Conditional Dominance criteria: definition and application to risk-management.

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Abstract: We define the concept of conditional dominance and use it for the obtention of bounds on the hedging prices of random variables. These bounds depend only on the characteristics of the financial market and the random variables to hedge. Moreover, they are coherent with the equilibrium and tighter than the ones obtained by the classical super-replication approach, significantly in some cases. This approach can be applied in static as well as dynamic frameworks.

Key words: hedging, incomplete markets, risk management, stochastic ordering.

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1 Introduction

Over the last years, the hedging of random variables within an incomplete financial framework is a crucial topic in Finance and Insurance. In Finance, there are some famous papers devoted to the hedging of a risky position (see e.g. the references). More recently in Insurance, a debate has been started in France about the possible creation of pension funds and about the ways to use the financial markets to hedge the risks associated with the retirement problem. In this paper, we concentrate on the private values of the risks to be hedged (which means the value that an agent is willing to pay in order to remove the risk) and we provide a theoretical contribution by showing how to obtain bounds on them.

There are several ways to associate prices to random variables which are not priced by the financial market; the more popular ones are quadratic hedging and super-replication.

Quadratic hedging is introduced by Föllmer and Sondermann (1986) and extended by Föllmer and Schweizer (1991), Schweizer (1991,1992) and Duffie and Richardson (1991). Under the quadratic hedging approach, the price of any random variable is put equal to the L^2-projection on the space of the random variables priced by the financial market. This approach is very practical since it provides a unique price which can be easily computed by using the theory of square-integrable random variables. Nevertheless, this method has an important weakness since the part of the random variable which is orthogonal to the space of the random variables priced by the financial market, is valorized by zero and this part (also called tracking error in literature) is precisely the risk that cannot be replicated by the financial markets.

Super-replication (see e.g. El Karoui and Quenez 1995) proposes an interval of prices which contains the price of the random variable to be hedged. This interval is determined by an upper and lower bound which are defined as the infimum (resp. supremum) of the prices of the random variables priced by the financial market that dominate (resp. are dominated by) the random variable to hedge almost surely. Unfortunately, this approach leads to very large intervals (see e.g. Soner, Shreve and Cvitanic 1995).

Another method to determine prices in an incomplete framework is introduced by Hodges and Neuberger (1989), continued by Davis, Panas and Zariphopoulou (1993). Under this approach, the price of the random variable to be hedged, is valorized as the private value of the agent. Therefore, the proposed price depends on the characteristics of the agent and especially on

its utility function, which is very hard to model in practice.

In this paper, we show how to determine bounds on the private value of any random variable which do not depend on the characteristics of the agents. The bounds are only depending on the characteristics of the financial market on one hand, and on the random variable to be hedged on the other hand. We prove that our bounds are strictly improving the bounds provided by super-replication.

We obtain those bounds by using a stochastic dominance approach. Stochastic dominance has been used by Levy (1985) in order to find bounds on the prices of European options, but he needed a strong restriction on the financial portfolios of the agents. In our paper, we do not need such quite unrealistic hypotheses, but our approach uses a slightly stronger criterion in comparison with the second stochastic dominance criterion used by Levy (1985).

The paper is organized as follows: in section 2, we introduce the framework and the general notion of hedging induced by conditional dominance. Section 3 is devoted to the study of upper and lower bounds on the private values of any random variable, and this by using conditional dominance. We prove that the interval we obtain, is included in the interval given by the super-replication approach. Some examples of the conditional dominance upper bound are computed in section 4, namely for the case of log-normal and multinomial random variables. These bounds are compared with the ones determined by super-replication. In section 5, we consider a dynamic version of the conditional dominance. Section 6 concludes the paper.

2 Framework and Notations

Throughout the paper, we work on a probability space $(\Omega, \mathcal{F}, P)$. The information structure is described by a filtration $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions. It is assumed that $\mathcal{F}$ is generated by a d-dimensional Markovian process G. The state of the world at date t is then characterized by G_t.

2.1 The financial market

We assume the existence of a financial market with two assets: a riskless asset, with a constant rate of return taken equal to zero for the ease of exposition, and a risky asset whose price at time t is denoted by S_t, an $\mathcal{F}_t$ measurable random variable. In the following, we assume that the financial

market is able to supply any quantity of the two assets, that the price of any portfolio is obtained linearly from the unitary prices and that there are no transaction costs. We consider an agent who can trade at discrete dates $\mathcal{T} = \{0, 1, .., T - 1\}$, where $T \in \mathbb{N}$ is his time-horizon. This agent is endowed by an initial wealth $x \in \mathbb{R}$ and a Von Neumann-Morgenstern utility function u which is strictly concave and strictly increasing. The problem of the agent is to hedge and price the random variable $Y \in L^1(\Omega, \mathcal{F}_T, P)$.

We define a financial strategy as a random process $\{\theta_t\}_{0 \le t \le T-1}$, where the $\mathcal{F}_t$ measurable random variable θ_t denotes the number of units of the risky asset chosen at date t. We denote by Θ the set of the financial strategies. A random variable $X \in L^1(\Omega, \mathcal{F}_T, P)$ is attainable at date T if there exist a real number x and a financial strategy $\{\theta_t\}_{0 \le t \le T-1}$ such that

$$X = x + \sum_{t=0}^{T-1} \theta_t (S_{t+1} - S_t). \tag{1}$$

Remark that x in this notation is the initial amount to invest and by no-arbitrage arguments, this is exactly the price of X. In the sequel, we need to consider the function which maps the attainable contingent claims into the line of real numbers, by associating its price to the random variable. This function is denoted by $V(.)$ and using the notations of (1): $V(X) = x$. We denote by $\mathcal{A}_T \subset L^1(\Omega, \mathcal{F}_T, P)$ the subset of the random variables attainable at date T.

The optimization program of the agent is then

$$\max_{X \in \mathcal{A}_T} \mathbb{E}u\left(X + Y\right) \tag{2}$$

under the constraint that the price of X equals a fixed amount x. We denote by $J(x, Y)$ the optimal value of $\mathbb{E}u\left(X + Y\right)$ when the initial wealth is equal to x.

The hedging price of the random variable Y is the real number $\pi(Y)$ such that $J(x, Y) = J(x + \pi(Y), 0)$

The meaning of this definition is that $-\pi(Y)$ is the amount of money that the agent is willing to pay in order to remove the risk Y. It is adapted from the notion of certainty equivalent of one risk in the presence of others risks (see Pratt (1988)). This definition is not easy to manage, because it depends on the characteristics of the agent, and especially on the utility function u which cannot be estimated easily in practice. However, if $Y \in \mathcal{A}_T$, it is well-known that $\pi(Y) = V(Y)$, see e.g. Davis, Panas and Zariphopoulou (1993).

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In this way, the hedging price is an extension (depending on the agent) of V to the set $L^1(\Omega, \mathcal{F}_T, P)$.

2.2 Conditional dominance

In this subsection, we define conditional dominance and link this concept to first and second stochastic dominances and to $P - a.e.$ dominance.

Let $(X, Y) \in L^1(\Omega, \mathcal{F}, P) \times L^1(\Omega, \mathcal{F}, P)$. We say that

- X dominates Y in the sense of conditional dominance, and we denote $X \succeq_{CD} Y$, if and only if there exists a random variable ε such that $Y = X + \varepsilon \; P - a.e$ and $\mathbb{E}(\varepsilon \mid X) \leq 0 \; P - a.e.$

- X dominates Y in the sense of first stochastic dominance, and we denote $X \succeq_{FSD} Y$, if and only if there exists $X' \in L^1(\Omega, \mathcal{F}, P)$ such that X and X' are identically distributed and $X' \geq Y \; P - a.e.$

- X dominates Y in the sense of second stochastic dominance, and we denote $X \succeq_{SSD} Y$, if and only if there exists $Y' \in L^1(\Omega, \mathcal{F}, P)$ such that Y and Y' are identically distributed and $X \succeq_{CD} Y'$.

First and second stochastic dominances are defined and discussed for instance in Huang and Litzenberger (1988) or Levy (1992), while their use in insurance is shown e.g. in Goovaerts *et al.* (1990). By definition, conditional dominance is a reinforcement of the second stochastic dominance. The following Lemma is obvious and the proof is omitted.

For any $(X, Y) \in L^1(\Omega, \mathcal{F}, P) \times L^1(\Omega, \mathcal{F}, P)$,
(i) $X \geq Y \; P - a.e$ implies $X \succeq_{FSD} Y$
(ii) $X \geq Y \; P - a.e$ implies $X \succeq_{CD} Y$
(iii) $X \succeq_{FSD} Y$ implies $X \succeq_{SSD} Y$
(iv) $X \succeq_{CD} Y$ implies $X \succeq_{SSD} Y$
(v) $X \succeq_{FSD} Y$ does not imply necessarily $X \succeq_{CD} Y$
(vi) $X \succeq_{CD} Y$ does not imply necessarily $X \succeq_{FSD} Y$

The next Lemma provides an equivalent characterization of the CD-dominance:

Let $(X, Y) \in L^1(\Omega, \mathcal{F}, P) \times L^1(\Omega, \mathcal{F}, P)$. Then

$$X \succeq_{CD} Y \quad if and only if \quad \mathbb{E}(Y \mid X) \leq X \; P - a.e. \tag{3}$$

"If" part. Let us define $\varepsilon := Y - X$. Then $Y = X + \varepsilon$ $P - a.e$ and $\mathbb{E}(\varepsilon \mid X) = \mathbb{E}(Y - X \mid X) = \mathbb{E}(Y \mid X) - X \leq 0$ $P - a.e.$

"Only If" part. $Y = X + \varepsilon$ $P - a.e$ and $\mathbb{E}(\varepsilon \mid X) \leq 0$ $P - a.e$; then $\mathbb{E}(\varepsilon \mid X) = \mathbb{E}(Y - X \mid X) = \mathbb{E}(Y \mid X) - X \leq 0$ $P - a.e.$

3   Static pricing

In section 3 and 4, we consider only two dates: the initial date 0 and the time-horizon $T = 1$. Transactions are allowed at date 0 only and the number of units of the risky asset chosen at date 0 is denoted by θ_0. A contingent claim attainable at date 1 is then a random variable $X \in L^1(\Omega, \mathcal{F}_1, P)$ such that:

$$\exists (x, \theta_0) \in \mathbb{R} \times \mathbb{R}, \qquad X = x + \theta_0(S_1 - S_0). \tag{4}$$

Our aim is to use conditional dominance in order to obtain an upper bound for the hedging price of any variable $Y \in L^1(\Omega, \mathcal{F}_1, P)$, which should be more accurate than the one obtained by the $P - a.e.$ dominance, namely the usual super-replication cost.

Let $Y \in L^1(\Omega, \mathcal{F}_1, P)$. We define:

$P_{ae}(Y) := \inf\{V(X), X \in \mathcal{A}_1 \text{ and } X \geq Y \; P - a.e.\}$

$P_{CD}(Y) := \inf\{V(X), X \in \mathcal{A}_1 \text{ and } X \succeq_{CD} Y\}$

with the convention $\inf\{\emptyset\} = +\infty$.

The quantity $P_{ae}(Y)$ is known in the literature as the super-replication cost of Y. Thanks to Lemma 3, we get easily $P_{ae}(Y) \geq P_{CD}(Y)$.

Note that one could define P_{FSD} and P_{SSD} in an analoguous way. However, it is very easy to construct an equilibrium where $V(X) > P_{FSD}(X)$ (and consequently $V(X) > P_{SSD}(X)$) for some $X \in \mathcal{A}_1$. This is not in contradiction with the definition of the stochastic dominance. It implies only that it is not optimal for all the agents present on the market to buy X only. Then, it is hopeless to use P_{FSD} and P_{SSD} in order to obtain upper bounds on the hedging price.

In order to give an example of an equilibrium with $V(X) > P_{FSD}(X)$, let us consider the simple case that Ω is finite and that all possible states of the world have the same probability: $\Omega = \{\omega_1, .., \omega_n\}$, $P(\omega_i) = 1/n$ for $i = 1, ..., n$. Let for all $i = 1, ..., n$, X_i denote the Arrow-Debreu contingent claim associated to the state ω_i (which means that X_i pays 1 unit if the state of the world tomorrow is ω_i and 0 elsewhere).

Assume that for all $i = 1, ..., n$, $X_i \in \mathcal{A}_1$ and that the price at date 0 of X_i equals x_i. An immediate computation shows that for all $i = 1, ..., n$

$$\inf\{V(X), \, X \in \mathcal{A}_1, X \succeq_{FSD} X_i\} = 1 \leq i \leq n\inf\{x_i\},$$

which is a quantity independent of i. Let us now consider when $V(X) \leq \inf\{V(Y), \, Y \in \mathcal{A}_1, Y \succeq_{FSD} X\}$ for all X. Since $V(X_i) = x_i$, we must have in any case that $V(X_i) = x_i \leq 1 \leq i \leq n\inf\{x_i\}$. The fact that all states have the same probability implies then that all x_i should be equal.

In conlusion, in order to have $V(X) \leq \inf\{V(Y), \, Y \in \mathcal{A}_1, Y \succeq_{FSD} X\}$ for all X, all Arrow-Debreu state prices should be equal, which is of course not satisfied in all equilibria.

The following Proposition establishes the relationship between conditional dominance and the hedging price.

Let $X = x + \theta_0(S_1 - S_0) \in \mathcal{A}_1$ such that $\theta_0 \neq 0$ and $X \succeq_{CD} Y$. Then $V(X) \geq \pi(Y)$.

Suppose $V(X) < \pi(Y)$. Let X^* be the optimal contingent claim chosen by the agent a in the presence of the risk Y and $\widehat{X}$ the optimal contingent claim chosen by the agent a when the risk Y is not present. By the definition of $\pi(Y)$, we must have

$$\mathbb{E}u\left(\widehat{X}\right) = \mathbb{E}u\left(X^* + Y\right) and V\left(\widehat{X}\right) = V(X^*) + \pi(Y)$$

Since $X = x + \theta_0(S_1 - S_0)$, $\theta_0 \neq 0$, we get $\mathbb{E}(X^* + Y \mid X^* + X) = \mathbb{E}(X^* + Y \mid X)$ (because that the σ-algebra generated by $X^* + X$ coincides with the one generated by X for $\theta_0 \neq 0$). Therefore, $X \succeq_{CD} Y$ leads to $X^* + X \succeq_{CD} X^* + Y$. This implies $X^* + X \succeq_{SSD} X^* + Y$ which provides $\mathbb{E}u(X^* + X) \geq \mathbb{E}u(X^* + Y)$. But $V(X^* + X) = V(X^*) + V(X) < V(X^*) + \pi(Y) = V\left(\widehat{X}\right)$. This contradicts the optimality of $\widehat{X}$.

The last Proposition leads us to the construction of our upper bound. Since it requires that $X = x + \theta_0(S_1 - S_0)$ with $\theta_0 \neq 0$, we must introduce another pricing functional P_{CD}^*, which is a slight modification of the natural candidate P_{CD}. First we exclude

$$\{X = x + \theta_0(S_1 - S_0), \theta_0 = 0 and X \succeq_{CD} Y\}$$

from the set on which the infimum is taken. Afterwards, since it is easily checked that $X \in \mathcal{A}_1$ and $X \geq Y$ $P - a.e.$ implies $V(X) \geq \pi(Y)$, we can

improve the accuracy of the upper bound by adding

$$\{X = x + \theta_0(S_1 - S_0), \theta_0 = 0 \ and \ X \geq Y P - a.e.\}$$

Finally, we propose the following definition.

Let $Y \in L^1(\Omega, \mathcal{F}_1, P)$. We define:

$$P_{CD}^*(Y) := \inf\{V(X), \quad X = x + \theta_0(S_1 - S_0) \ and$$
$$if \theta_0 \neq 0, X \succeq_{CD} Y$$
$$if \theta_0 = 0, X \geq Y P - a.e.\}$$

As an immediate consequence of the last Proposition, we get:

$$for all Y \in L^1(\Omega, \mathcal{F}_1, P), P_{ae}(Y) \geq P_{CD}^*(Y) \geq \pi(Y).$$

This result yields to an upper bound for the hedging price $\pi(Y)$ which depends only on Y and not on the other characteristics of the agent a. Moreover, it is lower than the super-replication cost. Before comparing the two bounds on some examples in the next section, we deal with the obtention of a lower bound.

Following the literature on pricing functionals, it would be natural to define the lower bound as the supremum of the contingent claims' values dominated by Y in the CD sense. However, this notion is not suited here because for a non attainable Y, $Y \succeq_{CD} X$ does not imply $X' + Y \succeq_{CD} X' + X$ for any $X' \in \mathcal{A}_T$, since the σ-algebra generated by Y does not necessarily equal the one generated by $X' + Y$. This point shows that the lines of the proof of Proposition 6 cannot be followed in case of the lower bound.

When the hedging price is sublinear, we can nevertheless propose a lower bound. Sublinearity of the pricing functionals is often assumed in the literature as an axiom, see e.g. the discussion in Jouini (1997).

Assume that $\forall Y, Y' \in L^1(\Omega, \mathcal{F}_1, P)$, $\pi(Y + Y') \leq \pi(Y) + \pi(Y')$. Define $Q_{CD}^*(Y) = -P_{CD}^*(-Y)$, and $\overline{Q}_{ae}(Y) = \sup\{V(X), X \in \mathcal{A}_1 \text{ and } Y \geq X P - a.e\}$. Then $\pi(Y) \geq Q_{CD}^*(Y) \geq \overline{Q}_{ae}(Y)$.

From the definition of the hedging price, we obviously have $\pi(0) = 0$. Then, sublinearity implies $\forall Y \in L^1(\Omega, \mathcal{F}_1, P)$, $\pi(Y) \geq -\pi(-Y)$, and $\pi(Y) \geq Q_{CD}^*(Y)$ holds, thanks to Proposition 6. The second part of the inequality comes again from Proposition 6, since $\overline{Q}_{ae}(Y) = -P_{ae}(-Y)$.

4 Examples

4.1 The log-normal case

In this subsection we investigate the special case where $S_1 = S_0 \exp(\sigma W)$ and $Y = Y_0 \exp(\eta Z)$ with S_0, Y_0, σ and η in $\mathbb{R}^+$ and $(W, Z) \sim N_2 \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right)$.
This example will be useful in the dynamic case, where we will illustrate our general results by studying the hedging in discrete time of a geometric Brownian motion when the hedging variable is a geometric Brownian motion too. The following Proposition compares the values of the upper bounds $P_{ae}(Y)$ and $P^*_{CD}(Y)$.

If $\rho = 0$ or if ($\rho > 0$ and $\frac{\sigma}{\rho\eta} > 1$),
$$\text{then } P^*_{CD}(Y) = Y_0 \exp\left(\frac{\eta^2(1-\rho^2)}{2} \right) \text{and else } P^*_{CD}(Y) = +\infty.$$

If ($\rho = 1$ and $\sigma = \eta$), then $P_{ae}(Y) = Y_0$ and else $P_{ae}(Y) = +\infty$.

The bound given by the super-replication approach is thus strictly improved when the hedging variable is positively correlated with and more volatile than the random variable to hedge.

- First, it is easily checked that $P_{ae}(Y) = Y_0$ for ($\rho = 1$ and $\sigma = \eta$) and that $P_{ae}(Y) = +\infty$ elsewhere since the support of a log-normal random variable is $[0, +\infty)$. For the same reason, it is not possible to find a constant x such that $x \geq Y \; P - a.e.$, and therefore

$$P^*_{CD}(Y) = \inf_{\theta_0} \{ x \in \mathbb{R} : \exists \theta_0 \in \mathbb{R}\backslash\{0\} \, such \, that \, x + \theta_0(S_1 - S_0) \succeq_{CD} Y \}.$$

- It is well-known that if X and Y are two random variables such that $(\ln X, \ln Y) \sim N_2 \left(\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \begin{pmatrix} \sigma_1 & \widetilde{\rho} \\ \widetilde{\rho} & \sigma_2 \end{pmatrix} \right)$, then:

$$\mathbb{E}[Y \mid X] = X^{\widetilde{\rho}\frac{\sigma_2}{\sigma_1}} \exp\{\frac{1}{2}\sigma_2^2(1 - \widetilde{\rho}^2) + m_2 - \widetilde{\rho}m_1\frac{\sigma_2}{\sigma_1}\} \; P - a.e.$$

Let us now take X of the form $X = x + \theta_0(S_1 - S_0)$, with $\theta_0 \neq 0$. Then:

$$\mathbb{E}[Y \mid X = w] = \mathbb{E}[Y \mid S_1 = S_0 + \frac{w - x}{\theta_0}] = \frac{1}{\lambda}(1 + \frac{w - x}{S_0\theta_0})^{\rho\frac{\eta}{\sigma}}$$

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with $\frac{1}{\lambda} = Y_0 \exp\left(\frac{\eta^2\left(1-\rho^2\right)}{2}\right) > 0$. From Lemma 4 we have that $X \succeq_{CD} Y$ if and only if $\mathbb{E}[Y \mid X] \leq X \ P - a.e.$ This provides:

$$X \succeq_{CD} Y \Leftrightarrow (1 + \frac{w - x}{S_0 \theta_0})^{\rho \frac{\eta}{\sigma}} \leq w\lambda, \ \forall w \in Im_X(\Omega). \tag{5}$$

- Let us first study the case that $\rho = 0$. Then

$$X \succeq_{CD} Y \Leftrightarrow Y_0 \exp\left(\frac{\eta^2}{2}\right) \leq x + \theta_0(S_1 - S_0) \ P - a.e. \tag{6}$$

Therefore, we determine θ_0 such that the infimum of

$$\left\{x \in \mathbb{R} : \exists \theta_0 \in \mathbb{R}^* \ such\,that\, Y_0 \exp\left(\frac{\eta^2}{2}\right) \leq x + \theta_0(S_1 - S_0) \ P - a.e.\right\}$$

is obtained. It is clear that we can exclude the case $\theta_0 < 0$ since the support of S_1 is $[0, +\infty)$. Thus, we look for the infimum of x such that there exists a $\theta_0 > 0$ such that $x \geq Y_0 \exp\left(\frac{\eta^2}{2}\right) + \theta_0 S_0$. Therefore, in this case

$$P_{CD}^*(Y) = Y_0 \exp\left(\frac{\eta^2}{2}\right). \tag{7}$$

- Let us now turn to the general case of $\rho \neq 0$. In order to find θ_0 that minimizes $\{x \in \mathbb{R} : \exists \theta_0 \in \mathbb{R}^* \text{ such that } x + \theta_0(S_1 - S_0) \succeq_{CD} Y\}$, we exclude again the case $\theta_0 < 0$. Indeed, if $\theta_0 < 0$, then $Im_X(\Omega) = \{w : 1 + \frac{w-x}{S_0 \theta_0} > 0\} = (-\infty, x - S_0 \theta_0)$, and the inequality (5) becomes:

$$\forall w \in (-\infty, x - S_0 \theta_0), \ \left(1 + \frac{w - x}{S_0 \theta_0}\right)^{\rho \frac{\eta}{\sigma}} \leq w\lambda$$

The left-hand side of the inequality term is always positive, and the right-hand side is negative for $w \leq 0$, so there are no solutions.

We further concentrate on the case $\theta_0 > 0$. Then $Im_X(\Omega) = \{w \text{ such that } 1 + \frac{w-x}{S_0 \theta_0} > 0\} = (x - S_0 \theta_0, +\infty)$. Notice that for θ_0 such that $x - S_0 \theta_0 < 0$, the inequality (5) becomes:

$$\forall w \in (x - S_0 \theta_0, +\infty), \ \left(1 + \frac{w - x}{S_0 \theta_0}\right)^{\rho \frac{\eta}{\sigma}} \leq w\lambda$$

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But the left-hand side of this inequality is positive for $w > x - S_0\theta_0$, whereas the right-hand side is negative for $w \in (x - S_0\theta_0, 0)$. Therefore we just need to consider θ_0 such that $x - S_0\theta_0 \geq 0$, i.e. θ_0 must lie in the interval $\left(0, \frac{x}{S_0}\right]$. Let us denote $\delta := \frac{\sigma}{\rho\eta}$, $k := (\lambda)^\delta$ and $\theta := S_0\theta_0$. We first consider the case where ρ is negative.

- Case $\rho < 0$: $X \succeq_{CD} Y$ is equivalent to $1 + \frac{w-x}{\theta} \geq w^\delta k$. The left-hand side of the inequality tends to zero when w tends to $(x - \theta)$, whereas the right-hand side tends to a positive value. Then, there are no solutions in that case.

- Case $\rho > 0$: $X \succeq_{CD} Y$ is equivalent to $\forall w \in (x - \theta, +\infty)$, $\varphi(w) \geq 0$, where the function $\varphi(.)$ is defined as follows: $\varphi(w) := \theta k w^\delta - w + x - \theta$.

    - If $\delta < 1$, there are no solutions as $\varphi(+\infty) = -\infty$.
    - If $\delta = 1$, $\forall w \in (x - \theta, +\infty)$, $\varphi(w) \geq 0$ on $(x - \theta, +\infty)$ is equivalent to $\varphi(x - \theta) \geq 0$ and $\theta k - 1 \geq 0$. We have then two conditions: $\theta \geq \frac{1}{k}$ and $\theta \in (0, x]$, so the infimum of x is obtained at $\frac{1}{k}$ and $P_{CD}^*(Y) = \frac{1}{k} = Y_0 \exp\left(\frac{\eta^2\left(1-\rho^2\right)}{2}\right)$.
    - If $\delta > 1$, we need to study more precisely the behaviour of the function $\varphi(.)$ on $(x - \theta, +\infty)$. It is easily checked that:

$$\varphi(+\infty) = +\infty$$
$$\varphi(x - \theta) = \theta k (x - \theta)^\delta > 0$$
$$\varphi'(\overline{w}) = 0 \Leftrightarrow \overline{w} = \left(\frac{1}{\delta k \theta}\right)^{\frac{1}{\delta - 1}}$$

      Then $\varphi(w) \geq 0 \; \forall w \in (x - \theta, +\infty)$ if and only if $\overline{w} \geq x - \theta$ and $\varphi(\overline{w}) \geq 0$. But $\varphi(\overline{w}) \geq 0$ is equivalent to $x - \theta \geq \overline{w}\frac{\delta-1}{\delta}$. Therefore, our requirements are equivalent to $\overline{w}\frac{\delta-1}{\delta} \leq x - \theta \leq \overline{w}$. Setting $\beta = \frac{\delta-1}{\delta}\left(\frac{1}{\delta k}\right)^{\frac{1}{\delta-1}} > 0$ and $\gamma = \frac{1}{\delta-1} > 0$, we get $\varphi(\overline{w}) = -\beta\theta^{-\gamma} + x - \theta$. We consider this last expression as a function of θ: $f(\theta) := -\beta\theta^{-\gamma} + x - \theta$. Since we have: $f(0) = -\infty$ and $f(x) = -\beta x^{-\gamma} < 0$, there exists a $\theta \in (0, x]$ such that $f(\theta) \geq 0$ only if there exists a $\overline{\theta} \in (0, x)$ such that $f'(\overline{\theta}) = 0$ and $f(\overline{\theta}) \geq 0$. As $f'(\theta) = 0$ at $\overline{\theta} = \left(\frac{1}{\beta\gamma}\right)^{-\frac{1}{\gamma+1}}$, it turns out that

$$f(\overline{\theta}) = x - \left(\frac{1}{\beta\gamma}\right)^{-\frac{1}{\gamma+1}}\left[1 + \frac{1}{\gamma}\right] \geq 0$$

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if and only if

$$x \geq (\frac{1}{\beta\gamma})^{-\frac{1}{\gamma+1}}[1 + \frac{1}{\gamma}].$$

And then, the minimal x such that $f(\theta)$ remains positive is the upper bound given by:

$$P_{CD}^*(Y) = (1+\gamma)\gamma^{-\frac{\gamma}{\gamma+1}}\beta^{\frac{1}{\gamma+1}} = \frac{1}{\lambda}$$

which provides the announced result.

The proof of this proposition tells us that in order to hedge a risk Y in a static way, we have to invest

$$\overline{\theta} = (\frac{1}{\beta\gamma})^{-\frac{1}{\gamma+1}} = \frac{\rho\eta}{\sigma}Y_0 \exp\left(\frac{\eta^2 (1-\rho^2)}{2}\right) \qquad (8)$$

in the risky asset if $\frac{\sigma}{\rho\eta} \geqslant 1$, thus if there is a positive correlation between the risk Y and the risky asset S_1 and if $\sigma \geqslant \eta$. In the non-correlated case, the total investment should be done in the riskless asset, i.e. $\overline{\theta} = 0$.

4.2 The finite state case

The aim of this section is to show that the CD criterion may strictly improve the upper bound given by the super-replication approach in a case where Ω is finite. As a simple example, let us consider the case where there are $n \geq 3$ possible future states of the world, but where the risky financial asset can take only two values: u and d.

Formally, let $n \geq 3$, $\Omega = \{\omega_1, .., \omega_n\}$, $p_i = P(\omega_i) > 0$ and consider the risky asset with $S_0 = 1$, $S_1(\omega_i) = u > 1$, $i = 1, .., n_1 < n$, and $S_1(\omega_i) = d < 1$, $i = n_1 + 1, .., n$. Let Y be the risky to be hedged, with $Y(\omega_i) \neq Y(\omega_j), \forall\, i \neq j$.

In order to compute the super-replication upper bound $P_{ae}(Y)$ we can find the set Π of equivalent martingale measures and notice that $P_{ae}(Y)$ is also given (see e.g. El Karoui and Quenez 1995) by

$$P_{ae}(Y) = ess\sup_{\pi\in\Pi} \mathbb{E}^\pi(Y). \qquad (9)$$

By no arbitrage arguments it is easy to check that the set Π is characterized by the following conditions:

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$$\Pi = \{\pi \in (0,1)^n : \sum_{i=1}^n \pi_i = 1, \sum_{i=1}^{n_1} \pi_i = \frac{1-d}{u-d}, \sum_{i=n_1+1}^n \pi_i = \frac{u-1}{u-d}\}, \qquad (10)$$

so that $P_{ae}(Y)$ is given by

$$P_{ae}(Y) = \frac{1-d}{u-d}\max\{y_1,..,y_{n_1}\} + \frac{u-1}{u-d}\max\{y_{n_1+1},..,y_n\}. \qquad (11)$$

where $y_i = Y(\omega_i)$, $i = 1,..,n$.

Let us now turn our attention to the upper bound given by the CD criterion.

From (3) and the definition of S_t and Y we obtain

$$x + \theta_0(S_1 - S_0) \succeq_{CD} Y \iff \begin{cases} \frac{y_1 p_1 + .. + y_{n_1} p_{n_1}}{p_1 + .. + p_{n_1}} \leq x + \theta_0(u-1) \\ \frac{y_{n_1+1} p_{n_1+1} + .. + y_n y_n}{p_{n_1+1} + .. + p_n} \leq x + \theta_0(d-1) \end{cases}$$
$$\iff \begin{cases} x \geq \overline{y}_u - \theta_0(u-1) \\ x \geq \overline{y}_d + \theta_0(1-d), \end{cases}$$

where $\overline{y}_u = \sum_{i=1}^{n_1}\left(\frac{y_i p_i}{\sum_{j=1}^{n_1} p_j}\right)$ and $\overline{y}_d = \sum_{i=n_1+1}^n\left(\frac{y_i p_i}{\sum_{j=n_1+1}^n p_j}\right)$.

Now, by definition, $P_{CD}(Y) = \inf\{x \in \mathbb{R} : \exists\theta_0 \in \mathbb{R}^* \text{ such that } x + \theta_0(S_1 - S_0) \succeq_{CD} Y\} \wedge \max\{y_1,...,y_n\}$ and the infimum is obtained with

$$\theta_0^* = \frac{1}{u-d}(\overline{y}_u - \overline{y}_d), \qquad (12)$$
$$P_{CD}^*(Y) = x^* = \frac{1-d}{u-d}\overline{y}_u + \frac{u-1}{u-d}\overline{y}_d.$$

since $\max\{y_1,..,y_{n_1}\} = \sum_{i=1}^{n_1}\left(\frac{\max\{y_1,..,y_{n_1}\}p_i}{\sum_{j=1}^{n_1} p_j}\right) > \sum_{i=1}^{n_1}\left(\frac{y_i p_i}{\sum_{j=1}^{n_1} p_j}\right) = \overline{y}_u$ and analogously $\max\{y_{n_1+1},..,y_n\} > \overline{y}_d$.

This leads also to the conclusion that $P_{CD}^*(Y)$ is strictly lower than $P_{ae}(Y)$.

Notice that the crucial point in our example is that the number of all possible values that can be obtained by S_1 is smaller than the number of all possible values obtained by Y, otherwise the CD upper bound coincides with the super-replication one.

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5 Dynamic pricing

5.1 The general result

In this section we come back to the general dynamic model presented in section 2. We show how to use the CD criterion in order to obtain an upper bound for the hedging price when dynamic strategies are allowed. This upper bound is constructed by using recursively the reasoning made in the static framework. For intuition, let us consider the case that Ω is finite, in which we can describe the informational structure by a tree. We then introduce repeatly a static price functional between dates t and $t+1$ which is a straightforward generalization of the static price functional P_{CD}^* of definition 7, but started from the node which is attained at date t. Thus we apply the functionals conditional to the information until then. In a general markovian setting we describe the history until time t by the state variable G_t (see section 2), and we introduce the following notations.

For $k \in \mathbb{R}^d$, we denote by $\Omega_t(k) := \{\omega \in \Omega \, such \, that \, G_t(\omega) = k\}$. Let L_t be the space of the integrable random variables which are measurable w.r.t. $\mathcal{F}_t$. For $A \in L_t$ and $k \in Im(G_t)$, we denote by $A_{|\Omega_t(k)}$ the restriction of A to $\Omega_t(k)$.

For any $t \in \mathcal{T}$, $U \in L_{t+1}$, we define the static price functional between dates t and $t+1$ $P_{CD}^{*t,t+1}(U) \in L_t$ as follows:

$$
P_{CD}^{*t,t+1}(U)(\omega) = \inf\{ \quad x \in \mathbb{R} \colon \exists \theta_t \in \mathbb{R} \, such \, that
$$
$$
if \, \theta_t \neq 0, [x + \theta_t(S_{t+1} - S_t)]_{|\Omega_t(G_t(\omega))} \succeq_{CD} U_{|\Omega_t(G_t(\omega))}
$$
$$
if \, \theta_t = 0, x \geqslant U_{|\Omega_t(G_t(\omega))} P - a.e.\}
$$

Notice that $P_{CD}^{*t,t+1}(\cdot)$ maps L_{t+1} into L_t. Using the static price functionals, we now turn to a dynamical quantity by backward reasoning.

Let $Y \in L_T$ and define $P_{CD}^{*Dyn}(Y) \in \mathbb{R}$ as follows:

$$
P_{CD}^{*Dyn}(Y) := P_{CD}^{*0,1}(P_{CD}^{*1,2}(...P_{CD}^{*T-2,T-1}(P_{CD}^{*T-1,T}(Y))...)). \tag{13}
$$

In the following proposition, we prove that the dynamical conditional dominance quantity $P_{CD}^{*Dyn}(Y)$ is an upper bound of the hedging price of $Y \in L_T$. Moreover, it is easy to check that $P_{CD}^{*Dyn}(Y)$ is lower than the upper bound given by the super-replication approach. The proof of this result is

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left to the reader; the arguments are similar to the ones used in the static case.

$\forall\, Y \in L_T,\ P_{CD}^{*Dyn}(Y) \geq \pi\,(Y).$

First, for any $t \in \mathcal{T}$ and $U \in L_{t+1}$ let us define $\pi_{t,t+1}(U) \in L_t$ as follows:

$$\max_{\theta_t,\theta_{t+1},..,\theta_{T-1}} \mathbb{E}\left[u\left(x + \sum_{t=0}^{T-1}\theta_t(S_{t+1} - S_t) + U\right) | \mathcal{F}_t\right]$$

$$= \max_{\theta_t,\theta_{t+1},..,\theta_{T-1}} \mathbb{E}\left[u\left(x + \sum_{t=0}^{T-1}\theta_t(S_{t+1} - S_t) + \pi_{t,t+1}(U)\right) | \mathcal{F}_t\right].(14)$$

Notice that $\pi_{t,t+1}(\cdot)$ maps L_{t+1} into L_t. Thanks to Proposition 6, we get:

$$\forall U \in L_{t+1}, P_{CD}^{*t,t+1}(U) \geq \pi_{t,t+1}(U) \quad P - a.e.$$

Using this inequality recursively yields:

$$
\begin{aligned}
P_{CD}^{*Dyn}(Y) &= P_{CD}^{*0,1}(P_{CD}^{*1,2}(...P_{CD}^{*T-2,T-1}(P_{CD}^{*T-1,T}(Y))...) \\
&\geq P_{CD}^{*0,1}(P_{CD}^{*1,2}(...P_{CD}^{*T-2,T-1}(\pi_{T-1,T}(Y))...) \\
&\geq \pi_{0,1}(...\pi_{T-2,T-1}(\pi_{T-1,T}(Y))...).
\end{aligned}
$$

By the dynamic programming principle, we obtain for $X = x + \sum_{t=0}^{T-1}\theta_t(S_{t+1} - S_t)$:

$$
\begin{aligned}
\max_{\theta_0,\theta_1,..,\theta_{T-1}} \mathbb{E}u\,(X + Y) &= \max_{\theta_0,\theta_1,..,\theta_{T-2}} \mathbb{E}[\max_{\theta_{T-1}} \mathbb{E}[u\,(X + Y)\,|\,\mathcal{F}_{T-1}]] \\
&= \max_{\theta_0,\theta_1,..,\theta_{T-2}} \mathbb{E}[\max_{\theta_{T-1}} \mathbb{E}[u\,(X + \pi_{T-1,T}\,(Y))\,|\,\mathcal{F}_{T-1}]] \\
&= \max_{\theta_0,\theta_1,..,\theta_{T-3}} \mathbb{E}\max_{\theta_{T-2},\theta_{T-1}} \mathbb{E}[u\,(X + \pi_{T-2,T-1}(\pi_{T-1,T}(Y)))\,|\,\mathcal{F}_{T-2}] \\
&= \max_{\theta_0,\theta_1,..,\theta_{T-1}} \mathbb{E}u\,(X + \pi_{0,1}(...\pi_{T-2,T-1}(\pi_{T-1,T}(Y))...)).
\end{aligned}
$$

Since by definition

$$\max_{\theta_0,\theta_1,..,\theta_{T-1}} \mathbb{E}u\,(X + Y) = \max_{\theta_0,\theta_1,..,\theta_{T-1}} \mathbb{E}u\,(X + \pi(Y)),$$

we conclude that $\pi_{0,1}(...\pi_{T-2,T-1}(\pi_{T-1,T}(Y))...) = \pi\,(Y)$ and the proof is complete.

5.2 Discrete-time hedging of a geometric Brownian motion

In this subsection we apply the dynamic pricing rule given by (13) to the case where $(S_t)_{t\geq 0}$ and $Y = (Y_t)_{t\geq 0}$ are two correlated geometric Brownian motions. The problem is to use the financial market in order to hedge Y_T.

For an economic interpretation, let us consider a model with two agents: the employer and the employee. The employee works between dates 0 and T for the employer and receives a wage process $(\widetilde{Y}_t)_{0\leq t\leq T}$. This process is adapted w.r.t. the filtration $\mathcal{F}$, but cannot be expressed in general as a linear combination of the financial assets. At his retirement date T, the employee receives also a fixed amount Y_T which is in fact a defined pension paid out at once at the date T. As the pension usually depends on the wage history at a fixed percentage β, the problem of the employer, who has the charge of paying the contributions, is to hedge $Y_T = \beta \widetilde{Y}_T$ by using the financial market.

Therefore, we consider a continuous time framework, where the filtration $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ is assumed to be generated by the two-dimensional standard brownian motion $(G_t)_{t\geq 0} = (W_t, \overline{W}_t)_{t\geq 0}$, and where transactions are allowed only at discrete dates $\mathcal{T} = \{0, 1, .., T-1\}, T \in \mathbb{N}$. The processes $(S_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are assumed to evolve stochastically according to the following stochastic differential equations:

$$
\begin{aligned}
dS_t &= \alpha S_t dt + \sigma S_t dW_t, &(15)\\
dY_t &= \mu Y_t dt + \eta Y_t(\rho dW_t + \sqrt{1-\rho^2}d\overline{W}_t) = \mu Y_t dt + \eta Y_t dZ_t,
\end{aligned}
$$

with $(\alpha, \sigma), (\mu, \eta) \in \mathbb{R} \times \mathbb{R}^+$ and where $(Z_t)_{t\geq 0}$ is a standard brownian motion with $< W_t, Z_t >= \rho t$.

If $\sigma \geq \eta$ and $\rho \geq 0$, then $P^{*Dyn}_{CD}(Y_T) = P^*_{CD}(Y_T) = Y_0 \exp[T(\mu - \frac{\eta^2\rho^2}{2} - \rho\frac{\eta}{\sigma}(\alpha - \frac{\sigma^2}{2}))]$.

From (15) it results that for any $t \in \mathcal{T}$, S_t and Y_t are distributed as:

$$
S_t \overset{d}{=} S_{t-1}\exp(\alpha - \frac{\sigma^2}{2} + \sigma W_1) \quad and \quad Y_t \overset{d}{=} Y_{t-1}\exp(\mu - \frac{\eta^2}{2} + \eta Z_1)
$$

with $(W_1, Z_1) \sim N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right)$.

Applying Proposition 9 to the case of the $\mathcal{F}_t-$measurable random variable Y_t, we obtain

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$$P_{CD}^{*t-1,t}(Y_t) = Y_{t-1} \exp[\mu - \frac{\eta^2 \rho^2}{2} - \rho\frac{\eta}{\sigma}(\alpha - \frac{\sigma^2}{2})], \qquad (16)$$

which, by a recursive argument, gives the result.

The assumption $\sigma \geq \eta$ is equivalent to say that the process $(S_t)_{t \geq 0}$ of the financial market is more volatile than the process $(Y_t)_{t \geq 0}$ to be hedged, a very natural requirement. Notice that, if $\sigma < \rho\eta$ or $\rho < 0$ from Proposition 9 we get $P_{CD}^{*Dyn}(Y_T) = +\infty$, and our approach is equivalent to the super-replication one. The hedging strategy $\{\overline{\theta}_t\}_{0 \leq t \leq T-1}$ obtained in the last Proposition depends on the process $(Y_t)_{t \geq 0}$. Indeed, we have:

$$\overline{\theta}_t = \frac{\rho\eta}{\sigma}Y_t \exp[\mu - \frac{\eta^2 \rho^2}{2} - \rho\frac{\eta}{\sigma}(\alpha - \frac{\sigma^2}{2})]. \qquad (17)$$

At last, it is very simple to generalize our approach to the case where one wants to hedge a discrete random process $(Y_t)_{t \in \{1,..,T\}}$ instead of a single amount Y_T. This case is analogous: it is enough to replace (13) by:

$$P_{CD}^{*Dyn}((Y_t)_{t \in \{1,..,T\}}) := P_{CD}^{*0,1}(Y_1 + P_{CD}^{*1,2}(...Y_{T-2} + P_{CD}^{*T-2,T-1}(Y_{T-1} + P_{CD}^{*T-1,T}(Y_T))...).$$

6   Conclusion

In an incomplete market setting, the pricing and hedging of risky positions is a difficult problem. Using super-replication, one obtains intervals for the hedging price of a risk, but these intervals turn out to be too large in general to be used in practice.

In order to determine tighter intervals, we have defined conditional dominance, which can be related with first and second order stochastic dominance. We have proved that we indeed obtain upper and lower bounds for the hedging price which are compatible with the equilibrium.

In some cases the use of conditional dominance improves the super-replication approach in a significant way. For example, in the case of log-normal random variables (which is an interesting case for pension funds), the super-replication yields $+\infty$ on upper bound, whereas our calculations result in explicit formulae. We have also provided an example in case of multinomial variables.

In the first part of this paper we have concentrated on a static situation with only two dates of interest: an initial date and a time horizon. Afterwards, we turned our attention to a generalization of this approach to a

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dynamic model by backward optimization. In case of geometric Brownian motion, the dynamical upper bound has been derived. This example has been motivated by an economical interpretation in pension funds.

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