Moment matching approximation of Asian basket option prices

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Abstract

In this paper we propose some moment matching pricing methods for European-style discrete arithmetic Asian basket options in a Black & Scholes framework. We generalize the approach of [5] and of [8] in several ways. We create a framework that allows for a whole class of conditioning random variables which are normally distributed. We moment match not only with a lognormal random variable but also with a log-extended-skew-normal random variable. We also improve the bounds of [9]. Numerical results are included and on the basis of our numerical tests, we explain which method we recommend depending on moneyness and time-to-maturity.

Key words: Asian basket option, sum of non-independent random variables, moment matching, log-extended-skew-normal
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1. Introduction

In this paper we propose pricing methods for European-style discrete arithmetic Asian basket options in a Black & Scholes framework.

We consider a basket consisting of n assets with prices $S_i(t)$, $i = 1, \ldots, n$, which are described, under the risk neutral measure $\mathbb{Q}$ and with r some risk-neutral interest rate, by

$$dS_i(t) = rS_i(t)dt + \sigma_i S_i(t)dW_i(t),$$

where $\{W_i(t),\ t > 0\}$ are standard Brownian motions associated with the price of asset i. Further, we assume that the different asset prices are instantaneously correlated in a constant way i.e.

$$\text{corr}(dW_i, dW_j) = \rho_{ij}dt. \tag{1}$$

An Asian basket option is a path-dependent multi-asset option whose payoff combines the payoff structure of an Asian option with that of a basket option. The current time $t = 0$ price of a discrete arithmetic Asian basket call option with a fixed strike K, maturity T and m averaging dates is determined by

$$\text{ABC}(n, m, K, T) = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[(\mathbb{S} - K)_+\right] \tag{2}$$

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with

$$\mathbb{S} = \sum_{\ell=1}^{n} \sum_{j=0}^{m-1} a_\ell b_j S_\ell(0) e^{(r-\frac{1}{2}\sigma_\ell^2)(T-j)+\sigma_\ell W_\ell(T-j)} \tag{3}$$

where a_ℓ and b_j are positive coefficients which both sum up to 1, and where $(x)_+ = \max\{x, 0\}$. For $T \leq m-1$, the Asian basket call option is said to be in progress and for $T > m-1$, we call it forward starting. Throughout the paper we consider forward starting Asian basket call options but the methods apply in general. The prices of Asian basket put options follow from the call option prices by the call-put parity relation. Indeed, if the price of an Asian basket put option with a fixed strike K, maturity T and m averaging dates is denoted by ABP(n, m, K, T), static arbitrage arguments lead to the following call-put parity relation:

$$\text{ABC}(n, m, K, T) - \text{ABP}(n, m, K, T) = \sum_{\ell=1}^{n} \sum_{j=0}^{m-1} a_\ell b_j S_\ell(0) e^{-rj} - e^{-rT} K.$$

Determining the price of the Asian basket option is not a trivial task, because we do not have an explicit analytical expression for the distribution of the weighted sum of the assets. Dahl and Benth value such options in [6] and [7] by quasi-Monte Carlo techniques and singular value decomposition. In [9] we derived lower and upper bounds based on stop-loss premia for non-independent random variables as in [12] or [10], [11] and on conditioning variables as in [5], [16] or [14]. We also derived upper bounds for Asian basket options applying techniques as in [17] and [13]. A natural extension is to use approximation techniques which are easier to treat mathematically, as discussed in [4] and references therein. Indeed those working in financial institutions prefer an approximate analytical solution above a more accurate solution involving lengthy numerical calculations. In the case of a basket option Deelstra et al. [8] combine the conditioning approach as in [5] with some moment matching methods to derive an approximation. Zhou and Wang approximate in [20] the underlying portfolio by some log-extended-skew-normal variates, whose parameters are determined by moment matching methods and derive a closed form approximation formulae for pricing both Asian and basket options.

In this paper we focus on approximations for pricing Asian basket options. We review and extend approximation methods of [8] and [20] to the Asian basket case. The main innovation of this paper is to show how to extend the approximation methods of [5] and [8] to a broad class of normal conditioning random variables and to improve the upper bounds based on the Rogers and Shi approach reported in [9].

The paper is organized as follows. In section 2 we use methods of Curran [5] and we decompose the price of the Asian basket option in two parts; one of which is computed exactly. In section 3 we determine the class of normal conditioning random variables and corresponding integration bound to split the integral in the expression of the Asian basket option price. The remaining part of the integral is approximated using moment matching methods with a lognormal approximating random variable in section 4. In section 5 we generalize the approach of [20] based on a moment matching log-extended-skew-normal approximation. Further we adapt this approach to the extended Curran methods of section 4. In section 6 we consider some numerical results and discuss the qualitative behaviour of these approximations.

2. Splitting the price by conditioning

In this section we follow the lines of [8] based on the method of conditioning as in [5] and [16]. The price of the Asian basket option can be decomposed in two parts, one of which is computed exactly while the remaining part in the decomposition is approximated using moment matching methods. We will show how to improve the exact part found in [8]. At the same time we obtain an extension of the method of [5] to more general conditioning random variables and an improvement of the upper bound based on the Rogers and Shi approach.

Apply the tower property with a conditioning random variable Λ and suppose that there exists a $d_\Lambda \in \mathbb{R}$ such that $\Lambda \geq d_\Lambda$ implies that $\mathbb{S} \geq K$, then the option price (2) can be split in two parts:

$$\text{ABC}(n, m, K, T) := I_1 + I_2$$

with

$$I_1 = e^{-rT}\int_{d_\Lambda}^{\infty}\left(\mathbb{E}^{\mathbb{Q}}\left[\mathbb{S}\mid\Lambda=\lambda\right]-K\right)dF_\Lambda(\lambda) = e^{-rT}\int_{d_\Lambda}^{\infty}\mathbb{E}^{\mathbb{Q}}\left[\mathbb{S}\mid\Lambda=\lambda\right]dF_\Lambda(\lambda) - e^{-rT}K(1-F_\Lambda(d_\Lambda)) \quad (4)$$

$$I_2 = e^{-rT}\int_{-\infty}^{d_\Lambda}\mathbb{E}^{\mathbb{Q}}\left[(\mathbb{S}-K)_+\mid\Lambda=\lambda\right]dF_\Lambda(\lambda). \quad (5)$$

Theorem 1 *The first term I_1 (4) of the Asian basket option price ABC(n,m,K,T) in* (2) *with* $\mathbb{S}$ (3) *as underlying can be written explicitly if for all ℓ and j, $(W_\ell(T-j),\Lambda)$ is bivariate normally distributed with $\Lambda \sim \mathcal{N}(\mathbb{E}^{\mathbb{Q}}[\Lambda],\sigma_\Lambda)$:*

$$I_1 = \sum_{\ell=1}^{n}\sum_{j=0}^{m-1} a_\ell b_j S_\ell(0) e^{-rj}\Phi\left[r_{\ell,j}\sigma_\ell\sqrt{T-j}-d_\Lambda^*\right] - e^{-rT}K\Phi\left(-d_\Lambda^*\right) \quad (6)$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function , $d_\Lambda^* = \frac{d_\Lambda-\mathbb{E}^{\mathbb{Q}}[\Lambda]}{\sigma_\Lambda}$ and

$$r_{\ell,j} = \frac{cov(W_\ell(T-j),\Lambda)}{\sigma_\Lambda\sqrt{T-j}}. \quad (7)$$

Proof The conditional expectation in (6) is easily seen to be:

$$\mathbb{E}^{\mathbb{Q}}[\mathbb{S}\mid\Lambda=\lambda] = \sum_{\ell=1}^{n}\sum_{j=0}^{m-1} a_\ell b_j S_\ell(0) e^{(r-\frac{1}{2}r_{\ell,j}^2\sigma_\ell^2)(T-j)+r_{\ell,j}\sigma_\ell\sqrt{T-j}\frac{\lambda-\mathbb{E}^{\mathbb{Q}}[\Lambda]}{\sigma_\Lambda}}. \quad (8)$$

Elementary integral calculation and normalization of the random variable Λ lead to the required result. □

We will in the next section focus on the choice of Λ and the integration bound d_Λ.

3. Choice of Λ and d_Λ

In this section we treat the problem in a more general setting of sums of non-independent lognormal random variables. Indeed, the double sum (3) is a special case of

$$\mathbb{S} = \sum_{i=1}^{N} w_i\alpha_i e^{\beta_i+\gamma_i Y_i}, \quad (9)$$

where the weights w_i sum up to one, the coefficients $\alpha_i(>0)$, β_i, γ_i are deterministic and the normally distributed random variables Y_i have mean zero, variance $\sigma_{Y_i}^2$ and are correlated with $\rho_{ij} = \text{corr}(Y_i,Y_j)$.

In order to generalize the approach of [5] and of [8], we transform the sum $\mathbb{S}$ as follows:

$$\mathbb{S} = F\sum_{i=1}^{N}\tilde{w}_i e^{\beta_i-\ln\delta_i+\gamma_i Y_i} := F\mathbb{S}_F, \quad (10)$$

where

$$\tilde{w}_i = \frac{w_i\alpha_i\delta_i}{F}, \qquad F = \sum_{i=1}^{N} w_i\alpha_i\delta_i, \qquad \delta_i > 0, \quad i=1,\ldots,N.$$

The coefficients δ_i can be chosen arbitrarily. Different choices will be discussed below.
According to the approach of [8] we choose the conditioning random variable Λ as a linear combination of the random variables Y_i obtained as a linear transformation of a first order approximation of $\mathbb{S}$ (10). Thus

$$\mathbb{S} \geq F\sum_{i=1}^{N}\tilde{w}_i(1+\beta_i-\ln\delta_i+\gamma_i Y_i) = F + F\sum_{i=1}^{N}\tilde{w}_i(\beta_i-\ln\delta_i) + F\sum_{i=1}^{N}\tilde{w}_i\gamma_i Y_i \geq K := FK_F \quad (11)$$

provides

$$\Lambda := F\Lambda_F = F\sum_{i=1}^{N}\tilde{w}_i\gamma_i Y_i \quad (12)$$

and the integration bound

$$d_\Lambda = K - F - F\sum_{i=1}^{N}\tilde{w}_i(\beta_i - \ln\delta_i) = F[K_F - 1 - \sum_{i=1}^{N}\tilde{w}_i(\beta_i - \ln\delta_i)] := F d_{\Lambda_F}. \tag{13}$$

On the other hand applying the approach of [5] we approximate the arithmetic average $\mathbb{S}_F$ by its corresponding geometric average $\mathbb{G}_F$:

$$\mathbb{S} = F\mathbb{S}_F \geq F\mathbb{G}_F = F\prod_{i=1}^{N}\left(e^{\beta_i - \ln\delta_i + \gamma_i Y_i}\right)^{\tilde{w}_i} \geq K = FK_F. \tag{14}$$

Taking the logarithm and noting that in view of (13)

$$\Lambda_F = \ln\mathbb{G}_F - \mathbb{E}^{\mathbb{Q}}\left[\ln\mathbb{G}_F\right], \tag{15}$$

this reasoning leads to the integration bound

$$d_\Lambda = F d_{\Lambda_F} = F[\ln K_F - \sum_{i=1}^{N}\tilde{w}_i(\beta_i - \ln\delta_i)]. \tag{16}$$

The two approaches lead to the same expression for the conditioning random variable but to two different integration bounds d_Λ.
Since the function $f(x) = \ln x - (x-1)$ reaches its maximum value zero for $x = 1$, it is negative in $]0, +\infty[$ and the expression (16) for d_Λ will be smaller than the expression (13). Thus using (16) as integration bound when splitting the integral in (4) and (5) will provide an improvement compared to the use of (13) for d_Λ as was the case in [8].

This also implies that the upper bounds based on the Rogers and Shi approach, referred to UBRS in [9] calculated with (16) will give better results than using (13). Indeed, from theorem 3 in [9] one can see that UBRSΛ is an increasing function of d_Λ. From the numerical results in Table 4 in section 6, we can conclude that the so-called partially/exact comonotonic upper bound denoted by PECUB is also considerably improved when determined by (16).

For the Asian basket case (2)-(3), we list five conditioning random variables Λ (12) with corresponding integration bound d_Λ (16) based on the choices for Λ that can be found in literature, see e.g. [18] and [19]:

$$\text{FA}k = \sum_{\ell=1}^{n}\sum_{j=0}^{m-1} a_\ell b_j S_\ell(0)\delta_k(\ell,j)\sigma_\ell W_\ell(T-j), \tag{17}$$

$$d_{\text{FA}k} = F\ln K_F - \sum_{\ell=1}^{n}\sum_{j=0}^{m-1} a_\ell b_j S_\ell(0)\delta_k(\ell,j)[(r - \frac{1}{2}\sigma_\ell^2)(T-j) - \ln\delta_k(\ell,j)], \quad k = 1,\ldots,5, \tag{18}$$

with

$$F = \sum_{\ell=1}^{n}\sum_{j=0}^{m-1} a_\ell b_j S_\ell(0)\delta_k(\ell,j), \quad k = 1,\ldots,5, \tag{19}$$

and with

$$\delta_1(\ell,j) = e^{(r-\frac{1}{2}\sigma_\ell^2)(T-j)}, \quad \delta_2(\ell,j) = 1, \quad \delta_3(\ell,j) = e^{r(T-j)}, \quad \delta_4(\ell,j) = S_\ell(0)^{-1} \tag{20}$$

$$\delta_5(\ell,j) = e^{r(T-j) - \frac{1}{2}(r_{\ell,j}^{\text{FA3}}\sigma_\ell\sqrt{T-j} - \Phi^{-1}(p))^2}, \tag{21}$$

where $r_{\ell,j}^{\text{FA3}}$ is the correlation (7) for $\Lambda = \text{FA3}$ and $p \in]0,1[$ is the level of the conditional tail expectation used to locally optimize the choice of Λ (see [18]).
The integration bound of the form (13) equals the expression in (18) but with the first term $F\ln K_F$ replaced by $K - F$ and was used in [8] and [9].

4. Moment matching lognormal approximation

In this section we discuss approximations of the second part I_2 (5) based on the moment matching technique. First we note that the following expressions for this part I_2 are equivalent and can be used to start from for the moment matching approximation:

$$\begin{aligned} I_2 &= e^{-rT} \int_{-\infty}^{d_\Lambda} \mathbb{E}^{\mathbb{Q}}[(\mathbb{S} - K)_+ \mid \Lambda = \lambda] dF_\Lambda(\lambda) \\ &= e^{-rT} F \int_{-\infty}^{d_{\Lambda_F}} \mathbb{E}^{\mathbb{Q}}[(\mathbb{S}_F - K_F)_+ \mid \Lambda_F = \lambda] dF_{\Lambda_F}(\lambda) \\ &= e^{-rT} F \int_{0}^{K_F} \mathbb{E}^{\mathbb{Q}}[(\mathbb{S}_F - K_F)_+ \mid \mathbb{G}_F = g] dF_{\mathbb{G}_F}(g). \end{aligned}$$

We will further work with the first expression and transform it into:

$$I_2 = e^{-rT} \int_{-\infty}^{d_\Lambda} \mathbb{E}^{\mathbb{Q}}[((\mathbb{S} - f_s(\Lambda)) - (K - f_s(\Lambda)))_+ \mid \Lambda = \lambda] dF_\Lambda(\lambda) \tag{22}$$

and we will study three cases for $s = 1, 2, 3$:

$$f_1(\Lambda) = 0, \qquad f_2(\Lambda) = F(1 + \ln \mathbb{G}_F), \qquad f_3(\Lambda) = F\mathbb{G}_F, \tag{23}$$

where we recall that by relation (12) and (15) the dependence on Λ is equivalent to the dependence on $\mathbb{G}_F$. These choices for $f_s(\Lambda)$ are inspired by the approximations derived in (11) and in (14). These two cases were considered for basket options in [8] but with an integration bound d_Λ of the form (13) instead of (18) and only for the conditioning random variables FA1 and FA2 for the case $s = 2$ and FA4 for the case $s = 3$. The present approach allows for a broader class of conditioning random variables of the form (12).

We will approximate the conditioned random variable $\mathbb{S} - f_s(\Lambda) \mid \Lambda = \lambda$ by a lognormal random variable with parameters $\mu_s(\lambda)$ and $\sigma_s(\lambda)$ and with the same first two moments as $\mathbb{S} - f_s(\Lambda) \mid \Lambda = \lambda$, $s = 1, 2, 3$. Then the expression for the expectation in the integrand of (22) is well known and is similar to the Black-Scholes formula.
This reasoning leads to the following result:
Theorem 2 *A moment matching lognormal approximation to the part I_2* (22)-(23) *of the Asian basket option price ABC(n, m, K, T) in (2) written on $\mathbb{S}$ is*

$$e^{-rT} \int_{-\infty}^{d_\Lambda} [e^{\mu_s(\lambda) + \frac{1}{2}\sigma_s^2(\lambda)} \Phi(d_1(\lambda)) - (K - f_s(\lambda))\Phi(d_2(\lambda))] dF_\Lambda(\lambda), \quad s = 1, 2, 3, \tag{24}$$

with

$$d_1(\lambda) = \frac{\mu_s(\lambda) + \sigma_s^2(\lambda) - \ln(K - f_s(\lambda))}{\sigma_s(\lambda)}, \qquad d_2(\lambda) = d_1(\lambda) - \sigma_s(\lambda),$$

$f_s(\lambda)$ *defined in* (23) *and*

$$\begin{aligned} \mu_s(\lambda) + \frac{1}{2}\sigma_s^2(\lambda) &= \ln(\mathbb{E}^{\mathbb{Q}}[\mathbb{S} \mid \Lambda = \lambda] - f_s(\lambda)) \\ \mu_s(\lambda) + \sigma_s^2(\lambda) &= \frac{1}{2} \ln(\mathbb{E}^{\mathbb{Q}}[\mathbb{S}^2 \mid \Lambda = \lambda] - 2f_s(\lambda)\mathbb{E}^{\mathbb{Q}}[\mathbb{S} \mid \Lambda = \lambda] + f_s^2(\lambda)). \end{aligned}$$

For the Asian basket case with the underlying $\mathbb{S}$ given by (3), the conditional expectations above can be written out explicitly. $\mathbb{E}^{\mathbb{Q}}[\mathbb{S} \mid \Lambda = \lambda]$ was given in (8) while for $\mathbb{E}^{\mathbb{Q}}[\mathbb{S}^2 \mid \Lambda = \lambda]$ we find:

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}}[\mathbb{S}^2 \mid \Lambda = \lambda] \\ &= \sum_{\ell,u=1}^{n} \sum_{j,p=0}^{m-1} a_\ell a_u b_j b_p S_\ell(0) S_u(0) e^{(r - \frac{1}{2}\sigma_\ell^2)(T-j) + (r - \frac{1}{2}\sigma_u^2)(T-p) + \frac{1}{2}(1 - r_{\ell j,up}^2)\sigma_{\ell j,up}^2 + r_{\ell j,up}\sigma_{\ell j,up} \frac{\lambda - \mathbb{E}^{\mathbb{Q}}[\Lambda]}{\sigma_\Lambda}} \end{aligned} \tag{25}$$

with

$$\begin{aligned} \sigma_{\ell j,up}^2 &= \sigma_\ell^2(T-j) + \sigma_u^2(T-p) + 2\sigma_\ell\sigma_u\rho_{\ell u} \min(T-j, T-p) \\ r_{\ell j,up}\sigma_{\ell j,up} &= r_{\ell,j}\sigma_\ell\sqrt{T-j} + r_{u,p}\sigma_u\sqrt{T-p}, \end{aligned}$$

and the correlations $\rho_{\ell u}$ and $r_{\ell,j}$, $r_{u,p}$ defined in (1) and (7). When Λ is one of the FAk (17) then we moreover have for $k = 1, \dots, 5$:

$$\sigma_\Lambda^2 = \sum_{\ell,u=1}^{n} \sum_{j,p=0}^{m-1} a_\ell a_u b_j b_p S_\ell(0) S_u(0) \delta_k(\ell, j) \delta_k(u,p) \sigma_\ell \sigma_u \rho_{\ell u} \min(T-j, T-p) \tag{26}$$

$$r_{\ell,j} = \frac{1}{\sigma_\Lambda \sqrt{T-j}} \sum_{u=1}^{n} \sum_{p=0}^{m-1} a_u b_p S_u(0) \delta_k(u,p) \sigma_u \rho_{\ell u} \min(T-j, T-p). \tag{27}$$

5. Moment matching log-extended-skew-normal approximation

In the case of an Asian option and a basket option Zhou and Wang approximate in [20] the underlying portfolio by some log-extended-skew-normal (LESN) variates whose parameters are determined by the moment matching method, and derive certain closed form approximation formulae related to the standard extended-skew-normal distributions. In this section we first follow their idea and generalize this method to the Asian basket case (2). Second we extend their idea to the conditioning approach as in the previous section by following the idea of [5] and [8].

5.1. *Log-extended-skew-normal random variable*

The univariate skew normal distribution was introduced by Azzalini in [2]. In conjunction with coauthors, he extended this class to include the multivariate analog of the skew-normal. A survey of such models is given by Arnold and Beaver in [1]. For a recent discussion and applications of the skew-normal distribution see for example [3] or [15].

Definition 3 *A random variable Z is said to be standard extended-skew-normal distributed with the skewness parameters α and τ, denoted by $Z \sim ESN(\alpha, \tau)$, if Z has the distribution function*

$$\Psi(x, \alpha, \tau) = \int_{-\infty}^{x} \phi(z) \frac{\Phi\left(\tau\sqrt{1+\alpha^2} + \alpha z\right)}{\Phi(\tau)} dz, \quad x \in \mathbb{R},$$

where $\phi(\cdot)$ denotes the density function and $\Phi(\cdot)$ the standard normal cumulative distribution function. If $\tau = 0$, then we say that Z has a standard skew-normal distribution with the skewness parameter α, denoted by $Z \sim SN(\alpha)$. The standard extended-skew-normal distribution has the following property

$$1 - \Psi(x, \alpha, \tau) = \Psi(-x, -\alpha, \tau), \qquad \forall x, \alpha, \tau \in \mathbb{R}.$$

A more general form of a skew normal distribution is obtained by introducing a location parameter μ and a positive scale parameter σ:

Definition 4 *A random variable, Y, defined by $Y = \mu + \sigma Z$ with $Z \sim ESN(\alpha, \tau)$, is said to be extended-skew-normally distributed and denoted by $Y \sim ESN(\mu, \sigma, \alpha, \tau)$. The random variable, X, defined by $X := e^Y$ is then said to be log-extended-skew-normally distributed and we denote $X \sim LESN(\mu, \sigma, \alpha, \tau)$.*

Proposition 5 *The moment generating function of a random variable $Y \sim ESN(\mu, \sigma, \alpha, \tau)$ is given by*

$$\mathbb{E}[e^{Yt}] = \frac{\Phi(\tau + \gamma t)}{\Phi(\tau)} \exp\left[\mu t + \frac{1}{2}\sigma^2 t^2\right], \quad \textit{with } \gamma = \frac{\sigma\alpha}{\sqrt{1+\alpha^2}},$$

and provides the moments of the corresponding random variable $X = e^Y \sim LESN(\mu, \sigma, \alpha, \tau)$.

5.2. *Log-extended-skew-normal approximation of the underlying portfolio*

As suggested by Zhou and Wang in [20] we rewrite $\mathbb{S}$ (3) as $F\mathbb{S}_F$ cfr. (10) and approximate the sum $\mathbb{S}_F$ by some log-extended-skew-normal random variable, since the LESN distribution is not only close to the lognormal distribution, but also has the capability to capture the skew and kurtosis, besides the mean and variance.

Thus we approximate the sum $\mathbb{S}_F$ by assuming that it is log-extended-skew-normally distributed with parameters μ, σ, α and τ, and having the same four moments as the sum itself:

Theorem 6 *A moment matching LESN approximation to the Asian basket option price ABC*(n, m, K, T) (2) *written on the underlying* $\mathbb{S}$ *given by (3) is*

$$Fe^{-rT}M(1)\Psi\left(d_1, -\alpha, \tau+\gamma\right) - Ke^{-rT}\Psi\left(d_2, -\alpha, \tau\right) \tag{28}$$

with

$$d_1 = \frac{\mu + \sigma^2 - \ln\frac{K}{F}}{\sigma}, \qquad d_2 = d_1 - \sigma, \tag{29}$$

and F *defined in* (19), (20) *and* (21)*, and where the parameters* μ*,* σ*,* γ *and* τ *are solutions to the following system of equations*

$$\begin{cases} \ln\dfrac{\Phi(\tau+4\gamma)}{M(4)} - 6\ln\dfrac{\Phi(\tau+2\gamma)}{M(2)} + 8\ln\dfrac{\Phi(\tau+\gamma)}{M(1)} - 3\ln\Phi(\tau) = 0 \\ \ln\dfrac{\Phi(\tau+3\gamma)}{M(3)} - 3\ln\dfrac{\Phi(\tau+2\gamma)}{M(2)} + 3\ln\dfrac{\Phi(\tau+\gamma)}{M(1)} - \ln\Phi(\tau) = 0 \\ \mu = \dfrac{1}{2}\ln\dfrac{\Phi(\tau+2\gamma)}{M(2)} - 2\ln\dfrac{\Phi(\tau+\gamma)}{M(1)} + \dfrac{3}{2}\ln\Phi(\tau) \\ \sigma^2 = -\ln\dfrac{\Phi(\tau+2\gamma)}{M(2)} + 2\ln\dfrac{\Phi(\tau+\gamma)}{M(1)} - \ln\Phi(\tau) \end{cases}, \tag{30}$$

with $M(i)$*,* $t = i, \ldots, 4$ *the first four moments of* $\mathbb{S}/F$*.*

Proof Analogous to the proof given by Zhou and Wang [20]. □

For pricing Asian options and basket options, Zhou and Wang only considered the case that F in (19) is determined by $\delta_3(\ell, j) = e^{r(T-j)}$ from (20).

In fact any choice of F will provide the same result, since relation (28) for the approximate Asian basket option price is independent of F. Indeed the terms $\ln F$ disappear in the last relation of (30) providing σ^2. Also the terms $\ln F$ cancel out in the first two equations of (30) which determine the parameters τ and γ, and therefore also $\alpha = \frac{\gamma}{\sqrt{\sigma^2-\gamma^2}}$. Only in the expression for μ remains a term $-\ln F$ which however cancels out with the $+\ln F$ from $-\ln\frac{K}{F}$ in d_1 (29). Hence d_1 and d_2 are also independent of the factor F. Finally noting that $FM(1) = \mathbb{E}^{\mathbb{Q}}[\mathbb{S}]$ proves our claim.

5.3. *Log-extended-skew-normal approximation after splitting and conditioning*

We return to the case that we have split the option price in an exact part I_1 (4) and a part I_2 (22) that we approximate by a moment matching technique but now using a log-extended-skew-normal random variable with parameters μ, σ, α and τ that will depend on the conditioning random variable and on f_s, $s = 1, 2, 3$. We will not write this dependence explicitly in the notations of these parameters.

Theorem 7 *A moment matching LESN approximation to the part* I_2 (22)-(23) *of the Asian basket option price ABC*(n, m, K, T) *in (2) written on* $\mathbb{S}$ *given by (3) is*

$$e^{-rT}\int_{-\infty}^{d_\Lambda}\left[\mathbb{E}^{\mathbb{Q}}[\mathbb{S} - f_s(\Lambda) \mid \Lambda = \lambda]\Psi(d_1(\lambda), -\alpha, \gamma+\tau) - (K - f_s(\lambda))\Psi(d_2(\lambda), -\alpha, \tau)\right] dF_\Lambda(\lambda), \quad s = 1, 2, 3, \tag{31}$$

with

$$d_1(\lambda) = \frac{\mu + \sigma^2 - \ln\frac{K - f_s(\lambda)}{F}}{\sigma}, \qquad d_2(\lambda) = d_1(\lambda) - \sigma,$$

and F *defined in* (19), (20) *and* (21)*. The parameters* μ*,* σ*,* γ *and* τ *are solutions to the system* (30) *of equations where the moments* $M(i)$*,* $i = 1, \ldots, 4$*, are the first four moments of* $\frac{1}{F}(\mathbb{S} - f_s(\Lambda)) \mid \Lambda = \lambda$*.*

Proof Analogous to the proof of Theorem 6 but starting from (22). □

In numerical experiments we set τ equal to zero to simplify and speed up the calculations of the parameters. The system (30) of equations reduces in that case to the last three equations. The three moments $M(i)$, $i = 1, 2, 3$, are

stock	initial stock price	weight (in %)	volatility (in %)	dividend yield (in %)
BASF	42.55	25	33.34	2.59
Bayer	48.21	20	31.13	2.63
Degussa-Hüls	34.30	30	33.27	3.32
FMC	100.00	10	35.12	0.69
Schering	66.19	15	36.36	1.24

Table 1
Stock characteristics

$$
\begin{aligned}
M(1) &= \frac{1}{F}(\mathbb{E}^{\mathbb{Q}}[\mathbb{S} \mid \Lambda = \lambda] - f_s(\lambda)) \\
M(2) &= \frac{1}{F^2}(\mathbb{E}^{\mathbb{Q}}[\mathbb{S}^2 \mid \Lambda = \lambda] - 2f_s(\lambda)\mathbb{E}^{\mathbb{Q}}[\mathbb{S} \mid \Lambda = \lambda] + f_s^2(\lambda)) \\
M(3) &= \frac{1}{F^3}(\mathbb{E}^{\mathbb{Q}}[\mathbb{S}^3 \mid \Lambda = \lambda] - 3f_s(\lambda)\mathbb{E}^{\mathbb{Q}}[\mathbb{S}^2 \mid \Lambda = \lambda] + 3f_s^2(\lambda)\mathbb{E}^{\mathbb{Q}}[\mathbb{S} \mid \Lambda = \lambda] - f_s^3(\lambda)),
\end{aligned}
$$

with $\mathbb{E}^{\mathbb{Q}}[\mathbb{S} \mid \Lambda = \lambda]$ and $\mathbb{E}^{\mathbb{Q}}[\mathbb{S}^2 \mid \Lambda = \lambda]$ given by (8) and (25) and

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[\mathbb{S}^3 \mid \Lambda = \lambda] = \sum_{\ell,u,i=1}^{n} \sum_{j,p,h=0}^{m-1} & a_\ell a_u a_i b_j b_p b_h S_\ell(0) S_u(0) S_i(0) e^{(r-\frac{1}{2}\sigma_\ell^2)(T-j)+(r-\frac{1}{2}\sigma_u^2)(T-p)+(r-\frac{1}{2}\sigma_i^2)(T-h)} \\
& \times e^{\frac{1}{2}(1-r_{\ell j,up,ih}^2)\sigma_{\ell j,up,ih}^2 + r_{\ell j,up,ih}\sigma_{\ell j,up,ih}\frac{\lambda-\mathbb{E}^{\mathbb{Q}}[\Lambda]}{\sigma_\Lambda}},
\end{aligned}
$$

where σ_Λ^2 is given by (26) and

$$
\begin{aligned}
\sigma_{\ell j,up,ih}^2 &= \sigma_\ell^2(T-j) + \sigma_k^2(T-p) + \sigma_i^2(T-h) + 2\sigma_\ell\sigma_u\rho_{\ell u}\min(T-j, T-p) \\
&\quad + 2\sigma_\ell\sigma_i\rho_{\ell i}\min(T-j, T-h) + 2\sigma_i\sigma_u\rho_{iu}\min(T-h, T-p), \\
r_{\ell j,up,ih}\sigma_{\ell j,up,ih} &= r_{\ell,j}\sigma_\ell\sqrt{T-j} + r_{u,p}\sigma_u\sqrt{T-p} + r_{i,h}\sigma_i\sqrt{T-h}\,,
\end{aligned}
$$

with the correlations $\rho_{\ell u}$, ρ_{iu}, $\rho_{\ell i}$ defined in (1) and $r_{\ell,j}$, $r_{u,p}$, $r_{i,h}$ defined in (27).
Further, we only consider the case that $s = 3$ but for all conditioning random variables FAk (17), $k = 1, \ldots, 5$. When testing the two other cases for s we end up with non-real values for α.

Another possibility to deal with the cumbersome calculations in solving the system (30) of the four equations is to fix λ in the parameters instead of putting $\tau = 0$. Following the suggestion of [5] we make the moments constant by fixing λ on d_Λ which is equivalent to fixing the value of $\mathbb{G}_F$ on K_F. However in this way the quality of the approximation is much worse than the approach with putting $\tau = 0$ and keeping the parameters dependent on λ.

6. Numerical results

In this section we consider a numerical example for an Asian basket option in the Black & Scholes setting. In order to compare the approximations and upper bounds for the Asian basket option prices, we take a set of input data from [4] which are also used in [9]. The Asian basket option with monthly averaging is written on a fictitious chemistry-pharma basket that consists of the five German DAX stocks listed in Tables 1 and 2.

The annual risk-free interest rate r is equal to 6% and we compute approximations for options with three different maturity dates (half a year, one year and five years). The exercise prices are chosen in such a way that Tables 3 and 4 show results for in-the-money, at-the-money and out-of-the money options. The moneyness of the option is defined as

$$
\frac{K}{\sum_{\ell=1}^{n}\sum_{j=0}^{m-1} a_\ell b_j S_\ell(0) e^{r(T-j)}} - 1. \tag{32}
$$

The averaging period of all options is five months and starts five months before maturity.

	BASF	Bayer	Degussa-Hüls	FMC	Schering
BASF	1.00	0.84	−0.07	0.45	0.43
Bayer	0.84	1.00	0.08	0.62	0.57
Degussa-Hüls	−0.07	0.08	1.00	−0.54	−0.59
FMC	0.45	0.62	−0.54	1.00	0.86
Schering	0.43	0.57	−0.59	0.86	1.00

Table 2
Correlation structure

In Table 3 we list moment matching approximated option prices composed as the sum of (6) and (24) for the three cases (23) and for all five conditioning random variables FAk (17), where for FA5 we put $p = 0.95$. For the log-extended-skew-normal approximations based on Theorem 6 we only report one result since (28) is independent of the choice of F. The LESN approximations composed as the sum of (6) and (31) are computed for the case $f_3(\lambda)$ (23) — the other cases lead to complex values for α — but for all five conditioning random variables FAk. We reduced the computations by putting either $\tau = 0$ or by fixing λ in the moments.

From Table 3 we conclude that for short maturities the results are very similar for lognormal and for LESN approximations for the first three conditioning random variables. The results for FA4 are much worse while those for FA5 are slightly worse. It is also clear that the LESN approach with constant moments is not recommended for any maturity or moneyness. For long maturities the LESN approximations with $\tau = 0$ and with conditioning random variables FA1, FA2 and FA3 outperform all other approximations.

The values for the upper bounds UBRSΛ and PECUBΛ in Table 4 are clearly better when using the integration bounds $d_{\mathrm{FA}k}$ (18) than when using the corresponding integration bounds with $F \ln K_F$ replaced by $K - F$, cfr. (13), of [9]. We also added the column of the lower bound LBΛ (as explained in [9]) to show that the lower bound is a very precise bound, but that the approximations of Table 3 are much closer to the Monte Carlo simulations than the bounds.

In Fig. 1 - Fig. 3 we plot the pricing error of an approximation with respect to the moneyness (32) for different maturities T, by choosing in each figure the conditioning variable which leads to the best results. The pricing error expressed in basis points (bp) is defined as

$$\frac{\text{approximation} - \text{MC value}}{\sum_{\ell=1}^{n} a_\ell S_\ell(0)} 10\,000,$$

where the denominator equals here 50.498 according to the data in Table 1. The log-extended-skew-normal approximations for f_3 are only given for $\tau = 0$ since fixing $\lambda = d_\Lambda$ in the moments does not lead to good results.

From the Fig. 1 - Fig. 3, we see that the approximations sometimes overestimate and sometimes underestimate the price. The log-extended-skew-normal approximations are less fluctuating, certainly for long-term maturities. However, notice that the pricing-error scale is different in the different figures. The log-extended-skew-normal approximations based on f_3 clearly outperform the other approximations.

7. Conclusions

We derived moment matching pricing approximation methods for the price of European-style discrete arithmetic Asian basket call options by decomposing the option price into an exact and an approximating part. We generalized the results of [5], [8] and [20] in several ways: by considering a quite large class of normally distributed conditioning variables, by looking at better integration bounds, by moment matching several conditioned random variables not only by using the lognormal law, but also the log-extended-skew-normal law. These techniques can be applied to evaluating other instruments based on a sum of dependent random variables.

Based on our numerical tests, we recommend the reader to use the log-extended-skew-normal approximations based on f_3, especially for long-term maturities.

			lognormal			LESN			
	K	MC & SE	$\mathbb{S} - f_s(\Lambda) \mid \Lambda = \lambda$			$\mathbb{S}$	$\mathbb{S} - f_3(\Lambda) \mid \Lambda = \lambda$		FAk
T	(moneyness)	To compare	$s = 1$	$s = 2$	$s = 3$		$\tau = 0$	$\lambda = d_\Lambda$	k
$\frac{1}{2}$	40	MC:10.8462	10.8464	10.8463	10.8462	10.8466	10.8462	10.8457	1
	(−0.2181)	SE:0.0007	10.8464	10.8463	10.8462		10.8462	10.8458	2
			10.8462	10.8460	10.8466		10.8462	10.8457	3
			10.8478	10.8478	10.8460		10.8463	10.8496	4
			10.8467	10.8467	10.8461		10.8462	10.8440	5
	50	MC:2.7865	2.7861	2.7863	2.7864	2.7854	2.7864	2.7864	1
	(−0.0227)	SE:0.0005	2.7862	2.7862	2.7864		2.7864	2.7865	2
			2.7861	2.7862	2.7864		2.7864	2.7864	3
			2.7923	2.7922	2.7811		2.7863	2.8823	4
			2.7856	2.7857	2.7865		2.7864	2.7831	5
	60	MC:0.2342	0.2338	0.2341	0.2341	0.2347	0.2341	0.2360	1
	(0.1728)	SE:0.0001	0.2338	0.2341	0.2341		0.2341	0.2361	2
			0.2338	0.2341	0.2344		0.2341	0.2359	3
			0.2269	0.2270	0.2375		0.2340	0.3211	4
			0.2339	0.2339	0.2341		0.2341	0.2342	5
1	40	MC:11.7167	11.7177	11.7171	11.7158	11.7177	11.7166	11.7123	1
	(−0.2332)	SE:0.0008	11.7177	11.7172	11.7158		11.7166	11.7125	2
			11.7178	11.7174	11.7147		11.7166	11.7122	3
			11.7307	11.7306	11.7132		11.7174	11.7433	4
			11.7214	11.7216	11.7151		11.7167	11.6983	5
	50	MC:4.7362	4.7345	4.7348	4.7364	4.7341	4.7365	4.7371	1
	(−0.0415)	SE:0.0006	4.7347	4.7346	4.7363		4.7365	4.7375	2
			4.7344	4.7344	4.7366		4.7365	4.7368	3
			4.7529	4.7528	4.7193		4.7364	4.8874	4
			4.7318	4.7334	4.7366		4.7363	4.7229	5
	60	MC:1.4118	1.4099	1.4126	1.4113	1.4113	1.4113	1.4214	1
	(0.1502)	SE:0.0003	1.4099	1.4125	1.4113		1.4113	1.4222	2
			1.4099	1.4121	1.4126		1.4113	1.4210	3
			1.3978	1.3982	1.4035		1.4102	1.6530	4
			1.4078	1.4080	1.4121		1.4113	1.4093	5
5	40	MC:17.3142	17.3192	17.3191	17.2935	17.3208	17.3166	17.2876	1
	(−0.3346)	SE:0.0010	17.3949	17.3304	17.2946		17.3162	17.3104	2
			17.4026	17.3602	17.2787		17.3170	17.2735	3
			17.4937	17.4896	17.2782		17.3249	17.7599	4
			17.4562	17.3018	17.2934		17.3190	17.0853	5
	50	MC:12.6063	12.6250	12.5672	12.5846	12.6065	12.6035	12.6700	1
	(−0.1807)	SE:0.0009	12.6287	12.5676	12.5843		12.6041	12.7088	2
			12.6232	12.5785	12.5890		12.6033	12.6490	3
			12.5347	12.8179	12.8205		12.6161	13.5507	4
			12.6449	12.6046	12.5848		12.6039	12.4303	5
	60	MC:9.1438	9.1228	9.1117	9.1284	9.1420	9.1431	9.3520	1
	(−0.0168)	SE:0.0008	9.1325	9.0989	9.1269		9.1440	9.4135	2
			9.1168	9.0851	9.1513		9.1426	9.3230	3
			9.0517	9.3349	9.3351		9.1520	10.6943	4
			9.1310	9.1656	9.0927		9.1407	9.0478	5
	70	MC:6.6678	6.6347	6.6567	6.6549	6.6617	6.6656	7.0425	1
	(0.1470)	SE:0.0008	6.6447	6.6404	6.6530		6.6662	7.1321	2
			6.6282	6.6121	6.6913		6.6652	7.0048	3
			6.7980	6.7999	6.5679		6.6684	8.8520	4
			6.5807	6.6867	6.6596		6.6625	6.6399	5

Table 3
Comparing Approximations for Asian basket call option prices

T	K (moneyness)	MC & SE To compare	LBΛ	UBRSΛ with d_Λ (18)	UBRSΛ with d_Λ [9]	PECUBΛ with d_Λ (18)	PECUBΛ with d_Λ [9]	FAk k
$\frac{1}{2}$	40	MC:10.8462	10.8448	10.8517	10.8556	10.8821	10.9042	1
	(−0.2181)	SE:0.0007	10.8448	10.8516	10.8558	10.8820	10.9054	2
			10.8448	10.8518	10.8566	10.8824	10.9087	3
			10.8433	10.8578	11.1429	10.8974	11.1100	5
	50	MC:2.7865	2.7801	2.8939	2.8939	3.4911	3.4912	1
	(−0.0227)	SE:0.0005	2.7801	2.8932	2.8932	3.4915	3.4919	2
			2.7800	2.8947	2.8949	3.4911	3.4934	3
			2.7737	2.9488	3.0736	3.5180	4.0501	5
	60	MC:0.2342	0.2299	0.4556	0.4617	0.8908	0.9288	1
	(0.1728)	SE:0.0001	0.2299	0.4557	0.4613	0.8915	0.9272	2
			0.2300	0.4555	0.4599	0.8897	0.9212	3
			0.2320	0.5163	0.5319	0.8533	0.9626	5
1	40	MC:11.7167	11.6984	11.7857	11.8013	11.9971	12.0623	1
	(−0.2332)	SE:0.0008	11.6989	11.7846	11.8025	11.9964	12.0710	2
			11.6980	11.7864	11.8103	11.9973	12.0927	3
			11.6783	11.8340	12.3240	12.0563	12.5306	5
	50	MC:4.7362	4.7094	5.0154	5.0155	5.7643	5.7644	1
	(−0.0415)	SE:0.0006	4.7095	5.0127	5.0127	5.7661	5.7664	2
			4.7093	5.0175	5.0186	5.7636	5.7700	3
			4.6873	5.1353	5.4376	5.7811	6.4161	5
	60	MC:1.4118	1.3882	1.8935	1.9106	2.6327	2.7179	1
	(0.1502)	SE:0.0003	1.3875	1.8927	1.9080	2.6366	2.7092	2
			1.3887	1.8945	1.9038	2.6311	2.6800	3
			1.3911	2.0334	2.1417	2.5516	2.9013	5
5	40	MC:17.3142	16.9863	18.1377	18.1608	18.5559	18.5893	1
	(−0.3346)	SE:0.0010	17.0031	18.1194	18.1698	18.5676	18.6527	2
			16.9727	18.1508	18.3305	18.5464	18.7903	3
			16.7462	18.4094	18.4818	18.5481	18.6231	5
	50	MC:12.6063	12.2353	13.9176	13.9249	14.5404	14.5541	1
	(−0.1807)	SE:0.0009	12.2421	13.9031	13.9032	14.5676	14.5678	2
			12.2283	13.9336	13.9787	14.5276	14.6051	3
			12.0441	14.2167	14.4801	14.4048	14.7463	5
	60	MC:9.1438	8.7834	10.9200	11.0153	11.5385	11.6879	1
	(−0.0168)	SE:0.0008	8.7775	10.9175	10.9485	11.5779	11.6439	2
			8.7854	10.9327	10.9330	11.5237	11.5243	3
			8.6873	11.2343	11.6669	11.2745	11.8449	5
	70	MC:6.6678	6.3285	8.8451	9.0801	9.2663	9.5618	1
	(0.1470)	SE:0.0008	6.3127	8.8601	9.0051	9.3117	9.5048	2
			6.3376	8.8493	8.8720	9.2508	9.2873	3
			6.3166	9.1348	9.6583	8.9129	9.5241	5

Table 4
Comparing lower and upper bounds for Asian basket call option prices

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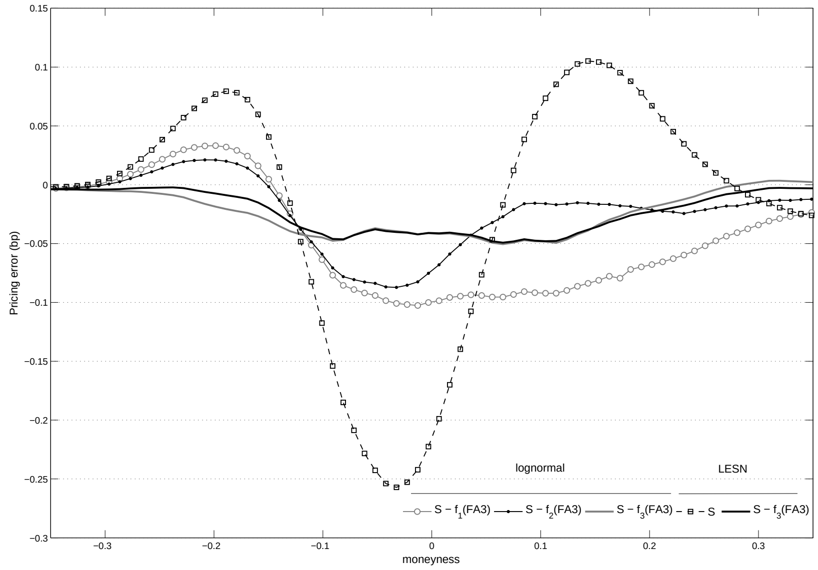

Fig. 1. Comparison of bounds for an Asian basket option value with $T = 0.5$ and by using $\mathrm{FA3}$.

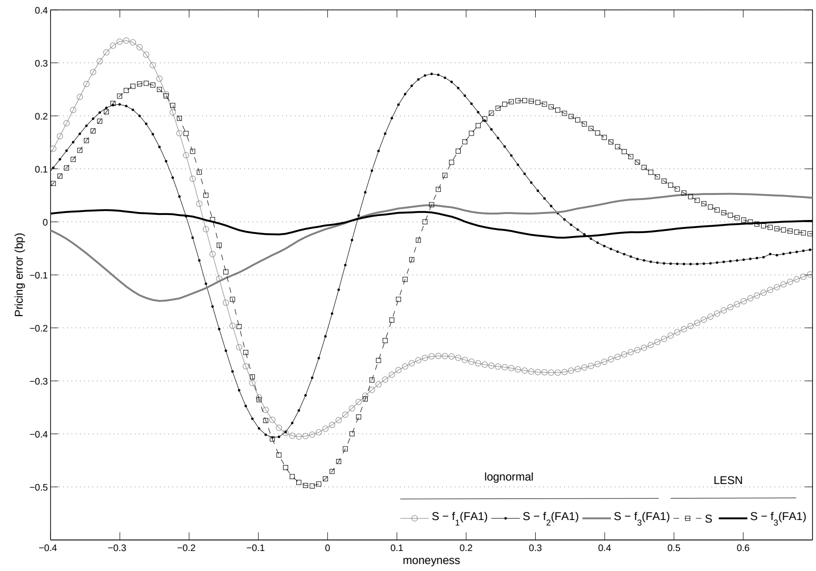

Fig. 2. Comparison of approximations for an Asian basket option with $T = 1$ and by using $\mathrm{FA1}$.

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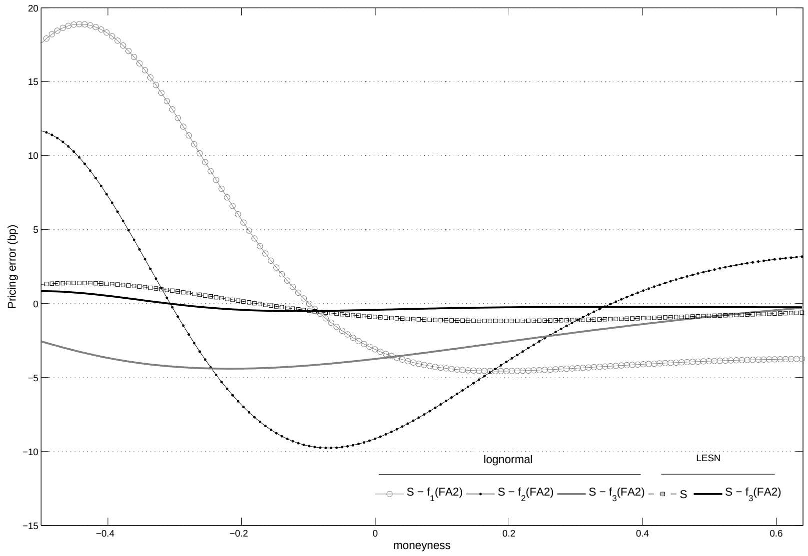

Fig. 3. Comparison of approximations for an Asian basket option with $T = 5$ and by using FA2.

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