(t, x+t)$ by μ_t and we will omit the superscript or argument x in the notations of $\bar{\mu}$, M and τ_M.

At this stage we remain generic on the process X: in the Appendix A we recall the definition of affine process as in Keller-Ressel and Mayerhofer (2015) who provide a unified approach including the classical affine and the Wishart specification for the process X.

We are interested in calculating an insurance contingent claim paying a single benefit C_T upon survival of the insured (with age x at time 0) at time T, which is $\mathcal{F}_T$-measurable. Using the risk-neutral pricing approach, this basic insurance contract depending upon (possibly correlated) mortality and interest rates has the following value at time t, which we denote by $SB_t(C_T;T)$ as in Biffis (2005)

$$SB_t(C_T;T) \quad = \quad \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} 1_{\{\tau_M > T\}} C_T | \mathcal{G}_t\right].$$

Using the law of iterated expectations and the facts that $\{r_t\}_t$ is adapted with respect to the filtration $\{\mathcal{F}_t\}_t$ and that C_T is $\mathcal{F}_T$-measurable, we can write:

$$SB_t(C_T;T) \quad = \quad \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} C_T \mathbb{E}^{\mathbb{Q}}\left[1_{\{\tau_M > T\}} | \mathcal{G}_t \vee \mathcal{F}_T\right] | \mathcal{G}_t\right].$$

From this, we note that $SB_t(C_T;T)$ is zero on the set $\{\tau_M \leq t\}$. Focusing on the set $\{\tau_M > t\}$, we exploit the doubly stochastic property, obtaining:

$$SB_t(C_T;T) \quad = \quad 1_{\{\tau_M > t\}} \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T (r_s + \mu_s) ds} C_T | \mathcal{G}_t\right].$$

The conditioning on $\mathcal{G}_t$ can be reduced to that on $\mathcal{F}_t$, as shown in Appendix C of Biffis (2005). As a conclusion, by using equations (2) and (3):

$$\begin{aligned} SB_t(C_T;T) &= 1_{\{\tau_M > t\}} \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T (r_s + \mu_s) ds} C_T | \mathcal{F}_t\right] \\ &= 1_{\{\tau_M > t\}} \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \bar{r} + \bar{\mu} + \langle R+M, X_s \rangle ds} C_T | \mathcal{F}_t\right]. \end{aligned} \tag{4}$$

In the following, $\mathbb{E}^{\mathbb{Q}}\left[\cdot | \mathcal{F}_t\right]$ is also denoted by $\mathbb{E}_t^{\mathbb{Q}}\left[\cdot\right]$.

This insurance product is a building block for more complex contingent claims based upon survival at a given date, as will be shown in Section 3.

In order to evaluate this elementary product, we will use a change of probability measure approach. Indeed, we will define the probability measure $\mathbb{Q}^{T,\mu}$ with the Radon-Nikodym derivative of $\mathbb{Q}^{T,\mu}$ with respect to $\mathbb{Q}$ as

$$\frac{d\mathbb{Q}^{T,\mu}}{d\mathbb{Q}} := \zeta_T = \frac{e^{-\int_0^T \bar{r}+\bar{\mu}+\langle R+M, X_s\rangle ds}}{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T \bar{r}+\bar{\mu}+\langle R+M, X_s\rangle ds}\right]},$$

and we define ζ_t^T as the conditional expectation under $\mathbb{Q}$ of ζ_T with respect to the filtration $\mathcal{F}_t$, which is by definition a martingale under $\mathbb{Q}$. Hence, we obtain for $t \leq T$

$$\zeta_t^T = \mathbb{E}^{\mathbb{Q}}[\zeta_T|\mathcal{F}_t] = \frac{e^{-\int_0^t \bar{r}+\bar{\mu}+\langle R+M, X_s\rangle ds}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \bar{r}+\bar{\mu}+\langle R+M, X_s\rangle ds}|\mathcal{F}_t\right]}{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T \bar{r}+\bar{\mu}+\langle R+M, X_s\rangle ds}\right]}.$$

Therefore, using Bayes' rule, the conditional expectation involving C_T under the new measure can be calculated by the following fraction

$$\mathbb{E}^{\mathbb{Q}^{T,\mu}}[C_T|\mathcal{F}_t] = \frac{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \bar{r}+\bar{\mu}+\langle R+M, X_s\rangle ds}C_T|\mathcal{F}_t\right]}{\tilde{P}(t,T)},$$

where

$$\tilde{P}(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T \bar{r}+\bar{\mu}+\langle R+M, X_s\rangle ds}\right]$$

denotes the price at time t of a pure endownment insurance or (to stress the similarities in finance) a *survival zero-coupon bond* (SZCB hereafter) with maturity T (for an insured of age x at time 0, who is still alive at time t). It is a life insurance product which gives one dollar at time T upon survival of the insured at that time[3].
As X is an affine process, it is well-known that this expectation can be obtained as follows

$$\begin{aligned}\tilde{P}(t,T) &= e^{-(\bar{r}+\bar{\mu})(T-t)}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \langle R+M, X_s\rangle ds}|\mathcal{F}_t\right] \\ &= e^{-(\bar{r}+\bar{\mu})(T-t)}e^{-\Phi_{(0,1)}(T-t,R+M)-\langle\Psi_{(0,1)}(T-t,R+M),X_t\rangle} \qquad (5)\end{aligned}$$

where the functions $\Phi_{(0,1)}(\cdot, R+M)$ and $\Psi_{(0,1)}(\cdot, R+M)$ satisfy the system of generalized Riccati ODEs (39) and (40) recalled in the Appendix.
Finally, formula (4) can be expressed as the following product

$$SB_t(C_T;T) = 1_{\{\tau_M>t\}}\tilde{P}(t,T)\mathbb{E}^{\mathbb{Q}^{T,\mu}}[C_T|\mathcal{F}_t]. \qquad (6)$$

Note that if $C_T = 1$, the (unitary) survival benefit is related to the SZCB in the following way

$$SB_t(1;T) = 1_{\{\tau_M>t\}}\tilde{P}(t,T), \qquad (7)$$

[3]Note that $\tilde{P}(t,T)$ can be used as a numéraire because it can be replicated by a strategy involving longevity bonds (see e.g. Lin and Cox (2005)), in analogy with the usual bootstrapping procedure used to find the zero rate curve starting by coupon bonds.

meaning that the unitary survival benefit is basically equivalent to a SZCB modulo the probability that the insured is alive at time t; in particular, $SB_0(1;T) = \tilde{P}(0,T)$.
The main advantage of this approach is that, as we can observe in (6), it allows us to split the survival benefit as the product of two expectations: the first one corresponds to the price of the SZCB (given by (5)) and the second one is related to the expectation of the benefit C_T under the probability measure $\mathbb{Q}^{T,\mu}$, which remains to be determined.

Remark 1. *We point out that our approach boils down to the ones of Liu et al. (2013, 2014) since they use as numéraire the unitary survival benefit of* (7) *and we use the SZCB. Our single-step change-of-measure approach is slightly different from the ones of Jalen and Mamon (2009) insofar the latters use a double-step procedure in order to simplify the initial expression. Indeed, Jalen and Mamon (2009) start by using the forward measure with the money market account as numéraire in order to price a unitary survival benefit (i.e. $SB_t(1;T)$). Then, in order to determine the GAO value, they introduce a new measure, which is associated to $SB_t(1;T)$ as numéraire. Even if from a theoretical point of view the two methods are equivalent, it is worth recalling that each time a change of measure is used, the drift of the processes in question changes according to the Girsanov theorem. For example, in the setting of Wishart processes, the two step change of measure approach leads to a Wishart process whose drift depends upon the solution of a time varying Riccati differential equation which needs to be solved numerically (typically by a Runge–Kutta method). On the contrary, within our approach we end up with a time homogeneous Riccati differential equation which can be solved by using the well-known linearization method (see e.g. Da Fonseca et al. (2008)).*

3 Annuities and Guaranteed Annuity Options

In this section we consider the pricing of two insurance contracts issued at time 0 for an insured aged x years. The first contract is a T_1-year deferred indexed annuity which provides a unit amount plus a percentage of the short interest rate at the beginning of each year from T_1 upon survival of the insured, see e.g. Biffis (2005). The second contract is a guaranteed annuity option (GAO), which is one of the most familiar embedded options in life insurances, see e.g. Hardy (2003). They began to be included in some UK pension policies in the 1950's and became very popular in the 1970's and 1980's in the UK. One of the most popular type of GAOs in the US and Japan provide the right to convert a policyholder's accumulated fund to a life annuity at a fixed rate when the policy matures (see e.g. Ballotta and Haberman (2003, 2006), Boyle and Hardy (2006), Deelstra and Rayée (2013)) and they are referred to as Guaranteed Minimum Income Benefit (GMIB). In this paper, we will consider a simple GAO type within traditional endownment policies such as studied e.g. in Pelsser (2003), Liu et al. (2013, 2014) and Zhu and Bauer (2011). According to Pelsser (2003), this form of GAO is very popular since it plays a crucial role in with-profit policies.

The fair values are derived by using the change of measure described in the previous section. For the indexed annuity, we also present an alternative way based on Fourier methods like in Duffie et al. (2000).

3.1 Indexed annuities

We start by recalling the definition of a whole life annuity due. A whole life annuity is an insurance product that gives the policyholder a predetermined periodic payoff until his death. Let us denote an immediate annuity paying a payoff C_j in the beginning of year $j = 0, 1, \dots$ in case of the insured's survival at each policy date by following the notation of Biffis (2005), namely by:

$$\ddot{a}_x(C) = \sum_{j=0}^{\omega-x-1} SB_0(C_j; j),$$

where ω is the largest possible survival age. Therefore, the present value at time 0 of a whole life annuity due with yearly payments of one unit from 0 on for a person aged x at time 0 is given by

$$\ddot{a}_x := \ddot{a}_x(1) = \sum_{j=0}^{\omega-x-1} \tilde{P}(0, j),$$

where we use the standard notation[4] for the whole life annuity due. Following the notations of (Liu et al., 2014), the present value at time T of a whole life annuity due with yearly payments of one unit from T on for a person aged x at time 0, will be denoted by $\ddot{a}_x(T)$ if there is no confusion that T is is referring to time and not to a payoff; and is given by

$$\ddot{a}_x(T) = \sum_{j=0}^{\omega-(x+T)-1} \tilde{P}(T, T+j). \tag{8}$$

We now consider a T_1-years deferred whole life annuity which turns out a monetary unit plus a percentage γ of the short rate at each policy date $T_1, T_1+1, T_1+2, \dots$ upon survival of the insured. Following the notation of Biffis (2005), we denote this T_1-years deferred indexed annuity by $SB_0(\ddot{a}_{x+T_1}(1+\gamma r); T_1)$, where the insured has age x at time 0. Therefore

$$\begin{aligned} SB_0(\ddot{a}_{x+T_1}(1+\gamma r); T_1) &= \sum_{h=T_1}^{\omega-x-1} SB_0(1+\gamma r_h; h) \\ &= \sum_{h=T_1}^{\omega-x-1} \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^h (r_s+\mu_s)ds}(1+\gamma r_h)\right]. \end{aligned} \tag{9}$$

[4]We refer to e.g. Dickson et al. (2013) for an excellent introduction to actuarial notions.

We provide two different approaches for evaluating this indexed annuity product in the setting of our general affine model.

The first expression is obtained by using the change of numéraire defined in the previous section. The second one follows from the methodology derived in Duffie et al. (2000).

Proposition 1. *The present value of a T_1-years deferred life annuity which pays out a monetary unit plus a percentage γ of the short rate at each policy date $T_1, T_1+1, T_1+2, \ldots$ upon survival of the insured is given by*

$$SB_0(\ddot{a}_{x+T_1}(1+\gamma r); T_1) = \sum_{h=T_1}^{\omega-x-1} \tilde{P}(0,h)\left(1+\gamma\bar{r}+\gamma\left\langle R, \mathbb{E}^{\mathbb{Q}^{h,\mu}}[X_h]\right\rangle\right) \quad (10)$$

$$= \sum_{h=T_1}^{\omega-x-1}\Big((1+\gamma\bar{r})\tilde{P}(0,h) + \gamma e^{-(\bar{r}+\bar{\mu})h} L^0_\nu(0,h,-(R+M),0,\nu R)\Big) \quad (11)$$

where L^0_ν denotes the derivative wrt $\nu \in \mathbb{R}$ at $\nu = 0$ of the following function

$$L(t,T,\theta_1,\theta_2,\nu\theta_3) = \mathbb{E}^{\mathbb{Q}}_t\left[e^{\int_t^T \langle\theta_1, X_u\rangle du + \langle(\theta_2+\nu\theta_3), X_T\rangle}\right], \quad (12)$$

with $(t,T,\theta_1,\theta_2,\nu\theta_3) \in [0,T]\times\mathbb{R}_+\times\mathbb{R}^{3d}$ for which the transform is well-defined and interchangeability between the derivative and the expectation is allowed[5].

Remark 2. *From* (10) *it follows that if we know the dynamics of X under the measure $\mathbb{Q}^{h,\mu}$, it will be possible to calculate $\mathbb{E}^{\mathbb{Q}^{h,\mu}}[X_h]$ and we can get an explicit pricing formula. On the other hand, in* (11) *we have applied the Fourier method as in Duffie et al. (2000), which does not need any change of probability measure.*

Proof. i) [Change of measure approach] By using the change of numéraire associated with a SZCB, we can express the indexed annuity (9) as follows

$$SB_0(\ddot{a}_{x+T_1}(1+\gamma r); T_1) = \sum_{h=T_1}^{\omega-x-1} \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^h (r_s+\mu_s)ds}(1+\gamma r_h)\right]$$

$$= \sum_{h=T_1}^{\omega-x-1} \tilde{P}(0,h)\mathbb{E}^{\mathbb{Q}^{h,\mu}}[1+\gamma r_h]$$

$$= \sum_{h=T_1}^{\omega-x-1} \tilde{P}(0,h)\left(1+\gamma\bar{r}+\gamma\left\langle R, \mathbb{E}^{\mathbb{Q}^{h,\mu}}[X_h]\right\rangle\right),$$

where we used the equation (3) in the last equality.

[5] Such regularity assumptions can e.g. be found in Duffie et al. (2000) for the CIR case and in Cuchiero et al. (2011) for the Wishart case.

ii) [Fourier method] By computing the derivative of (12) with respect to ν and evaluating at $\nu = 0$, we get

$$\partial_\nu L(t, T, \theta_1, \theta_2, \nu\theta_3)|_{\nu=0} = \mathbb{E}_t^{\mathbb{Q}} \left[e^{\int_t^T \langle \theta_1, X_u \rangle du + \langle \theta_2, X_T \rangle)} \langle \theta_3, X_T \rangle \right].$$

Simple calculations lead to the following equalities

$$\begin{aligned} SB_0(\ddot{a}_{x+T_1}(1+\gamma r); T_1) &= \sum_{h=T_1}^{\omega-x-1} SB_0(1+\gamma r_h; h) \\ &= \sum_{h=T_1}^{\omega-x-1} \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_0^h (r_s+\mu_s) ds} (1+\gamma r_h) \right] \\ &= \sum_{h=T_1}^{\omega-x-1} e^{-(\bar{r}+\bar{\mu})h} \Big((1+\gamma\bar{r}) \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_0^h \langle R+M, X_s \rangle ds} \right] \\ &\quad + \gamma \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_0^h \langle R+M, X_s \rangle ds} \langle R, X_h \rangle) \right] \Big). \end{aligned}$$

We notice that

$$\mathbb{E}^{\mathbb{Q}} \left[e^{-\int_0^h \langle R+M, X_s \rangle ds} \right] = e^{(\bar{r}+\bar{\mu})h} \tilde{P}(0, h)$$

and that, by adopting the notation of Chiarella et al. (2014)

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_0^h \langle R+M, X_s \rangle ds} \langle R, X_h \rangle \right] &= \partial_\nu L(0, h, -(R+M), 0, \nu R)|_{\nu=0} \\ &= L_\nu^0(0, h, -(R+M), 0, \nu R). \end{aligned}$$

Collecting terms gives the result. □

3.2 Guaranteed Annuity Options

Consider an x year old policyholder at time 0 who disposes at (retirement) age R_x of a unitary payoff. A GAO gives to the policyholder the right to choose at time $T \equiv R_x - x$ between an annual payment of g, where g is a fixed rate called the Guaranteed Annuity rate, or a cash payment equal to the capital 1. As already mentioned, this contract is an example of an embedded option in life insurance policies. In the 1970s and 1980s, when long term interest rates were high, the option was far out of the money, to the point that the most popular guaranteed rate for a male aged 65 in the UK, was an annuity cash value ratio of $g = 1/9$ (see e.g. Bolton et al. (1997)). Afterwards, when in the 1990's long term interest rates started to fall, the value of guaranteed annuity options increased. Nowadays, due to the long term nature of these contracts, the change in financial and mortality variables seems to be a challenging issue for life insurance compagnies.

We first concentrate on the evolution of a whole life annuity due for a person aged x at time 0, which gives an annual payment of one unit amount at the beginning of each year from time T on and this conditional on survival. As noticed before in (8), such an annuity can be seen as a sum of SZCBs and therefore its value at time T boils in the general affine approach down to

$$\begin{aligned}\ddot{a}_x(T) &= \sum_{j=0}^{\omega-(x+T)-1} \tilde{P}(T, T+j) \\ &= \sum_{j=0}^{\omega-R_x-1} \mathbb{E}_T^{\mathbb{Q}} \left[e^{-\int_T^{T+j} (\bar{r}+\bar{\mu})+\langle (R+M), X_s \rangle ds} \right] \\ &= \sum_{j=0}^{\omega-R_x-1} e^{-(\bar{r}+\bar{\mu})(j)} e^{-\Phi_{(0,v)}(j,R+M)-\langle \Psi_{(0,v)}(j,R+M), X_T \rangle}\end{aligned}$$

where $\Phi_{(0,v)}(j, R+M)$ and $\Psi_{(0,v)}(j, R+M)$ are resp. given by (39) and (40) in the Appendix.

At time T the value of the contract with the embedded GAO can be expressed by the following decomposition

$$\begin{aligned}V(T) &= \max(g\ddot{a}_x(T), 1) \\ &= 1 + g \max\left(\ddot{a}_x(T) - \frac{1}{g}, 0\right).\end{aligned}$$

Applying the risk neutral valuation procedure, we find the value at time $t = 0$ of the second term, which is called the GAO option price entered by an x-year policyholder at time $t = 0$:

$$C(0, x, T) = \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_0^T (r_s+\mu_s) ds} g \max\left(\ddot{a}_x(T) - \frac{1}{g}, 0\right) \right]. \tag{13}$$

When using the probability measure $\mathbb{Q}^{T,\mu}$ defined in (2), the GAO option price decomposes into the following product

$$C(0, x, T) = g\tilde{P}(0, T)\mathbb{E}^{\mathbb{Q}^{T,\mu}} \left[\max\left(\ddot{a}_x(T) - \frac{1}{g}, 0\right) \right]. \tag{14}$$

As the payoff of the GAO is very similar to the one of a basket option, for most models the values of (13) and (14) can only be performed by Monte Carlo simulations. In Liu et al. (2014) numerical experiments in the Gaussian setting have shown that equation (14) is a bit more precise and in particular it is less time-consuming than the implementation of equation (13). In Section 5 we will investigate this issue in different affine models such as the multi-CIR and the Wishart cases.

4 Models

The Gaussian framework has been deeply investigated in the insurance literature on derivative products (see e.g. Jalen and Mamon (2009), Liu et al. (2014), Luciano and Vigna (2008) and references therein), leading to simple formulas, but also to the possibility to get negative rates. In this section we focus on more general affine models and we investigate the influence in pricing of their dependence structure, which reduces to be constant in the Gaussian setting. In particular, we concentrate on factor models based upon multi-CIR and Wishart processes. In the former we circumvent the typical trivial correlation structure of the multi-CIR case (where positive factors must be uncorrelated in order to preserve the affine property of the model) by assuming that mortality and interest rates are driven by systematic and idiosyncratic factors. The Wishart case is more interesting as it allows for more sophisticated dependence between the positive factors.

4.1 The multi-CIR case

In this subsection, we model X by an n-dimensional affine process whose independent components evolve according to the CIR risk neutral dynamics

$$dX_{it} = k_i(\theta_i - X_{it})dt + \sigma_i\sqrt{X_{it}}dW^{\mathbb{Q}}_{it}, \quad i = 1, .., n.$$

To see that this example fits into the general affine framework described in the Appendix, it is enough to take

$$a(x) = \begin{pmatrix} \sigma_1^2 x_1 & 0 & \dots & 0 \\ 0 & \sigma_2^2 x_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \sigma_n^2 x_n \end{pmatrix}$$

and

$$b(x) = \begin{pmatrix} k_1\theta_1 \\ k_2\theta_2 \\ \vdots \\ k_n\theta_n \end{pmatrix} + \begin{pmatrix} -k_1 & 0 & \dots & 0 \\ 0 & -k_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & -k_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

in equation (38). The price of the SZCB $\tilde{P}(t, T)$ can be easily derived in this framework:

$$\begin{aligned} \tilde{P}(t,T) &= \mathbb{E}^{\mathbb{Q}}_t \left[e^{-\int_t^T (\bar{r}+\bar{\mu}) + \langle (R+M), X_s \rangle ds} \right] \\ &= e^{-(\bar{r}+\bar{\mu})(T-t)} \prod_{i=1}^{n} \mathbb{E}^{\mathbb{Q}}_t \left[e^{-\int_t^T (R_i+M_i) X_{is} ds} \right] \\ &= e^{-(\bar{r}+\bar{\mu})(T-t)} \prod_{i=1}^{n} e^{-\phi_i(T-t, R_i+M_i) - \psi_i(T-t, R_i+M_i) X_{it}} \qquad (15) \end{aligned}$$

where ϕ_i and ψ_i are solution of the Riccati equations (39)-(40) which become (see e.g. Duffie et al. (2000)):

$$\begin{aligned}\frac{\partial \psi_i(\tau,u_i)}{\partial \tau} &= 1-k_i\psi_i(\tau,u_i)-\frac{u_i\sigma_i^2}{2}\psi_i(\tau,u_i)^2,\\ \frac{\partial \phi_i(\tau,u_i)}{\partial \tau} &= k_i\theta_i u_i\psi_i(\tau,u_i),\end{aligned}$$

with $\tau = T-t$, $u_i = R_i + M_i$ and initial conditions $\psi_i(0,u_i)=0$ and $\phi_i(0,u_i)=0$. The solutions of this system are given by

$$\psi_i(\tau,u_i) = \frac{2u_i}{\zeta(u_i)+k_i}-\frac{4u_i\zeta(u_i)}{\zeta(u_i)+k_i}\frac{1}{(\zeta(u_i)+k_i)\exp[\zeta(u_i)\tau]+\zeta(u_i)-k_i}, \tag{16}$$

$$\begin{aligned}\phi_i(\tau,u_i) &= -\frac{k_i\theta_i}{\sigma_i^2}[\zeta(u_i)+k_i]\tau+\frac{2k_i\theta_i}{\sigma_i^2}\log[(\zeta(u_i)+k_i)\exp[\zeta(u_i)\tau]+\zeta(u_i)-k_i]\\ &\quad -\frac{2k_i\theta_i}{\sigma_i^2}\log(2\zeta(u_i)),\end{aligned} \tag{17}$$

where $\zeta(u_i)=\sqrt{k_i^2+2u_i\sigma_i^2}$.

We will now specialize the price of indexed annuities and GAOs in this framework.

4.1.1 Indexed annuity in the multi-CIR case

Let us first concentrate upon the T_1-years deferred annuity in the multidimensional CIR framework. As in the general setting, we apply two approaches which give different formulas.

Proposition 2 (Change of measure approach)**.** *In the multidimensional CIR model, the present value of a T_1-years deferred life annuity which pays a monetary unit plus a percentage γ of the short rate at each policy date $T_1, T_1+1, T_1+2, \ldots$ upon survival of the insured can be written as follows:*

$$SB_0(\ddot{a}_{x+T_1}(1+\gamma r);T_1) = \sum_{h=T_1}^{\omega-x-1}\tilde{P}(0,h)\left(1+\gamma\bar{r}+\gamma\sum_{i=0}^{n}R^i\mathbb{E}^{\mathbb{Q}^{h,\mu}}[X_{ih}]\right) \tag{18}$$

where the expectation of X_{ih} under the measure $\mathbb{Q}^{h,\mu}$ is given by

$$\mathbb{E}^{\mathbb{Q}^{h,\mu}}[X_{ih}] = \left(x_i e^{\frac{\sigma_i^2}{k_i\theta_i u_i}\phi_i(h,u_i)}+k_i\theta_i\int_0^h \exp\left(\frac{\sigma_i^2}{k_i\theta_i u_i}\phi_i(h-s,u_i)+k_i s\right)ds\right)\exp(-k_i h).$$

Proof. Plugging $r_h=\bar{r}+\sum_{i=0}^n R_i X_{ih}$ into equation (10) gives immediately (18). From Girsanov's theorem it follows that

$$dW_t^{\mathbb{Q}^{T,\mu}} = dW_t^{\mathbb{Q}}+\sqrt{a(X_t)}\psi(T-t,R+M)dt,$$

where $\psi(\tau,\cdot) = (\psi_1(\tau,\cdot),\ldots,\psi_n(\tau,\cdot))^\top$ is given in equation (16). We have

$$dX_{it} \;=\; (k_i(\theta_i - X_{it}) - \sigma_i^2\psi_i(T-t, R_i+M_i)X_{it})dt + \sigma_i\sqrt{X_{it}}dW_{it}^{\mathbb{Q}^{T,\mu}}$$

for $i = 1,\ldots,n$, so that the expectation $y_i(t) := \mathbb{E}^{\mathbb{Q}^{T,\mu}}[X_{it}]$ is solution of the following ODE

$$\frac{dy_i(t)}{dt} = k_i\theta_i - (\sigma_i^2\psi_i(T-t,u_i) + k_i)y_i(t), \qquad y_i(0) = x_i$$

that can be solved by standard methods. □

Proposition 3 (Fourier approach)**.** *In the multidimensional CIR model, the present value of a T_1-years deferred life annuity which pays a monetary unit plus a percentage γ of the short rate at each policy date $T_1, T_1+1, T_1+2,\ldots$ upon survival of the insured is given by*

$$SB_0(\ddot{a}_{x+T_1}(1+\gamma r);T_1) \;=\; \sum_{h=T_1}^{\omega-x-1}\Big((1+\gamma\bar{r})\tilde{P}(0,h) + \gamma e^{-(\bar{r}+\bar{\mu})h}L^0_\nu(0,h,-(R+M),0,\nu R)\Big)$$

where

$$L^0_\nu(t,T,\theta_1,\theta_2,\nu\theta_3) \;=\; \left[\sum_{i=1}^n \partial_\nu\widetilde{L}_i(t,T,\theta_{i1},\theta_{i2},\nu\theta_{i3})\prod_{\substack{j=1\\ j\neq i}}^n \widetilde{L}_j(t,T,\theta_{j1},\theta_{j2},\nu\theta_{j3})\right]\Bigg|_{\nu=0},$$

with

$$\widetilde{L}_k(t,T,\theta_{k1},\theta_{k2},\nu\theta_{k3}) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{\int_t^T\theta_{k1}X_{ku}du + (\theta_{k2}+\nu\theta_{k3})X_{kT}}\right]. \tag{19}$$

Proof. We repeat equality (11) from the previous section

$$SB_0(\ddot{a}_{x+T_1}(1+\gamma r);T_1) \;=\; \sum_{h=T_1}^{\omega-x-1}\Big((1+\gamma\bar{r})\tilde{P}(0,h) + \gamma e^{-(\bar{r}+\bar{\mu})h}L^0_\nu(0,h,-(R+M),0,\nu R)\Big)$$

where

$$L^0_\nu(0,h,-(R+M),0,\nu R) \;=\; \partial_\nu L(0,h,-(R+M),0,\nu R)|_{\nu=0}$$

with

$$L(t,T,\theta_1,\theta_2,\nu\theta_3) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{\int_t^T\sum_{i=0}^n\theta_{1i}X_{iu}du + \sum_{i=0}^n(\theta_{i2}+\nu\theta_{i3})X_{iT}}\right].$$

Now

$$\begin{aligned}
\partial_\nu L(t,T,\theta_1,\theta_2,\nu\theta_3)|_{\nu=0} &= \partial_\nu\left(\prod_{i=1}^n\mathbb{E}_t^{\mathbb{Q}}\left[e^{\int_t^T\theta_{i1}X_{iu}du + (\theta_{i2}+\nu\theta_{i3})X_{iT}}\right]\right)\Bigg|_{\nu=0} \\
&= \left[\sum_{i=1}^n\partial_\nu\widetilde{L}_i(t,T,\theta_{i1},\theta_{i2},\nu\theta_{i3})\prod_{\substack{j=1\\ j\neq i}}^n\widetilde{L}_j(t,T,\theta_{j1},\theta_{j2},\nu\theta_{j3})\right]\Bigg|_{\nu=0}
\end{aligned}$$

where $\widetilde{L}_k$ is given by

$$\widetilde{L}_k(t,T,\theta_{k1},\theta_{k2},\nu\theta_{k3}) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{\int_t^T \theta_{k1}X_{ku}du+(\theta_{k2}+\nu\theta_{k3})X_{kT}}\right].$$

□

We notice that Lemma 1 in Appendix A gives explicit results for (19).

4.1.2 Guaranteed Annuity Options in the multi-CIR case

We start from the explicit expression for the annuity:

$$\begin{aligned}\ddot{a}_x(T) &= \sum_{j=0}^{\omega-(x+T)-1} \mathbb{E}_T^{\mathbb{Q}}\left[e^{-\int_T^{T+j} \bar{r}+\bar{\mu}+\sum_{i=0}^n (R_i+M_i)X_{is}ds}\right] \\ &= \sum_{j=0}^{\omega-(x+T)-1} e^{-(\bar{r}+\bar{\mu})j}\prod_{i=1}^n e^{-\phi_i(j,R_i+M_i)-\psi_i(j,R_i+M_i)X_{iT}},\end{aligned}$$

where we used equation (15) and where $\phi_i(j,R_i+M_i)$ and $\psi_i(j,R_i+M_i)$ are resp. given by (16) and (17).

The expression of the annuity can now be substituted in (13) or (14) in order to obtain the GAO value, which is given by resp.

$$C(0,x,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T (r_s+\mu_s)ds} g \max\left(\ddot{a}_x(T) - \frac{1}{g}, 0\right)\right] \tag{20}$$

or

$$C(0,x,T) = g\tilde{P}(0,T)\mathbb{E}^{\mathbb{Q}^{T,\mu}}\left[\max\left(\ddot{a}_x(T) - \frac{1}{g}, 0\right)\right]. \tag{21}$$

Note that the dynamics of X are known under both $\mathbb{Q}$ and $\mathbb{Q}^{T,\mu}$, so that we have two alternative expressions to which we can apply a Monte Carlo simulation in order to evaluate the GAO option price.

4.2 The Wishart case

In this section we assume that the affine process $(X_t)_{t\geq 0}$ is a d-dimensional Wishart process. Given a filtered probability space $(\Omega, \mathcal{G}, \mathcal{F}_t, \mathbb{Q})$ as in Section 2 and a $d\times d$ matrix Brownian motion W (i.e. a matrix whose entries are independent Brownian motions), the Wishart process X_t (without jumps) is defined as the solution of the $d\times d$-dimensional stochastic differential equation

$$dX_t = (\beta Q^\top Q + HX_t + X_t H^\top)dt + \sqrt{X_t}dW_t Q + Q^\top dW_t^\top \sqrt{X_t}, \quad t\geq 0, \tag{22}$$

where $X_0 = x \in S_d^+$, $\beta \geq d-1$, $H \in M_d$ (the set of real $d \times d$ matrices) $Q \in GL_d$ (the set of invertible real $d \times d$ matrices) and $Q^\top$ its transpose. Bru (1991) proved existence and uniqueness of a weak solution for equation (22). In the case where $\beta \geq d+1$, Bru (1991) showed that there exists a unique strong solution which takes values in S_d^{++}, i.e. the interior of the cone of positive semidefinite symmetric $d \times d$ matrices denoted by S_d^+ (see also Cuchiero et al. (2011)).

The price of a SZCB in the Wishart case can be derived as follows

$$\begin{aligned}\tilde{P}(t,T) &= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T \bar{r}+\bar{\mu}+\mathrm{Tr}((R+M)X_s)ds}\right] \\ &= e^{-(\bar{r}+\bar{\mu})(T-t)}e^{-\phi(T-t,R+M)-\mathrm{Tr}[\psi(T-t,R+M)X_t]},\end{aligned} \tag{23}$$

where ϕ and ψ solve the following system of ODE's ($\tau = T-t$)

$$\begin{cases} \frac{\partial \phi}{\partial \tau} = Tr[\beta Q^\top Q \psi(\tau, R+M)], \\ \phi(0, R+M) = 0, \\ \frac{\partial \psi}{\partial \tau} = \psi(\tau, R+M)H + H^\top \psi(\tau, R+M) - 2\psi(\tau, R+M)Q^\top Q \psi(\tau, R+M) + R + M, \\ \psi(0, R+M) = 0. \end{cases} \tag{24}$$

As proposed in Grasselli and Tebaldi (2008) and Da Fonseca et al. (2008), matrix Riccati equations can be linearized by doubling the dimension of the problem. Indeed, defining

$$\begin{pmatrix} A_{11}(\tau) & A_{12}(\tau) \\ A_{21}(\tau) & A_{22}(\tau) \end{pmatrix} = \exp\left(\tau \begin{pmatrix} H & 2Q^\top Q \\ R+M & -H^\top \end{pmatrix}\right)$$

it turns out that it is possible to express $\psi(\tau, R+M)$ as follows:

$$\psi(\tau, R+M) = A_{22}^{-1}(\tau) A_{21}(\tau),$$

and for $\phi(\tau, R+M)$ we get

$$\phi(\tau, R+M) = \frac{\beta}{2}\left(\log(\det(A_{22}(\tau))) + \tau Tr[H^\top]\right).$$

4.2.1 Indexed annuity in the Wishart case

We first derive the dynamics of the Wishart process under the measure $\mathbb{Q}^{T,\mu}$. To do that, we find the dynamics of the SZCB price $\tilde{P}(t,T)$ that can be found by applying Ito's lemma to the expression (23):

$$\begin{aligned}\frac{d\tilde{P}(t,T)}{\tilde{P}(t,T)} &= (\bar{r} + \bar{\mu} - \mathrm{Tr}[(R+M)X_t])dt - \mathrm{Tr}[\psi(\tau, R+M)\sqrt{X_t}dW_tQ] \\ &\quad - \mathrm{Tr}[\psi(\tau, R+M)Q^\top (dW_t)^\top \sqrt{X_t}].\end{aligned}$$

Girsanov's theorem gives the link between the $d \times d$ matrix Brownian motions under $\mathbb{Q}^{T,\mu}$ and $\mathbb{Q}$:

$$dW_t^{\mathbb{Q}^{T,\mu}} = dW_t + \sqrt{X_t}\psi(T-t, R+M)Q^\top dt.$$

As a consequence, the dynamics of X_t under $\mathbb{Q}^{T,\mu}$ are given by

$$\begin{aligned} dX_t &= \beta Q^\top Q dt + \left(H - Q^\top Q\psi(\tau, R+M)\right) X_t dt + X_t \left(H^\top - \psi(\tau, R+M) Q^\top Q\right) dt \\ &\quad + \sqrt{X_t}\left(dW_t^{\mathbb{Q}^{T,\mu}}\right) Q + Q^\top \left(dW_t^{\mathbb{Q}^{T,\mu}}\right)^\top \sqrt{X_t}. \end{aligned} \tag{25}$$

We now recall a useful result of Kang and Kang (2013) on the distribution of a Wishart process with time-varying linear drift.

Proposition 4. *(Kang and Kang (2013), Proposition A.6.) Let X be a Wishart process with time-varying linear drift which solves the following stochastic differential equation*

$$dX_t = (\beta Q^\top Q + H(t)X_t + X_t H(t)^\top)dt + \sqrt{X_t}dW_t Q + Q^\top dW_t^\top \sqrt{X_t}, \quad X_0 = x,$$

where $\beta \geq d-1$, Q is a $d \times d$ matrix, x is a symmmetric positive semidefinite matrix, $H(.)$ is a $d \times d$ matrix valued continuous function, and W is a standard $d \times d$ matrix Brownian motion. Then X_T has noncentral Wishart distribution $\mathcal{W}_d(\beta, V(0), V(0)^{-1}\tilde{\Psi}(0)^\top x \tilde{\Psi}(0))$, where $V(t)$ and $\tilde{\Psi}(t)$ solve the following system of ODEs

$$\begin{aligned} \frac{d}{dt}\tilde{\Psi}(t) &= -H(t)^\top \tilde{\Psi}(t), \\ \frac{d}{dt}V(t) &= -\tilde{\Psi}(t)^\top Q^\top Q \tilde{\Psi}(t), \end{aligned}$$

with terminal conditions $\tilde{\Psi}(T) = I_d$ and $V(T) = 0$.

We can now state the following proposition, which gives the price of a T_1-years deferred indexed annuity product by using the change of measure described in Section 2.

Proposition 5 (Change of measure approach)**.** *In the Wishart model, the present value of a T_1-years deferred life annuity which pays out a monetary unit plus a percentage γ of the short rate at each policy date $T_1, T_1+1, T_1+2, \dots$ upon survival of the insured is given by*

$$SB_0(\ddot{a}_{x+T_1}(1+\gamma r); T_1) = \sum_{h=T_1}^{\omega-x-1} \tilde{P}(0,h)\left(1 + \gamma\bar{r} + n\gamma \operatorname{Tr}[RV(0)] + \gamma \operatorname{Tr}\left[R\tilde{\Psi}(0)^\top x \tilde{\Psi}(0)\right]\right),$$

where $V(t)$ and $\tilde{\Psi}(t)$ are solutions of the following system of ordinary differential equations

$$\begin{aligned} \frac{d}{dt}\tilde{\Psi}(t) &= -\widetilde{H}(h-t, R+M)^\top \tilde{\Psi}(t), \\ \frac{d}{dt}V(t) &= -\tilde{\Psi}(t)^\top Q^\top Q \tilde{\Psi}(t), \end{aligned}$$

with $\widetilde{H}(h-t, R+M) = H - Q^\top Q\psi(h-t, R+M)$ *and terminal conditions* $\tilde{\Psi}(T) = I_d$ *and* $V(T) = 0$.

Proof. In order to obtain the fair value of this insurance product, we first determine the term $\mathbb{E}^{\mathbb{Q}^{h,\mu}}[X_h]$. Introducing the notation $\widetilde{H}(h-t, R+M) = H - Q^\top Q\psi(h-t, R+M)$, the dynamics of X_t under the measure $\mathbb{Q}^{h,\mu}$ in (25) are given by

$$\begin{aligned} dX_t &= \left(\beta Q^\top Q + \widetilde{H}(h-t, R+M)X_t + X_t\widetilde{H}(h-t, R+M)^\top\right)dt \\ &\quad + \sqrt{X_t}(dW_t^{\mathbb{Q}^{h,\mu}})Q + Q^\top(dW_t^{\mathbb{Q}^{h,\mu}})^\top\sqrt{X_t}. \end{aligned}$$

Hence, under $\mathbb{Q}^{h,\mu}$, X is a Wishart process with time-varying linear drift. By virtue of Proposition 4, X_h has a noncentral Wishart distribution

$$X_h \sim \mathcal{W}_d(\delta, V(0), V(0)^{-1}\tilde{\Psi}(0)^\top x\tilde{\Psi}(0)).$$

The expectation of X_h under $\mathbb{Q}^{h,\mu}$ is then given by (see Appendix B)

$$\mathbb{E}^{\mathbb{Q}^{h,\mu}}[X_h] = nV(0) + \tilde{\Psi}(0)^\top x\tilde{\Psi}(0),$$

which gives the result. □

We now concentrate on the method of Duffie et al. (2000) in the Wishart framework.

Proposition 6 (Fourier approach)**.** *In the Wishart model, the present value of a* T_1*-years deferred life annuity which pays a monetary unit plus a percentage* γ *of the short rate at each policy date* $T_1, T_1+1, T_1+2, \ldots$ *upon survival of the insured is given by*

$$\begin{aligned} SB_0(\ddot{a}_{x+t}(1+\gamma r); T_1) &= \sum_{h=T_1}^{\omega-x-1}\Big((1+\gamma\bar{r})\tilde{P}(0,h) \\ &\quad + \gamma e^{-(\bar{r}+\bar{\mu})h}\left(\mathrm{Tr}(a_\nu^0(h)X_0) + c_\nu^0(h)\right)e^{\mathrm{Tr}(a^0(h)X_0)+c^0(h)}\Big) \end{aligned} \qquad (26)$$

where

$$\begin{cases} a^0(h) = A_{22}(h)^{-1}A_{21}(h) \\ c^0(h) = -\frac{1}{2}\,\mathrm{Tr}\left[(Q^\top Q)^{-1}\beta Q^2\log(A_{22}(h))\right] - \frac{h}{2}\,\mathrm{Tr}\left[(Q^\top Q)^{-1}\beta Q^2 H^\top\right] \\ a_\nu^0(h) = -(A_{22}(h))^{-1}RA_{12}(h)a^0(h) + (A_{22}(h))^{-1}RA_{11}(h) \\ c_\nu^0(h) = -\frac{1}{2}\,\mathrm{Tr}\left(\beta D_{\log,\,A_{22}(h)}(RA_{12}(h))\right) \end{cases}$$

with

$$\begin{pmatrix} A_{11}(h) & A_{12}(h) \\ A_{21}(h) & A_{22}(h) \end{pmatrix} = \exp\left(h\begin{pmatrix} H & 2Q^\top Q \\ R+M & -H^\top \end{pmatrix}\right),$$

and where $D_{\log,\,A}(E)$ *represents the Fréchet derivative of the logarithm function computed for the matrix* A *in the direction* $E \in M_n$.

We recall that the Fréchet derivative of a function $f : M_n \mapsto M_n$ at a point $A \in M_n$ is a linear functional $E \in M_n$ to $D_{f,\,A}(E) \in M_n$ such that

$$f(A+E) - f(A) - D_{f,\,A}(E) = o(||E||).$$

If $D_{f,\,A}(E)$ exists then it is unique. We refer to Al-Mohy et al. (2013) for an efficient algorithm for computing the Fréchet derivative of the matrix logarithm.

Proof. As the Wishart process X_t is affine, it is well-known that there exist some deterministic functions such that the Laplace transform of the process and its integral is an exponential of the affine function of X_t and these deterministic functions showing up as coefficient, see e.g. Bru (1991) or Da Fonseca et al. (2007), Da Fonseca et al. (2008). The moment generating function L turns out to be

$$L(t, T, \theta_1, \theta_2, \nu\theta_3) = \exp(\mathrm{Tr}(a(\tau)X_t) + c(\tau)) \tag{27}$$

with $\tau = T - t$ and where $a(\tau)$, $c(\tau)$ solve the Riccati ODEs

$$\begin{cases} \frac{\partial a}{\partial \tau} = a(\tau)H + H^\top a(\tau) + 2a(\tau)Q^2 a(\tau) + \theta_1, \\ a(0) = \theta_2 + \nu\theta_3, \\ \frac{\partial c}{\partial \tau} = \mathrm{Tr}[\beta Q^\top Q a(\tau)], \\ c(0) = 0, \end{cases}$$

and then can be expressed as follows:

$$\begin{cases} a(\tau) = ((\theta_2 + \nu\theta_3)A_{12}(\tau) + A_{22}(\tau))^{-1}((\theta_2 + \nu\theta_3)A_{11}(\tau) + A_{21}(\tau)) \\ c(\tau) = -\frac{1}{2}\,\mathrm{Tr}\left[(Q^\top Q)^{-1}\beta Q^2 \log((\theta_2 + \nu\theta_3)A_{12}(\tau) + A_{22}(\tau))\right] - \frac{\tau}{2}\,\mathrm{Tr}\left[(Q^\top Q)^{-1}\beta Q^2 H^\top\right]. \end{cases}$$

Furthermore, by computing the derivative of (27) with respect to ν at $\nu = 0$, one obtains

$$L^0_\nu(t, T, \theta_1, \theta_2, \nu\theta_3) = \left(\mathrm{Tr}(a^0_\nu(\tau)X_t) + c^0_\nu(\tau)\right)\exp\left(\mathrm{Tr}(a^0(\tau)X_t) + c^0(\tau)\right)$$

where $a^0(\tau) = a(\tau)|_{\nu=0}$, $c^0(\tau) = c(\tau)|_{\nu=0}$, $a^0_\nu(\tau) = \partial_\nu a(\tau)|_{\nu=0}$ and $c^0_\nu(\tau) = \partial_\nu c(\tau)|_{\nu=0}$. According to Chiarella et al. (2014), these expressions are known and are given by

$$\begin{aligned} a^0_\nu(\tau) &= -(\theta_2 a_{12}(\tau) + a_{22}(\tau))^{-1}\theta_3 a_{12}(\tau)a^0(\tau) \\ &\quad + (\theta_2 a_{12}(\tau) + a_{22}(\tau))^{-1}\theta_3 a_{11}(\tau), \\ c^0_\nu(\tau) &= -\frac{1}{2}\,\mathrm{Tr}\left(\beta D_{\log,\,\theta_2 a_{12}(\tau)+a_{22}(\tau)}(\theta_3 a_{12}(\tau))\right). \end{aligned}$$

□

4.2.2 Guaranteed Annuity Options in the Wishart model

We start by adapting the formula of the annuity expression $\ddot{a}_x(T)$ in (8) to the Wishart framework

$$\begin{aligned}\ddot{a}_x(T) &= \sum_{j=0}^{\omega-(x+T)-1} \tilde{P}(T, T+j) \\ &= \sum_{j=0}^{\omega-R_x-1} e^{-(\bar{r}+\bar{\mu})(j)} e^{-\phi(j,R+M)-Tr[\psi(j,R+M)X_T]}\end{aligned}$$

where $\phi(j, R+M)$ and $\psi(j, R+M)$ are solutions of the PDE system (24), with $\tau = j$.

As before, by applying the risk neutral valuation procedure, we can write the value of the GAO option price entered by an x-year policyholder at time $t = 0$ as

$$C(0, x, T) = \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_0^T (r_s+\mu_s)ds} g \max\left(\ddot{a}_x(T) - \frac{1}{g}, 0 \right) \right], \tag{28}$$

or, by using the probability measure $\mathbb{Q}^{T,\mu}$ defined in (2), as

$$C(0, x, T) = g\tilde{P}(0, T) E^{\mathbb{Q}^{T,\mu}} \left[\max\left(\ddot{a}_x(T) - \frac{1}{g}, 0 \right) \right]. \tag{29}$$

As the dynamics of the Wishart process X are available under $\mathbb{Q}$ and $\mathbb{Q}^{T,\mu}$ (see (22) and (25)) and $\tilde{P}(0, T)$ is given by (23), we can perform Monte Carlo simulations under both measures in order to find the price of the GAO.

5 Numerical illustrations

In this section, we will present several numerical experiments for the insurance products and affine models considered in Sections 3 and 4. Similarly to the work of Liu et al. (2014) in the Gaussian framework, we are interested in explaining the value of insurance contracts in terms of the dependence structure between the interest rate and the mortality process.

5.1 Multidimensional CIR model

In this subsection, we consider a 3-dimensional CIR process, having independent components $X_t = (X_{1t}, X_{2t}, X_{3t})$. This corresponds to the most parsimonious choice that allows for some dependence. We assume that the interest rate process $(r_t)_{t\geq 0}$ and the mortality process $(\mu_t)_{t\geq 0}$ are described by

$$r_t = \bar{r} + X_{1t} + X_{2t}, \quad \mu_t = \bar{\mu} + m_2 X_{2t} + m_3 X_{3t}, \tag{30}$$

with $\bar{r}$, $\bar{\mu}$, m_2 and m_3 constants. Hence, X_1 and X_3 correspond to idiosyncratic factors and X_2 to a systematic factor. In our illustration the coefficient m_2 is fixed and m_3 is chosen such that

$$\mathbb{E}^{\mathbb{Q}}[\mu_T] = C_x(T), \tag{31}$$

meaning that the expectation of the mortality is fixed to a given level $C_x(T)$ corresponding to the mortality rate, predicted by e.g. a Gompertz-Makeham model (see e.g. Dickson et al. (2013)), at age $x+T$ for an individual aged x at time 0. This condition is equivalent to

$$\bar{\mu} + m_2 \mathbb{E}^{\mathbb{Q}}[X_{2,T}] + m_3 \mathbb{E}^{\mathbb{Q}}[X_{3,T}] = C_x(T) \tag{32}$$

where

$$\mathbb{E}^{\mathbb{Q}}[X_{i,T}] = X_{i,0} e^{-k_i T} + \theta_i \left(1 - e^{-k_i T}\right).$$

We have chosen this example since this interest rate model was completely calibrated in Chiarella et al. (2016). This example implies always positive interest rates, which is not necessarily satisfied lately. We admit that in practice it would be better to fix positive linear coefficients m_2 and m_3 for the mortality model, leading to positive mortality rates[6] and to adapt another example of our family of interest rates with linear coefficients r_1 and r_2 so that interest rates can be potentially negative.

In the setting (30), the linear pairwise correlation between $(r_t)_{t\geq 0}$ and $(\mu_t)_{t\geq 0}$, denoted by ρ_t, forms a stochastic process given by

$$\rho_t = \frac{m_2 \sigma_2^2 X_{2t}}{\sqrt{\sigma_1^2 X_{1t} + \sigma_2^2 X_{2t}} \sqrt{m_2^2 \sigma_2^2 X_{2t} + m_3^2 \sigma_3^2 X_{3t}}}. \tag{33}$$

We generate different values for the linear pairwise correlation by varying the parameter m_2 (and therefore also m_3 by the constraint (31)).

Following Liu et al. (2014), we consider an individual of $x = 50$ years old at time 0 and we assume $\omega = 100$. We divide the time period into $l = 1500$ equal sub-intervals and we simulate $100,000$ sample paths. The parameters of the insurance products are given in Table 1. We choose $\bar{r} = -0.12332$ and $\bar{\mu} = 0$, in other words, the interest rate process $(r_t)_{t\geq 0}$ and the mortality process $(\mu_t)_{t\geq 0}$ are modeled by

$$r_t = -0.12332 + X_{1t} + X_{2t}, \quad \mu_t = m_2 X_{2t} + m_3 X_{3t},$$

where the parameters of the 3-dimensional CIR process are given in Table 2. The parameters of the interest rate process are taken from Chiarella et al. (2016), in particular $\bar{r}$ and the parameters for X_1 and X_2. The parameters of the mortality rate are chosen rather arbitrarily but in line with results about mortality tables and insurance products

[6] Our numerical study although showed that our mortality process almost always stayed positive during the 35 years of simulations.

in literature.[7] The expected value in (32) is fixed to the level $C_{50}(15) = 0.014$. For this choice, we applied the formula for mortality rates and the chosen parameters by the Belgian regulator ("Arrêté Vie 2003") for the pricing of life annuities purchased by males. Indeed, for an individual of age x, the mortality rate is given by:

$$\mu(x) = a_\mu + b_\mu . c_\mu^x \qquad a_\mu = -\ln(s_\mu) \qquad b_\mu = -\ln(g_\mu). \ln(c_\mu)$$

where the parameters s_μ, g_μ, c_μ take the values given in Table 3.

Product	Parameters	
GAO	$g = 0.111$	$T = 15$
Indexed annuity	$\gamma = 0.06$	$T_1 = 15$

Table 1: Parameter values of the insurance contracts.

CIR process	Parameters			
X_1	$k_1 = 0.3731$	$\theta_1 = 0.074484$	$\sigma_1 = 0.0452$	$X_1(0) = 0.0510234$
X_2	$k_2 = 0.011$	$\theta_2 = 0.245455$	$\sigma_2 = 0.0368$	$X_2(0) = 0.0890707$
X_3	$k_3 = 0.01$	$\theta_3 = 0.0013$	$\sigma_3 = 0.0015$	$X_3(0) = 0.0004$

Table 2: Parameter values of the 3-dimensional CIR process.

s_μ:	0.999441703848
g_μ:	0.999733441115
c_μ :	1.101077536030

Table 3: Belgian legal parameters for modelling mortality rates, for life insurance products, targetting a male population.

Figure 1 shows the Monte Carlo estimates for the expectation of the process of linear pairwise correlation ρ_t given by (33) for different values of m_2 for $t \in [0, 15]$. We observe that the correlation estimates remain relatively stable over time.

5.1.1 Sensitivity with respect to volatilities

We are interested in seeing how prices of GAOs and an indexed annuities are affected by changes in the volatility level. In Figure 2 and 3, we observe that the price of each insurance product is increasing with respect to the volatility of the idiosyncratic factor of the mortality and the one of the interest rate process.

[7] In practice, an elaborated calibration of the mortality parameters would be possible by following the approach in Russo et al. (2011) if the necessary data are available.

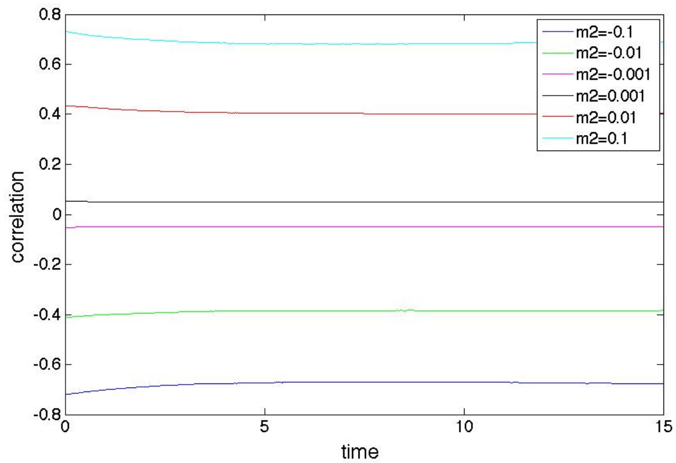

Figure 1: Expectation of ρ_t for different values of m_2.

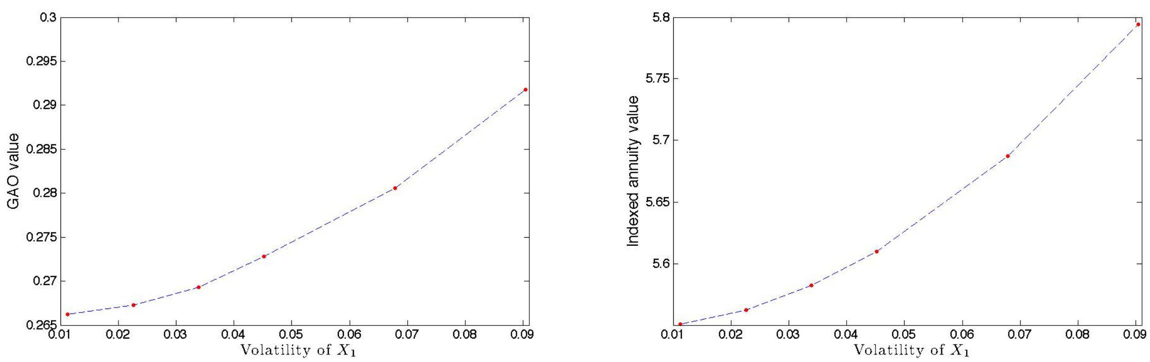

Figure 2: Fair value of a GAO and an indexed annuity as a function of σ_1

5.1.2 Sensitivity study with respect to the correlation between mortality and interest rates

In Table 4 and Table 5 we give the Monte Carlo estimates for the expectation of the interest rate and the mortality rate at three different times $t = 1, 15$ and 30 year and for different values of m_2. We observe that, at each different time t, the expected values of the rates remain relatively stable and are not influenced by the change of correlation. This fact is important because it allows us to study the behavior of the price as a function of solely the correlation.

Table 6 presents the prices of GAOs and indexed annuities for different values of m_2, and therefore for different values of the initial pairwise linear correlation coefficient ρ_0. We observe that when ρ_0 increases then both the value of the GAO and the indexed annuity increase as well, in line with Liu et al. (2014) in a Gaussian framework. Finally, as in Liu et al. (2014), we also find that the prices computed by the formula based upon the change of measure (formula (21)) are more precise than the once computed

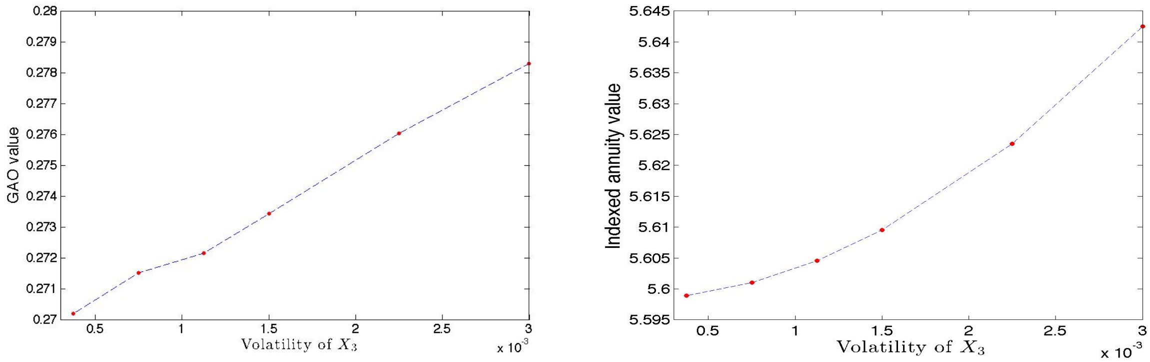

Figure 3: Fair value of a GAO and an indexed annuity as a function of σ_3

m_2	ρ_0	$\mathbb{E}^{\mathbb{Q}}[r_1]$	$\mathbb{E}^{\mathbb{Q}}[r_{15}]$	$\mathbb{E}^{\mathbb{Q}}[r_{30}]$
-0.1	-0.7196	0.0257508 (0.0000449)	0.0638806 (0.0001403)	0.0842519 (0.0001898)
-0.01	-0.4128	0.0258241 (0.0000448)	0.0639768 (0.0001405)	0.0842502 (0.0001904)
-0.001	-0.0516	0.0257827 (0.0000447)	0.0641229 (0.0001402)	0.0844225 (0.0001900)
0.001	0.0520	0.0258079 (0.0000448)	0.0637467 (0.0001397)	0.0842712 (0.0001902)
0.01	0.4355	0.0258816 (0.0000448)	0.0639790 (0.0001405)	0.0842924 (0.0001908)
0.1	0.7310	0.0258550 (0.0000447)	0.0642154 (0.0001406)	0.0846799 (0.0001905)

Table 4: Expectation of the interest rate process in the multi-CIR specification for different values of m_2. Numbers in parentheses represent the standard deviation of the corresponding MC estimate.

by formula (20), as can be noticed by the standard deviations of the Monte Carlo estimations. A possible explanation lies in the fact that Equation (29) is the product of the price of a survival bond, which can be calculated in an exact way, and the expectation of a relatively easy payoff; whereas formula (28) is the expectation of a product which is more involved and needs some more trajectories in order to obtain an equivalent precision.

5.2 Wishart model

In this section we assume that X follows a Wishart process. As previously, our goal consists in describing the behaviour of the fair value of GAOs and indexed annuities in terms of the dependence between the interest and the mortality rates. We recall that the mortality process $(\mu_t)_{t\geq 0}$ and the interest rate process $(r_t)_{t\geq 0}$ are modeled by

$$r_t = \bar{r} + \mathrm{Tr}(RX_t), \qquad \mu_t = \bar{\mu} + \mathrm{Tr}(MX_t), \qquad t \geq 0. \tag{34}$$

m_2	ρ_0	$\mathbb{E}^{\mathbb{Q}}[\mu_1]$	$\mathbb{E}^{\mathbb{Q}}[\mu_{15}]$	$\mathbb{E}^{\mathbb{Q}}[\mu_{30}]$
-0.1	-0.7196	0.0106074 (0.0000058)	0.0140120 (0.0000222)	0.0171573 (0.0000311)
-0.01	-0.4128	0.0108656 (0.0000028)	0.0139928 (0.0000107)	0.0168927 (0.0000151)
-0.001	-0.0516	0.0108978 (0.0000026)	0.0139911 (0.0000099)	0.0168495 (0.0000140)
0.001	0.0520	0.0109016 (0.0000025)	0.0140022 (0.0000097)	0.0168837 (0.0000137)
0.01	0.4355	0.0109253 (0.0000023)	0.0140005 (0.0000091)	0.0168563 (0.0000129)
0.1	0.7310	0.0111908 (0.0000035)	0.0140301 (0.0000135)	0.0166159 (0.0000187)

Table 5: Expectation of the mortality process in the multi-CIR specification for different values of m_2. Numbers in parentheses represent the standard deviation of the corresponding MC estimate.

m_2	ρ_0	GAO with formula (20)	GAO with formula (21)	Indexed annuity
-0.1	-0.7196	0.2246406 (0.0008418)	0.2257942 (0.0005775)	5.8269507
-0.01	-0.4128	0.2531518 (0.0009962)	0.2531801 (0.0006618)	6.1072984
-0.001	-0.0516	0.2567771 (0.0010047)	0.2571203 (0.0006748)	6.1387679
0.001	0.0520	0.2579532 (0.0010129)	0.2588907 (0.0006766)	6.1458521
0.01	0.4355	0.2638415 (0.0010419)	0.2611032 (0.0006890)	6.1781468
0.1	0.7310	0.3000678 (0.0012543)	0.3003570 (0.0008096)	6.5415269

Table 6: Fair values for the GAO and the indexed annuity in the multi-CIR specification.

For arbitrary $d \times d$ matrices R and M, the infinitesimal quadratic covariation between $(\mu_t)_{t\geq 0}$ and $(r_t)_{t\geq 0}$ is given by

$$\begin{aligned} d\langle r, \mu\rangle_t &= d\langle \operatorname{Tr}(RX), \operatorname{Tr}(MX)\rangle_t \\ &= 4\operatorname{Tr}(M^\top Q^\top Q R X_t)dt, \end{aligned}$$

where we used the fact that for $A, B \in M_d$ and a $d \times d$ matrix Brownian motion W, one has $d\langle \operatorname{Tr}(AW), \operatorname{Tr}(BW)\rangle_t = \operatorname{Tr}(AB^\top)dt$. The infinitesimal quadratic variation of $(r_t)_{t\geq 0}$ and $(\mu_t)_{t\geq 0}$ can be deduced similarly

$$\begin{aligned} d\langle r\rangle_t &= 4\operatorname{Tr}(Q^\top Q R X_t R^\top)dt, \\ d\langle \mu\rangle_t &= 4\operatorname{Tr}(Q^\top Q M X_t M^\top)dt. \end{aligned}$$

In the following, we assume[8] that $d = 2$ and we will make the simple choice of R and M:

$$M = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad R = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{35}$$

[8]We assume this very simple model for the ease of exposition. Wishart processes with a dimension strictly larger than (3×3) are difficult to implement, in the same way as high-dimensional numerical methods like FFT become difficult. Wishart processes with dimension $d \leq 3$ are however already very interesting since their flexibility for modeling dependence structures goes far beyond e.g. the CIR case.

For this choice, the stochastic correlation between $(r_t)_{t\geq 0}$ and $(\mu_t)_{t\geq 0}$ is given by

$$\rho_t = \frac{(Q_{11}Q_{12} + Q_{22}Q_{21})X_t^{12}}{\sqrt{(Q_{11}^2 + Q_{21}^2)X_t^{11}(Q_{22}^2 + Q_{12}^2)X_t^{22}}}. \tag{36}$$

As one can see from (36), modelling X by a Wishart process allows (even in this simple case) for a more general dependence structure than in the multidimensional CIR specification. Therefore, we will analyse the sensitivity in two ways. The first sensitivity study consists in varying the off-diagonal parameters of the initial Wishart process X_0 and seeing the impact on the prices. The second sensitivity study is based upon a change in the off-diagonal components of the matrix Q, which describe the dependence structure of the diagonal elements.

We use the Monte Carlo simulation method to obtain the fair value of a GAO. We generate 20,000 sample paths. Following Liu et al. (2014), we consider an individual of $x = 50$ years old at time 0 and we assume $\omega = 100$. The contract specifications are (still) the ones given in Table 1. In the remainder of this section the parameters R and M for the mortality process $(\mu_t)_{t\geq 0}$ and the interest rate process $(r_t)_{t\geq 0}$ are given by (35) and we choose $\bar{r} = 0.04$ and $\bar{\mu} = 0$ in (34).

5.2.1 Impact of a change in the initial value of the process

We are interested to see the impact on the value of a GAO and the indexed annuity when the off-diagonal components of the initial Wishart process X_0 (i.e. X_0^{12}) change. We consider two examples. In Example 1 (resp. Example 2), the off-diagonal elements of the volatility matrix Q are negative (resp. positive). We will observe that the fair value is either increasing or decreasing according to the initial correlation.

Example 1

In this experiment, we consider the following Wishart process:

$$H = \begin{pmatrix} -0.5 & 0.4 \\ 0.007 & -0.008 \end{pmatrix}, \quad Q = \begin{pmatrix} 0.06 & -0.0006 \\ -0.06 & 0.006 \end{pmatrix}, \quad X_0 = \begin{pmatrix} 0.01 & X_0^{12} \\ X_0^{12} & 0.001 \end{pmatrix}, \quad \beta = 3.$$

Similarly to the CIR specification, we checked the constancy of the expected values of the interest rate and the mortality intensity with respect to a fluctuating correlation. As before, we consider the rates at 3 distinct maturities: 1, 15 and 30 years. Expectations are computed via the Monte Carlo simulation method with 1500 sample paths. At each period, it turns out that the Monte Carlo estimates remain relatively stable despite the change of X_0^{12}. In Figure 4 we plot the Monte Carlo estimation for the expectation of the linear pairwise correlation process ρ_t given by formula (36). According to Figure 4, note that each path converges to the same value. Indeed, we recall that the expected long-term matrix of the Wishart process $X_\infty := \lim_{t\to\infty} \mathbb{E}[X_t]$ is given by the solution of the following (Lyapunov) equation:

$$HX_\infty + X_\infty H^\top + \beta Q^\top Q = 0,$$

and therefore is independent of the initial value.

Table 7 contains the fair values for the GAO (computed via formula (28) and (29)) and the indexed annuity (computed via formula (26)) derived according to the method of Duffie et al. (2000). We observe that when the correlation between the mortality and the interest rate grows, all prices increase, in line with Liu et al. (2014). Also in this setting, the GAO prices computed by the formula based upon the change of measure (formula (28)) are more precise than the once computed by formula (29), as can be noticed by the standard deviations of the Monte Carlo estimations.

X_0^{12}	ρ_0	GAO with formula(28)	GAO with formula (29)	Indexed annuity
-0.002	0.4894936	0.2448621 (0.0003981)	0.2451137 (0.0002435)	5.7801950
-0.0015	0.3671202	0.2437137 (0.0004092)	0.2443471 (0.0002408)	5.7729164
-0.0005	0.1223734	0.2436714 (0.0004018)	0.2437706 (0.0002430)	5.7583871
0	0	0.2431196 (0.0004078)	0.2435689 (0.0002410)	5.7511364
0.0005	-0.1223734	0.2424844 (0.0004001)	0.2429534 (0.0002398)	5.7438950
0.0015	-0.3671202	0.2412104 (0.0004056)	0.2420545 (0.0002440)	5.7294398
0.002	-0.4894936	0.2411214 (0.0004041)	0.2417495 (0.0002440)	5.7222261

Table 7: Fair values for the GAO and the indexed annuity in the Wishart specification, Example 1.

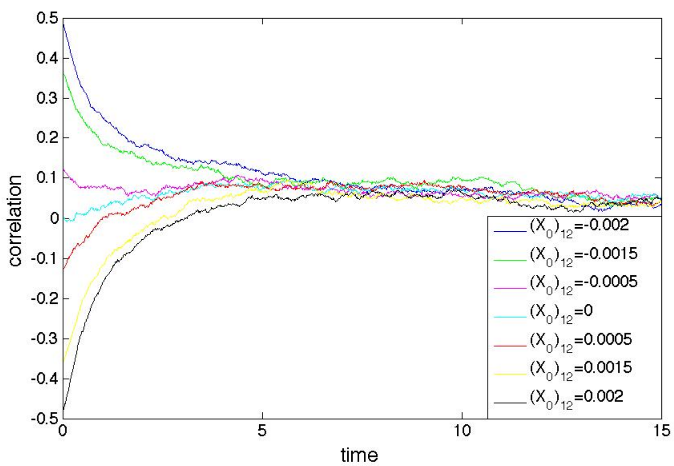

Figure 4: Monte Carlo estimate of the expectation of ρ_t for different values of X_0^{12} in the Wishart specification.

Example 2

In this second experiment, we consider the following Wishart process :

$$H = \begin{pmatrix} -0.5 & 0.4 \\ 0.007 & -0.008 \end{pmatrix}, \quad Q = \begin{pmatrix} 0.06 & 0.0006 \\ 0.06 & 0.006 \end{pmatrix}, \quad X_0 = \begin{pmatrix} 0.01 & X_0^{12} \\ X_0^{12} & 0.001 \end{pmatrix}, \quad \beta = 3.$$

This time the off-diagonal components of the matrix Q are the opposite numbers of those of Example 1. As previously, we perform the same stability test and in order to check that the expectation of the rates remain relatively stable despite the changes of X_0^{12}. The Monte Carlo estimate of ρ_t is presented in Figure 5. Table 8 contains the fair values for the GAO (computed by formula (28) and (29)) and the indexed annuity (computed by formula (26)).
Contrarily to Example 1, we observe from Table 8 that the price decreases when the initial correlation between mortality and the interest rate increases; this in contrast of the results of Liu et al. (2014) in the Gaussian setting. In the next subsection, we will give evidence that even other behaviors are possible.

X_0^{12}	ρ_0	GAO with formula (28)	GAO with formula (29)	Indexed annuity
-0.002	-0.4894936	0.1994176 (0.0005877)	0.1993275 (0.0003667)	5.2104471
-0.0015	-0.3671202	0.1990714 (0.0005945)	0.1987619 (0.0003767)	5.2045963
-0.0005	-0.1223734	0.1988364 (0.0006011)	0.1986171 (0.0003681)	5.1929144
0	0	0.1984553 (0.0005948)	0.1977835 (0.0003701)	5.1870834
0.0005	0.1223734	0.1984125 (0.0005943)	0.1976614 (0.0003675)	5.1812590
0.0015	0.3671202	0.1982640 (0.0005990)	0.1969242 (0.0003690)	5.1696300
0.002	0.4894936	0.1979702 (0.0005998)	0.1964036 (0.0003824)	5.1638254

Table 8: Fair values for the GAO and the indexed annuity in the Wishart framework, Example 2.

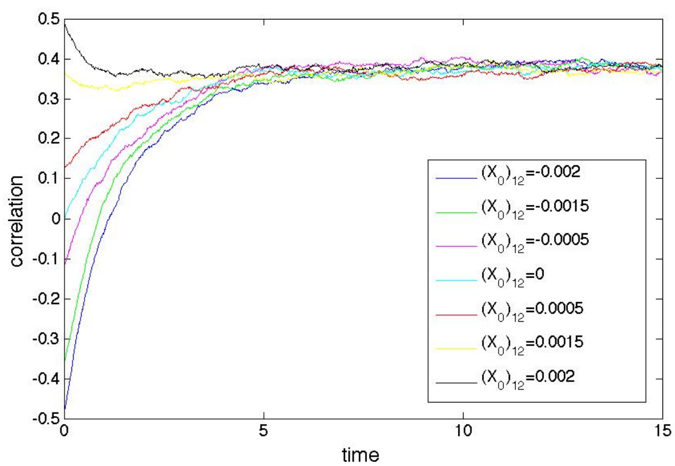

Figure 5: Monte Carlo estimate of the expectation of ρ_t for different values of X_0^{12} in the Wishart specification.

The results from Table 7 and 8 can be explained in the following way. By looking at the dynamics of the Wishart process, we see that the positive factors (i.e. X^{11} and

X^{22}) will be higher on average when the initial value X_0^{12} increases. Furthermore, since

$$r_t = \bar{r} + \mathrm{Tr}(RX_t) = X_t^{11} \qquad \mu_t = \bar{\mu} + \mathrm{Tr}(MX_t) = X_t^{22}, \qquad t \geq 0, \tag{37}$$

the exponential terms (which involve the interest rate and the mortality rate) in formula (28) and (29) decrease when X_0^{12} increases. Hence, as observed in Table 7 and 8, the GAO value decreases when X_0^{12} increases. Similar arguments hold for the indexed annuity.

5.2.2 Impact of a change in the volatility matrix Q

We now fix the initial value of the Wishart process (i.e. X_0) and we vary the off-diagonal elements of the volatility matrix Q which is chosen to be symmetric in this example.

Example 3
In this experiment, we consider the following parameters for the Wishart process:

$$H = \begin{pmatrix} -0.5 & 0.4 \\ 0.007 & -0.008 \end{pmatrix}, \quad Q = \begin{pmatrix} 0.06 & Q_{12} \\ Q_{12} & 0.006 \end{pmatrix}, \quad X_0 = \begin{pmatrix} 0.01 & 0.001 \\ 0.001 & 0.001 \end{pmatrix}, \quad \beta = 3.$$

Table 9 shows that the values of the insurance products are not monotone with respect to the linear correlation.

Q_{12}	ρ_0	GAO with formula (28)	GAO with formula (29)	Indexed annuity
-0.01	-0.2942210	0.2952542 (0.0008533)	0.2953898 (0.0007196)	6.6586982
-0.006	-0.2447468	0.3385183 (0.0006100)	0.3373131 (0.0005179)	7.0908734
-0.002	-0.1099389	0.3511130 (0.0005080)	0.3512829 (0.0003793)	7.1946104
0.002	0.1099389	0.3296504 (0.0006325)	0.3285171 (0.0004363)	6.9353738
0.006	0.2447468	0.2799801 (0.0008396)	0.2788112 (0.0006115)	6.3815167
0.01	0.2942210	0.2176668 (0.0010351)	0.2159984 (0.0007818)	5.6571110

Table 9: Fair values for the GAO and the indexed annuity in the Wishart specification, Example 3.

An interpretation of the results of Table 9 can be given by looking at the matrix $Q^\top Q$. We remark that the diagonal components of $Q^\top Q$ increase with the absolute value of Q_{12}. Therefore, by looking at the dynamics of the Wishart process (see equation (22)), we observe that the drift (and in particular the long term value) of its positive factors (and therefore the drift of the mortality and the interest rate process, see equation (37)) is an increasing function of the absolute value of Q_{12}. Hence the positive factors will be higher on average when the absolute value of Q_{12} increases. Consequently, we observe that the exponential terms (which involve the interest rate and the mortality rate) in formula (28) and (29) will decrease when the value of Q_{12} is moving away from zero and therefore the GAO value will also decrease. A similar argument holds for the indexed annuity.

6 Conclusion

In this paper, we have investigated the influence on pricing of the dependence structure between mortality and interest rates. Indeed, we have assumed that mortality and interest rates are driven by systematic and idiosyncratic factors, modeled by affine models which remain positive such as the multi-CIR and the Wishart models. In line with Liu et al. (2014), we applied a change of probability measure with the SZCB as numéraire to the valuation of a GAO and an indexed annuity. The interest of this change of probability measure is that it leads to rather simple formulas for the prices of the insurance products. We observed that for an advanced affine model (such as the Wishart one) that allows to reproduce a non-trivial dependence between the mortality and the interest rates, the values of a GAO or an indexed annuity cannot be explained only in terms of the initial pairwise linear correlation. This fact has important consequences on risk management in the presence of an unknown dependence.

It is clear that the dependence between mortality and interest rates has an important implication on the pricing of insurance products and that several behaviors are possible, depending on the model being used. The Wishart model seems to be the most flexible model (amongs those considered) producing the richest structure of dependence.

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A Affine specification: a unified approach

In this section we recall some background on affine processes. We will follow the unified approach as presented in Keller-Ressel and Mayerhofer (2015). Consider a time-homogeneous affine Markov process X taking values in a non-empty convex subset E of $\mathbb{R}^d$ $(d \geq 1)$, endowed with the inner product $\langle \cdot, \cdot \rangle$. The Markov process X is affine if it is stochastically continuous and its characteristic function has exponential-affine dependence on the initial state, i.e. there exist some deterministic functions $\tilde{\phi}_u : \mathbb{R}_+ \to \mathbb{C}$ and $\tilde{\psi}_u : \mathbb{R}_+ \to \mathbb{C}^d$ such that the semigroup P acts as follows:

$$\int_E e^{\langle u,w \rangle} P_t(x, dw) = e^{\tilde{\phi}_u(t) + \langle \tilde{\psi}_u(t), x \rangle}$$

for all $t \geq 0$, $x \in E$ and $u \in i\mathbb{R}^d$. It can be shown (see e.g. Cuchiero et al. (2011)) that the process X is a semimartingale with characteristics

$$\begin{aligned} A_t &= \int_0^t a(X_{s-})ds, \\ B_t &= \int_0^t b(X_{s-})ds, \\ \nu(\omega, dt, d\xi) &= K(X_{t-}(\omega), d\xi)dt, \end{aligned}$$

with $a(x), b(x), K(x, d\xi)$ affine functions:

$$\begin{aligned} a(x) &= a + x_1\alpha^1 + ... + x_d\alpha^d, \\ b(x) &= b + x_1\beta^1 + ... + x_d\beta^d, \\ K(x, d\xi) &= m(d\xi) + x_1\mu^1(d\xi) + ... + x_d\mu^d(d\xi), \end{aligned} \tag{38}$$

where $a(x)$ (the diffusion coefficient) is a positive semidefinite $d \times d$ matrix, a and α^i are $d \times d$ matrices, $b(x)$ (the vector of the drift), b and β^i are $\mathbb{R}^d$-vectors, $K(x, d\xi)$, $m(d\xi)$ and $\mu^i(d\xi)$ are Radon measures on $\mathbb{R}^d$ and $K(x, d\xi)$ is associated to the affine jump part and it is such that

$$\int_{\mathbb{R}^d} \left(\|\xi\|^2 \wedge 1\right) K(x, d\xi) < \infty$$

The deterministic functions $\tilde{\phi}_u(t), \tilde{\psi}_u(t)$ solve the *generalized Riccati equations*

$$\begin{aligned} &\frac{\partial}{\partial t}\tilde{\phi}_u(t) = \frac{1}{2}\langle\tilde{\psi}_u(t), a\tilde{\psi}_u(t)\rangle + \langle b, \tilde{\psi}_u(t)\rangle + \int_{\mathbb{R}^d\setminus\{0\}} \left(e^{-\langle\xi,\tilde{\psi}_u(t)\rangle} - 1 - \langle h(\xi), \tilde{\psi}_u(t)\rangle\right) m(d\xi), \\ &\quad \tilde{\phi}_u(0) = 0, \end{aligned}$$

and for all $i = 1, ..., d$:

$$\begin{aligned} &\frac{\partial}{\partial t}\tilde{\psi}_u^i(t) = \frac{1}{2}\langle\tilde{\psi}_u(t), \alpha^i\tilde{\psi}_u(t)\rangle + \langle \beta^i, \tilde{\psi}_u(t)\rangle + \int_{\mathbb{R}^d\setminus\{0\}} \left(e^{-\langle\xi,\tilde{\psi}_u(t)\rangle} - 1 - \langle h(\xi), \tilde{\psi}_u(t)\rangle\right) \mu^i(d\xi), \\ &\quad \tilde{\psi}_u(0) = u, \end{aligned}$$

where $h(\xi) = \mathbf{1}_{\{\|\xi\|\leq 1\}}\xi$ is a truncation function.

It is useful to consider the process $(X, Y^\gamma) := (X, \int_0^\cdot \langle\gamma, X_u\rangle du)$ which is an affine process with state space $E \times \mathbb{R}$ starting from $(X_0, 0)$. We now recall an interesting lemma.

Lemma 1. *Let $\tilde{P}^\gamma$ be the semigroup of the process (X, Y^γ). Then we have for every $u \in i\mathbb{R}^d$ and $v \in i\mathbb{R}$*

$$\int_{E\times\mathbb{R}} e^{\langle u,w\rangle + vz}\tilde{P}_t^\gamma((x, y), (dw, dz)) = e^{\Phi_{(u,v)}(t,\gamma) + \langle\Psi_{(u,v)}(t,\gamma),x\rangle + vy},$$

where the functions $\Phi_{(u,v)}(\cdot,\gamma)$ *and* $\Psi_{(u,v)}(\cdot,\gamma)$ *satisfy the following system of generalized Riccati ODEs*

$$\begin{aligned}\frac{\partial}{\partial t}\Phi_{(u,v)}(t,\gamma) &= \frac{1}{2}\langle\Psi_{(u,v)}(t,\gamma), a\Psi_{(u,v)}(t,\gamma)\rangle + \langle b, \Psi_{(u,v)}(t,\gamma)\rangle \\ &\quad + \int_{\mathbb{R}^d\setminus\{0\}} \left(e^{-\langle\xi,\Psi_{(u,v)}(t,\gamma)\rangle} - 1 - \langle h(\xi), \Psi_{(u,v)}(t,\gamma)\rangle\right) m(d\xi), \qquad (39)\\ \Phi_{(u,v)}(0,\gamma) &= 0,\end{aligned}$$

and for $i = 1, ..., d$

$$\begin{aligned}\frac{\partial}{\partial t}\Psi^i_{(u,v)}(t,\gamma) &= v\gamma^i + \frac{1}{2}\langle\Psi_{(u,v)}(t,\gamma), \alpha^i\Psi_{(u,v)}(t,\gamma)\rangle + \langle \beta^i, \Psi_{(u,v)}(t,\gamma)\rangle \\ &\quad + \int_{\mathbb{R}^d\setminus\{0\}} \left(e^{-\langle\xi,\Psi_{(u,v)}(t,\gamma)\rangle} - 1 - \langle h(\xi), \Psi_{(u,v)}(t,\gamma)\rangle\right) \mu^i(d\xi), \qquad (40)\\ \Psi_{(u,v)}(0,\gamma) &= u.\end{aligned}$$

Proof. Standard, see e.g. Grasselli and Miglietta (2016). □

B Noncentral Wishart distribution

Here we recall a result on noncentral Wishart distributions.

Theorem 1 (Theorem 3.5.1 in Gupta and Nagar (1999))**.** *Let* $X \sim \mathcal{N}_{p,n}(M, \Sigma \otimes I_n)$*,* $n \geq p$*, then* $S = XX^\top$ *is said to have a noncentral Wishart distribution with parameters* p*,* n*,* $\Sigma > 0$ *and* Θ*, written as* $S \sim \mathcal{W}_p(n, \Sigma, \Theta)$*, where* $\Theta = \Sigma^{-1}MM^\top$*.*

The matrix Θ is called the noncentrality parameter matrix. According to Gupta and Nagar (1999) [Theorem 3.5.7], the expectation of S is given by

$$\mathbb{E}[S] = n\Sigma + MM^\top.$$

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