

# Bounds for Asian basket options

Griselda Deelstra <sup>a,\*</sup>, Ibrahima Diallo <sup>b</sup>, Michèle Vanmaele <sup>c</sup>

<sup>a</sup>Department of Mathematics, ECARES, Université Libre de Bruxelles, CP 210, 1050 Brussels, Belgium

<sup>b</sup>ISRO, Université Libre de Bruxelles, CP 210, 1050 Brussels, Belgium

<sup>c</sup>Department of Applied Mathematics and Computer Science, Ghent University, Krijgslaan 281, building S9, 9000 Gent, Belgium

---

## Abstract

In this paper we propose pricing bounds for European-style discrete arithmetic Asian basket options in a Black & Scholes framework. We start from methods used for basket options and Asian options. First we use the general approach for deriving upper and lower bounds for stop-loss premia of sums of non-independent random variables as in Kaas et al. (2000) or Dhaene et al. (2002a). We generalize the methods in Deelstra et al. (2004) and Vanmaele et al. (2006). Afterwards we show how to derive an analytical closed-form expression for a lower bound in the non-comonotonic case. Finally, we derive upper bounds for Asian basket options by applying techniques as in Thompson (1999a) and Lord (2006). Numerical results are included and on the basis of our numerical tests, we explain which method we recommend depending on moneyness and time-to-maturity.

*Key words:* Asian basket option, sum of non-independent random variables, non-comonotonic sum  
*MCS:* 91B28, 60E15, 60J65

---

## 1. Introduction

In this paper we propose pricing methods for European-style discrete arithmetic Asian basket options in a Black & Scholes framework.

We consider a basket with  $n$  assets whose prices  $S_i(t)$ ,  $i = 1, \dots, n$ , are described, under the risk neutral measure  $\mathbb{Q}$  and with  $r$  some risk-neutral interest rate, by

$$dS_i(t) = rS_i(t)dt + \sigma_i S_i(t)dW_i(t), \quad (1)$$

where  $\{W_i(t), t > 0\}$  are standard Brownian motions associated with the price of asset  $i$ . Further, we assume that the different asset prices are instantaneously correlated in a constant way i.e.

$$\text{corr}(dW_i, dW_j) = \rho_{ij}dt. \quad (2)$$

An Asian basket option is a path-dependent multi-asset option whose payoff combines the payoff structure of an Asian option with that of a basket option. The price of a discrete arithmetic Asian

---

\* Corresponding author. Tel.: +32-2-6504650; fax: +32-2-6505899

*Email addresses:* Griselda.Deelstra@ulb.ac.be (Griselda Deelstra), idiallo@ulb.ac.be (Ibrahima Diallo), Michele.Vanmaele@UGent.be (Michèle Vanmaele).

basket call option with a fixed strike  $K$  and maturity  $T$  on  $m$  averaging dates at current time  $t = 0$  is determined by

$$ABC(n, m, K, T) = e^{-rT} \mathbb{E}^{\mathbb{Q}} \left[ \left( \sum_{\ell=1}^n a_{\ell} \sum_{j=0}^{m-1} b_j S_{\ell}(T-j) - K \right)_+ \right] \quad (3)$$

with  $a_{\ell}$  and  $b_j$  positive coefficients, which both sum up to 1, and with  $(x)_+ = \max\{x, 0\}$ . For  $T \leq m - 1$ , the Asian basket call option is said to be in progress and for  $T > m - 1$ , we call it forward starting. Throughout the paper we consider forward starting Asian basket options but the methods apply in general.

Asian basket options are suitable for hedging exposure as their payoff depend on an average of asset prices at different times and of different assets. Indeed, averaging has generally the effect of decreasing the variance, therefore making the option less expensive. Moreover the Asian basket option takes the correlations between the assets in the basket into account. Asian basket options are especially important in the energy markets where most delivery contracts are priced on the basis of an average price over a certain period.

Within a Black and Scholes [3] setting, no closed-form solutions are available for Asian basket options involving the average of asset prices taken at different dates. Dahl and Benth value such options in [6] and [7] by quasi-Monte Carlo techniques and singular value decomposition. But as this approach is rather time-consuming, it would be ideal to have accurate analytical and easily computable bounds or approximations of this price.

In the setting of Asian options, an analytical lower and upper bound in the case of continuous averaging is obtained by the methods of conditioning in [5] and in [17]. Thompson [19] used a first order approximation to the arithmetic sum and derived an upper bound that sharpens those of Rogers and Shi. Lord [15] revised Thompson's method and proposed a shift lognormal approximation to the sums and he included a supplementary parameter which is estimated by an optimization algorithm. In [16], Nielsen and Sandmann applied the Rogers and Shi approach to arithmetic Asian option pricing by using one specific standardized normally distributed conditioning variable and only in a Black & Scholes setting. Simon et al. [18] derived an easy computable upper bound for the price of an arithmetic Asian option based on the results of Dhaene et al. [9]. Dhaene et al. [10] and [11] studied extensively convex upper and lower bounds for sums of lognormals, in particular of Asian options. Vanmaele et al. [24] used techniques based on comonotonic risks for deriving upper and lower bounds for stop-loss premia of sums of non-independent random variables, as explained in [14] and the already mentioned [10] and [11]. Vanmaele et al. [24] improved the upper bound that was based on the idea of Rogers and Shi [17], and generalized the approach of Nielsen and Sandmann [16] to a general class of normally distributed conditioning variables. In [8] these methods for Asian options were generalized to the case of basket options.

In this paper, we concentrate upon the derivation of bounds for Asian basket options. We start with extending the methods of [8] and [24] to the Asian basket case.

New is that also in the non-comonotonic case we are able to derive a simple analytical lower bound and an upper bound based on the Rogers and Shi [17] approach.

Finally, we generalize the method of Thompson [19] and of Lord [15] to the Asian basket case. In Thompson's approach, we include an additional parameter which is optimized by using an optimization algorithm as in [15]. Numerical results are included and based on several numerical tests, we give a conclusion which should help the reader to choose a precise bound according to

the situation of moneyness and time-to-maturity that she is confronted with.

The paper is organized as follows. In Section 2, we deal with procedures for obtaining the lower and upper bounds for prices, by using the concept of comonotonicity as explained in [14], [10] and [11], along the lines of [24] and [8]. In Section 3, we derive an analytical closed-form expression for a lower bound in a non-comonotonic situation, which is then used to obtain the upper bound in the Rogers and Shi approach. In Section 4, we generalize the upper bound based on the idea of Thompson [19] and the approach of Lord [15] to discrete arithmetic Asian basket options. In Section 5, we discuss the quality of all these bounds in some numerical experiments and give a guideline of which bound to use in which situation.

## 2. Bounds based on comonotonicity and conditioning

In this section we generalize the bounds of [8] and [24] to the Asian basket case. In these papers the pricing of discrete arithmetic basket and Asian options are studied by using the notion of comonotonicity, as explained in [14], [10] and [11]. They further improve the bounds by incorporating the ideas of Curran [5], Rogers and Shi [17] and Nielsen and Sandmann [16], and by looking for good conditioning variables.

### 2.1. Comonotonic upper bound

Remark that the double sum  $\mathbb{S} = \sum_{\ell=1}^n a_{\ell} \sum_{j=0}^{m-1} b_j S_{\ell}(T-j)$ , showing up in equation (3), is a sum of lognormal distributed variables and can be written as

$$\mathbb{S} = \sum_{i=1}^{mn} X_i = \sum_{i=1}^{mn} \alpha_i e^{Y_i} \quad (4)$$

with

$$\alpha_i = a_{\lceil \frac{i}{m} \rceil} b_{(i-1) \bmod m} S_{\lceil \frac{i}{m} \rceil}(0) e^{(r - \frac{1}{2} \sigma_{\lceil \frac{i}{m} \rceil}^2)(T - (i-1) \bmod m)} \quad (5)$$

and

$$Y_i = \sigma_{\lceil \frac{i}{m} \rceil} W_{\lceil \frac{i}{m} \rceil}(T - (i-1) \bmod m) \sim \mathcal{N}(0, \sigma_{\lceil \frac{i}{m} \rceil}^2 = \sigma_{\lceil \frac{i}{m} \rceil}^2 (T - (i-1) \bmod m)) \quad (6)$$

for all  $i = 1, \dots, mn$ , where  $\lceil x \rceil$  is the smallest integer greater than or equal to  $x$  and

$$y \bmod m = y - \lfloor y/m \rfloor m,$$

where  $\lfloor y \rfloor$  denotes the greatest integer less than or equal to  $y$ . As explained in [11], given the marginal distributions of the terms in a random variable  $\mathbb{S} = \sum_{i=1}^k X_i$ , we shall look at the joint distribution with a smaller resp. larger sum, in the convex order sense. In particular, the comonotonic counterpart  $\mathbb{S}^c$  of (4) leads to the so-called comonotonic upper bound, denoted by CUB, where we recall that a random vector  $(X_1^c, \dots, X_k^c)$  is *comonotonic* if each two possible outcomes  $(x_1, \dots, x_k)$  and  $(y_1, \dots, y_k)$  of  $(X_1^c, \dots, X_k^c)$  are ordered componentwise.

**Theorem 1** *Suppose the sum  $\mathbb{S}$  is given by (4)-(6). Then the comonotonic upper bound for the option price  $ABC(n, m, K, T)$  in (3) is determined by:*

$$\text{CUB} = \sum_{\ell=1}^n \sum_{j=0}^{m-1} a_{\ell} b_j S_{\ell}(0) e^{-rj} \Phi \left[ \sigma_{\ell} \sqrt{T-j} - \Phi^{-1}(F_{\mathbb{S}^c}(K)) \right] - e^{-rT} K (1 - F_{\mathbb{S}^c}(K)) \quad (7)$$

where the value  $F_{\mathbb{S}^c}(K)$  of the cumulative density function (cdf) of the comonotonic sum  $\mathbb{S}^c$  can be found by solving

$$\sum_{\ell=1}^n \sum_{j=0}^{m-1} a_{\ell} b_j S_{\ell}(0) \exp \left[ \left( r - \frac{1}{2} \sigma_{\ell}^2 \right) (T - j) + \sigma_{\ell} \sqrt{T - j} \Phi^{-1} (F_{\mathbb{S}^c}(K)) \right] = K \quad (8)$$

with  $\Phi(\cdot)$  the standard normal cdf.

**Interpretation of the comonotonic upper bound** Starting from the payoff of the Asian basket option and bounding the  $(\cdot)_+$ -function above followed by a no-arbitrage argument, we find that the time zero price of such Asian basket option should satisfy the following two relations:

$$ABC(n, m, K, T) \leq \sum_{\ell=1}^n a_{\ell} AC_{\ell}(m, K_{\ell}, T) \leq \sum_{\ell=1}^n \sum_{j=0}^{m-1} a_{\ell} b_j e^{-rj} C_{\ell}(K_{\ell j}, T - j) \quad (9)$$

$$ABC(n, m, K, T) \leq \sum_{j=0}^{m-1} b_j e^{-rj} BC(n, K_j, T - j) \leq \sum_{\ell=1}^n \sum_{j=0}^{m-1} a_{\ell} b_j e^{-rj} C_{\ell}(K_{\ell j}, T - j). \quad (10)$$

with

$$\sum_{\ell=1}^n a_{\ell} K_{\ell} = \sum_{j=0}^{m-1} b_j K_j = \sum_{\ell=1}^n \sum_{j=0}^{m-1} a_{\ell} b_j K_{\ell j} = K. \quad (11)$$

This means that the Asian basket call option can be superreplicated by a static<sup>1</sup> portfolio of vanilla call options  $C_{\ell}$  on the underlying assets  $S_{\ell}$  in the basket and with different maturities and strikes. Also an average of Asian options  $AC_{\ell}$  or a combination of basket options  $BC$  with different maturity dates form a superreplicating strategy. Since the weights  $a_{\ell}$  as well as  $b_j$  sum up to one, a possible choice for the strikes in the decompositions (9) is  $K_{\ell} = K_j = K_{\ell j} = K$ . However this will not provide optimal superreplicating strategies. In [18] and [1] it was noted that in the Asian option case the comonotonic upper bound can be interpreted as the price of an optimal static superreplicating strategy consisting of vanilla options. Hobson et al. [13] obtained a similar result for a basket option in a model free framework, while Chen et al. [4] extended this to a more general class of exotic options.

Since prices for basket options can be simulated very fast, the expression (10) as a combination of basket options with different maturity dates might be useful.

## 2.2. Comonotonic lower bound

A lower bound, in the sense of convex order, for  $\mathbb{S} = \sum_{i=1}^{mn} X_i$  is

$$\mathbb{S}^{\ell} = \mathbb{E} [\mathbb{S} \mid \Lambda]$$

where  $\Lambda$  is a normally distributed random variable. If  $\mathbb{E} [X_i \mid \Lambda]$  are all non-decreasing functions of  $\Lambda$  or all non-increasing functions of  $\Lambda$ ,  $\mathbb{S}^{\ell}$  is a sum of comonotonic variables and the reasoning of Dhaene et al. [10] and [11] for the stop-loss premium leads to Theorem 2 below where  $L\Lambda$  denotes ‘lower bound using the conditioning variable  $\Lambda$ ’ and stands for  $e^{-rT} \mathbb{E}^{\mathbb{Q}} \left[ (\mathbb{S}^{\ell} - K)_+ \right]$ . The non-comonotonic situation for Asian basket options is solved in this paper in Section 3.

<sup>1</sup> When exercising an option at a maturity  $T - j$  with  $j \in \{1, \dots, m - 1\}$ , one has in addition to invest the payoff in the risk free money-account.

**Theorem 2** Suppose the sum  $\mathbb{S}$  is given by (4)-(6) and  $\Lambda$  is a normally distributed conditioning variable such that  $(W_\ell(T-j), \Lambda)$  are bivariate normally distributed for all  $\ell$  and  $j$  and the correlation coefficients

$$r_{\ell,j} = \frac{\text{Cov}(W_\ell(T-j), \Lambda)}{\sigma_\Lambda \sqrt{T-j}} \quad (12)$$

have the same sign, when not zero, for all  $\ell$  and  $j$ . Then the comonotonic lower bound for the option price  $ABC(n, m, K, T)$  in (3) is given by

$$\begin{aligned} \text{LBA} = & \sum_{\ell=1}^n \sum_{j=0}^{m-1} a_\ell b_j S_\ell(0) e^{-rj} \Phi \left[ \text{sign}(r_{\ell,j}) \left( r_{\ell,j} \sigma_\ell \sqrt{T-j} - \Phi^{-1}(F_{\mathbb{S}^\ell}(K)) \right) \right] \\ & - e^{-rT} K \Phi \left[ -\text{sign}(r_{\ell,j}) \Phi^{-1}(F_{\mathbb{S}^\ell}(K)) \right] \end{aligned} \quad (13)$$

where the value  $F_{\mathbb{S}^\ell}(K)$  of the cdf of the comonotonic sum  $\mathbb{S}^\ell$  solves

$$\sum_{\ell=1}^n \sum_{j=0}^{m-1} a_\ell b_j S_\ell(0) \exp \left[ \left( r - \frac{1}{2} r_{\ell,j}^2 \sigma_\ell^2 \right) (T-j) + r_{\ell,j} \sigma_\ell \sqrt{T-j} \Phi^{-1}(F_{\mathbb{S}^\ell}(K)) \right] = K. \quad (14)$$

To judge the quality of the stochastic lower bound  $\mathbb{E}[\mathbb{S} | \Lambda]$ , we might look at its variance. To maximize it, i.e. to make it as close as possible to  $\text{Var}[\mathbb{S}]$ , the average value of  $\text{Var}[\mathbb{S} | \Lambda = \lambda]$  should be minimized. In other words, to get the best lower bound,  $\Lambda$  and  $\mathbb{S}$  should be as alike as possible. Recently, Vanduffel et al. provide in [23] a detailed discussion for the optimal choice of the conditioning variable and propose new locally optimal choices. In the present paper, however, we restrict ourselves to four global conditioning variables.

A first idea to choose conditioning variables is based on [14] and [10] and consists in looking at first order approximations of  $\mathbb{S}$ . We can take  $\Lambda = FA1$  or  $FA2$  such that for  $i = 1, 2$ :

$$FAi = \sum_{k=1}^n \sum_{p=0}^{m-1} a_k b_p c_i(k, p) \sigma_k S_k(0) W_k(T-p), \quad (15)$$

with

$$c_1(k, p) = e^{(r - \frac{1}{2} \sigma_k^2)(T-p)}, \quad c_2(k, p) = 1.$$

Vanduffel et al. [21] suggest to look at the conditioning variable such that the first order approximation of the variance of  $\mathbb{S}^\ell$  is maximized. For Asian basket options, this is the case when  $\Lambda$  is given by

$$FA3 = \sum_{k=1}^n \sum_{p=0}^{m-1} a_k b_p S_k(0) e^{r(T-p)} \left[ \left( r - \frac{1}{2} \sigma_k^2 \right) (T-p) + \sigma_k W_k(T-p) \right]. \quad (16)$$

Nielsen and Sandmann [16] suggest to look at the geometric average  $\mathbb{G}$  which in the Asian basket case is defined by

$$\mathbb{G} = \prod_{\ell=1}^n \prod_{j=0}^{m-1} S_\ell(T-j)^{a_\ell b_j} = \prod_{\ell=1}^n \left( \prod_{j=0}^{m-1} \left( S_\ell(0) e^{(r - \frac{1}{2} \sigma_\ell^2)(T-j) + \sigma_\ell W_\ell(T-j)} \right)^{b_j} \right)^{a_\ell}$$

and to consider its standardized logarithm as conditioning variable

$$GA = \frac{\ln \mathbb{G} - \mathbb{E}^{\mathbb{Q}}[\ln \mathbb{G}]}{\sqrt{\text{Var}[\ln \mathbb{G}]}} = \frac{\sum_{\ell=1}^n \sum_{j=0}^{m-1} a_\ell b_j \sigma_\ell W_\ell(T-j)}{\sqrt{\text{Var} \left[ \sum_{\ell=1}^n \sum_{j=0}^{m-1} a_\ell b_j \sigma_\ell W_\ell(T-j) \right]}}. \quad (17)$$

For all these choices of  $\Lambda$ , the correlation coefficients  $r_{\ell,j}$ , which enter the lower bound, are easy to calculate. Their expressions contain the instantaneous correlations  $\rho_{ij}$  (2), which influence the sign of the  $r_{\ell,j}$ . Only when the (non-zero) correlation coefficients  $r_{\ell,j}$  have the same sign for all  $\ell$  and  $j$  the comonotonic lower bound may be applied. Otherwise when the correlations have mixed signs, the lower bound  $\mathbb{E}[S|\Lambda]$  is not a comonotonic sum. Hence the expression (13) is not longer valid. The lower bound of the option price will now involve an integral (see (21)). In Section 3 it is shown that this integral can be simplified and that a closed-form expression is still available.

### 2.3. Bounds based on the Rogers and Shi approach

Rogers and Shi [17] derived an upper bound based on the lower bound starting from the following general inequality for any random variable  $Y$  and  $Z$ :

$$0 \leq \mathbb{E} \left[ \mathbb{E}(Y^+ | Z) - \mathbb{E}(Y | Z)^+ \right] \leq \frac{1}{2} \mathbb{E} \sqrt{\text{Var}(Y | Z)}.$$

According to an idea of Nielsen and Sandmann [16], we determine  $d_\Lambda \in \mathbb{R}$  for each of the four different  $\Lambda$ 's (15), (16) and (17) such that  $\Lambda \geq d_\Lambda$  implies that  $\mathbb{S} \geq K$ .

Combination of both techniques, as done in [8] and [24], results in the following upper bounds which are denoted by UBRSA with  $\Lambda$  being a conditioning variable:

**Theorem 3** *Let  $\mathbb{S}$  be given by (4)-(6) and  $\Lambda$  be a normally distributed conditioning variable such that  $(W_\ell(T-j), \Lambda)$  are bivariate normally distributed for all  $\ell$  and  $j$ . Further, suppose that there exists a  $d_\Lambda \in \mathbb{R}$  such that  $\Lambda \geq d_\Lambda$  implies that  $\mathbb{S} \geq K$ . Then an upper bound to the option price  $ABC(n, m, K, T)$  in (3) is*

$$\begin{aligned} \text{UBRSA} = e^{-rT} \mathbb{E}^{\mathbb{Q}} \left[ (\mathbb{S}^\ell - K)_+ \right] &+ \frac{1}{2} e^{-rT} \left\{ \Phi(d_\Lambda^*) \right\}^{\frac{1}{2}} \times \left\{ \sum_{\ell=1}^n \sum_{k=1}^n \sum_{j=0}^{m-1} \sum_{p=0}^{m-1} a_\ell a_k b_j b_p S_\ell(0) S_k(0) \times \right. \\ &\times e^{r(2T-j-p)} \left( e^{\sigma_\ell \sigma_k \rho_{\ell k} \min(T-j, T-p)} - e^{r_{\ell,j} r_{k,p} \sigma_\ell \sigma_k \sqrt{(T-j)(T-p)}} \right) \\ &\left. \times \Phi \left( d_\Lambda^* - r_{\ell,j} \sigma_\ell \sqrt{T-j} - r_{k,p} \sigma_k \sqrt{T-p} \right) \right\}^{\frac{1}{2}} \end{aligned} \quad (18)$$

with  $d_\Lambda^* = \frac{d_\Lambda - \mathbb{E}^{\mathbb{Q}}[\Lambda]}{\sigma_\Lambda}$ ,  $r_{\ell,j}$  and  $r_{k,p}$  the correlation coefficients (12) and  $\rho_{\ell k}$  the instantaneous correlations (2).

Remark that if the correlation coefficients  $r_{\ell,j}$  have the same sign, when not zero, for all  $\ell$  and  $j$ , then  $e^{-rT} \mathbb{E}^{\mathbb{Q}} \left[ (\mathbb{S}^\ell - K)_+ \right]$  equals the comonotonic lower bound  $\text{LBA}$  of Theorem 2. The explicit expression of  $e^{-rT} \mathbb{E}^{\mathbb{Q}} \left[ (\mathbb{S}^\ell - K)_+ \right]$  in the non-comonotonic situation will be derived in Section 3. Therefore, it is one of the merits of this paper, that it shows that even in a non-comonotonic situation the upper bound based on Rogers and Shi UBRSA can be obtained.

### 2.4. Partially exact/comonotonic upper bound

The so-called partially exact/comonotonic upper bound, denoted by PECUBA with  $\Lambda$  being a conditioning variable, consists of an exact part of the option price and some improved comonotonic upper bound for the remaining part, and can be derived as in [24]:

**Theorem 4** Let  $\mathbb{S}$  be given by (4)-(6) and  $\Lambda$  is a normally distributed conditioning variable satisfying the assumptions of Theorem 3. Then the partially exact/comonotonic upper bound to the option price  $ABC(n, m, K, T)$  in (3) has the following expression:

$$\begin{aligned} \text{PECUBA} &= \sum_{\ell=1}^n \sum_{j=0}^{m-1} a_{\ell} b_j S_{\ell}(0) e^{-jr} \Phi \left( r_{\ell,j} \sigma_{\ell} \sqrt{T-j} - d_{\Lambda}^* \right) - e^{-rT} K (1 - \Phi(d_{\Lambda}^*)) + \\ &+ \sum_{\ell=1}^n \sum_{j=0}^{m-1} a_{\ell} b_j S_{\ell}(0) e^{-jr - \frac{1}{2} r_{\ell,j}^2 \sigma_{\ell}^2 (T-j)} \int_0^{\Phi(d_{\Lambda}^*)} e^{r_{\ell,j} \sigma_{\ell} \sqrt{T-j} \Phi^{-1}(v)} \times \\ &\quad \times \Phi \left( \sigma_{\ell} \sqrt{(T-j)(1-r_{\ell,j}^2)} - \Phi^{-1}(F_{\mathbb{S}^u|V=v}(K)) \right) dv - \\ &- K e^{-rT} \left( \Phi(d_{\Lambda}^*) - \int_0^{\Phi(d_{\Lambda}^*)} F_{\mathbb{S}^u|V=v}(K) dv \right) \end{aligned} \quad (19)$$

where  $V = \Phi \left( \frac{\Lambda - \mathbb{E}^{\mathbb{Q}}[\Lambda]}{\sigma_{\Lambda}} \right)$  and  $F_{\mathbb{S}^u|V=v}(K)$  solves

$$\sum_{\ell=1}^n \sum_{j=0}^{m-1} a_{\ell} b_j S_{\ell}(0) e^{(r - \frac{1}{2} \sigma_{\ell}^2)(T-j) + r_{\ell,j} \sigma_{\ell} \Phi^{-1}(v) \sqrt{T-j} + \sigma_{\ell} \sqrt{(T-j)(1-r_{\ell,j}^2)} \Phi^{-1}(F_{\mathbb{S}^u|V=v}(K))} = K. \quad (20)$$

We stress that for practical applications  $F_{\mathbb{S}^u|V=v}(K)$  just has to be solved from equation (20). The notation  $F_{\mathbb{S}^u|V=v}$ , however, stands for the cdf of the so-called improved comonotonic sum and we refer the interested reader to [14] in which this notion is introduced.

### 3. Non-comonotonic lower bound and upper bound based on the Rogers and Shi approach

In this section, we consider the case where not all  $r_{\ell,j}$  of (12) have the same sign. Then,  $\mathbb{S}^{\ell}$  will not be a comonotonic sum of random variables, making the determination of the lower bound more complicated since it does not follow from the comonotonicity literature. To determine a lower bound, we follow the approach suggested in [15] for basket options. We know that the lower bound can be rewritten as

$$\begin{aligned} e^{-rT} \mathbb{E}^{\mathbb{Q}} \left[ (\mathbb{S}^{\ell} - K)_+ \right] &= e^{-rT} \mathbb{E}^{\mathbb{Q}} \left[ (\mathbb{E}^{\mathbb{Q}}[\mathbb{S} | \Lambda] - K)_+ \right] \\ &= e^{-rT} \mathbb{E}^{\mathbb{Q}} \left[ \left( \sum_{\ell=1}^n \sum_{j=0}^{m-1} a_{\ell} b_j S_{\ell}(0) e^{(r - \frac{1}{2} r_{\ell,j}^2 \sigma_{\ell}^2)(T-j) + r_{\ell,j} \sigma_{\ell} \sqrt{T-j} \frac{\Lambda - \mathbb{E}^{\mathbb{Q}}[\Lambda]}{\sigma_{\Lambda}}} - K \right)_+ \right] \\ &= e^{-rT} \int_0^1 \left( \sum_{\ell=1}^n \sum_{j=0}^{m-1} a_{\ell} b_j S_{\ell}(0) e^{(r - \frac{1}{2} r_{\ell,j}^2 \sigma_{\ell}^2)(T-j) + r_{\ell,j} \sigma_{\ell} \sqrt{T-j} \Phi^{-1}(v)} - K \right)_+ dv, \end{aligned} \quad (21)$$

with  $v = \Phi \left( \frac{\Lambda - \mathbb{E}^{\mathbb{Q}}[\Lambda]}{\sigma_{\Lambda}} \right)$ .

Let us denote

$$f(v) = \sum_{\ell=1}^n \sum_{j=0}^{m-1} a_{\ell} b_j S_{\ell}(0) e^{(r - \frac{1}{2} r_{\ell,j}^2 \sigma_{\ell}^2)(T-j) + r_{\ell,j} \sigma_{\ell} \sqrt{T-j} \Phi^{-1}(v)} - K. \quad (22)$$

Notice that  $f(v)$  is no longer a monotone function of  $v$  (as in the comonotonic situation) when not all  $r_{\ell,j}$  have the same sign. The derivative  $f'(v)$  with respect to  $v$  equals

$$f'(v) = \frac{1}{\varphi[\Phi^{-1}(v)]} \sum_{\ell=1}^n \sum_{j=0}^{m-1} a_{\ell} b_j S_{\ell}(0) r_{\ell,j} \sigma_{\ell} \sqrt{T-j} e^{(r-\frac{1}{2}r_{\ell,j}^2\sigma_{\ell}^2)(T-j)+r_{\ell,j}\sigma_{\ell}\sqrt{T-j}\Phi^{-1}(v)},$$

where  $\varphi(\cdot)$  is the standard normal density function. Obviously, the above denominator is strictly positive for  $v \in (0, 1)$ . The numerator, which we will denote by  $K(v)$ , is a non-decreasing function of  $v$  since its derivative with respect to  $v$  is positive. Moreover, this numerator has the following limits:

$$\lim_{v \rightarrow 0} K(v) = -\infty \quad \text{and} \quad \lim_{v \rightarrow 1} K(v) = +\infty.$$

Therefore, there exists a unique  $v^*$  such that  $K(v^*) = 0$  and consequently  $f'(v^*) = 0$ . Since moreover

$$\lim_{v \rightarrow 0^+} f(v) = +\infty \quad \text{and} \quad \lim_{v \rightarrow 1^-} f(v) = +\infty,$$

we conclude that  $f(v)$  is either positive upon whole the interval  $[0, 1]$ , or has a strictly negative minimum  $f(v^*)$ . Hence, in the latter case,  $f(v)$  stays positive before a certain value  $d_{\Lambda_1} \in ]0, 1[$ , is then negative until a value  $d_{\Lambda_2} \in ]d_{\Lambda_1}, 1[$  but has then again positive values on the interval  $[d_{\Lambda_2}, 1]$ . Therefore, the following theorem can easily be proved:

**Theorem 5** *Let  $\mathbb{S}$  be given by (4)-(6) and let  $\Lambda$  be a normally distributed conditioning variable such that  $(W_{\ell}(T-j), \Lambda)$  are bivariate normally distributed for all  $\ell$  and  $j$ . Suppose that not all  $r_{\ell,j}$  of (12) have the same sign and consider the function  $f$  introduced in (22). The non-comonotonic lower bound for the option price  $ABC(n, m, K, T)$  in (3) is such that*

a) if  $f(v) \geq 0$  for all  $v$ , then

$$\text{LBA} = \sum_{\ell=1}^n \sum_{j=0}^{m-1} a_{\ell} b_j S_{\ell}(0) e^{-rj} - K e^{-rT}, \quad (23)$$

b) if  $f(v^*) < 0$ , with  $v^*$  the solution of  $f'(v) = 0$ , then

$$\begin{aligned} \text{LBA} &= \sum_{\ell=1}^n \sum_{j=0}^{m-1} a_{\ell} b_j S_{\ell}(0) e^{-rj} \Phi\left(d_{\Lambda_1}^* - r_{\ell,j} \sigma_{\ell} \sqrt{T-j}\right) - K e^{-rT} \Phi\left(d_{\Lambda_1}^*\right) + \\ &+ \sum_{\ell=1}^n \sum_{j=0}^{m-1} a_{\ell} b_j S_{\ell}(0) e^{-rj} \Phi\left(r_{\ell,j} \sigma_{\ell} \sqrt{T-j} - d_{\Lambda_2}^*\right) - K e^{-rT} \Phi\left(-d_{\Lambda_2}^*\right) \end{aligned} \quad (24)$$

where, for  $i = 1, 2$ ,  $d_{\Lambda_i}^* = \frac{d_{\Lambda_i} - \mathbb{E}^{\mathbb{Q}}[\Lambda]}{\sigma_{\Lambda}}$  and  $d_{\Lambda_1} \leq d_{\Lambda_2}$  denote the two solutions of the following equation in  $x$ :

$$\sum_{\ell=1}^n \sum_{j=0}^{m-1} a_{\ell} b_j S_{\ell}(0) e^{(r-\frac{1}{2}r_{\ell,j}^2\sigma_{\ell}^2)(T-j)+r_{\ell,j}\sigma_{\ell}\sqrt{T-j}\frac{x-\mathbb{E}^{\mathbb{Q}}[\Lambda]}{\sigma_{\Lambda}}} = K. \quad (25)$$

**PROOF.** Case of  $f(v) \geq 0$  for all  $v$  is trivial.

Case of  $f(v^*) < 0$ :  $\Lambda \leq d_{\Lambda_1}$  or  $\Lambda \geq d_{\Lambda_2}$  imply that  $\mathbb{S}^{\ell} \geq K$  and  $d_{\Lambda_1} < \Lambda < d_{\Lambda_2}$  implies  $\mathbb{S}^{\ell} < K$ .  $\square$

### Remarks

- (i) This lower bound can be used in the Rogers and Shi approach, so the upper bound UBRS can also be derived in the non-comonotonic situation.
- (ii) As a basket option is a special case of an Asian basket option with  $m = 1$ , the reasoning above and formula (24) (with  $m = 1$ ) remains valid for basket options in the cases where  $\mathbb{S}^{\ell}$



is not a comonotonic sum, providing a much simpler lower bound than in [8]. No optimization algorithm is needed.

- (iii) The approach in this section is general and can also be used in other settings in which sums of non-comonotonic random variables show up with correlations with mixed signs. In [22], Vanduffel et al. deal with cash flows with mixed signs and obtain a result with a similar taste.

#### 4. Generalization of an upper bound based on the method of Thompson and of Lord

In his paper [19], Thompson used intuition and simple optimization to derive an upper bound which tightened Rogers and Shi's upper bound considerably for continuously sampled Asian options. His reasoning is based upon a first order approximation and is therefore referred to as *FA*. In his Ph.D. thesis [20], Thompson already suggested the idea of adding a supplementary parameter but he did not work it out. Thompson's approximation is only justified when  $\sigma_\ell W_\ell(T - j)$  has a small variance (i.e. when  $\sigma_\ell^2(T - j)$  is small).

In case of Asian options, Lord [15] approximates the arithmetic sum by a shifted lognormal variable — and therefore the results are referred to as *SLN* — and then adds according to the ideas of Thompson a supplementary parameter.

In this section, both the methods of Thompson [19] and Lord [15] will be generalized to the Asian basket case by taking into account a supplementary parameter. The numerical section 5 will show that these methods provide most of the times the best upper bounds.

**Theorem 6** *Let  $\mathbb{S}$  be given by (4)-(6) and  $\bar{\sigma} > 0$ , then upper bounds based on Thompson's method (for  $X$  being *FA*) and Lord's reasoning (for  $X$  being *SLN*) for the option price  $ABC(n, m, K, T)$  in (3) are given by*

$$ABC(n, m, K, T) \leq e^{-rT} \sum_{\ell=1}^n \sum_{j=0}^{m-1} a_\ell b_j \int_{-\infty}^{+\infty} \left\{ c_\ell^X(T - j, x, \bar{\sigma}) \Phi \left( \frac{c_\ell^X(T - j, x, \bar{\sigma})}{d_\ell(T - j, \bar{\sigma})} \right) + d_\ell(T - j, \bar{\sigma}) \varphi \left( \frac{c_\ell^X(T - j, x, \bar{\sigma})}{d_\ell(T - j, \bar{\sigma})} \right) \right\} \varphi(x) dx. \quad (26)$$

with  $\varphi$  the standard normal density function, and with  $c_\ell^X(T - j, x, \bar{\sigma})$  and  $d_\ell^2(T - j, \bar{\sigma})$  the conditional mean and variance:

$$c_\ell^X(T - j, x, \bar{\sigma}) = S_\ell(0) e^{(r - \frac{1}{2}\sigma_\ell^2)(T - j) + \sigma_\ell x \sqrt{T - j}} - \bar{\sigma} K \left( \frac{\mu_\ell^X(T - j)}{\bar{\sigma}} + \sigma_\ell x \sqrt{T - j} - \sum_{i=1}^n \sum_{k=0}^{m-1} a_i b_k \sigma_i \frac{\min(T - k, T - j)}{\sqrt{T - j}} x \right) \quad (27)$$

and

$$d_\ell^2(T - j, \bar{\sigma}) = \bar{\sigma}^2 K^2 \left( \sum_{i=1}^n \sum_{k=0}^{m-1} \sum_{h=1}^n \sum_{p=0}^{m-1} a_i b_k a_h b_p \sigma_i \sigma_h \rho_{ih} \min(T - k, T - p) - \frac{\left( \sum_{i=1}^n \sum_{k=0}^{m-1} a_i b_k \sigma_i \rho_{i\ell} \min(T - k, T - j) \right)^2}{T - j} \right). \quad (28)$$

In case of the generalization of Thompson's method:

$$\mu_\ell^{FA}(T-j) = \frac{1}{K} \left( S_\ell(0) e^{(r-\frac{1}{2}\sigma_\ell^2)(T-j)} + \gamma^{FA} \sqrt{\text{Var}(Y_\ell^{FA}(T-j))} \right) \quad (29)$$

with

$$\gamma^{FA} = \frac{K - \sum_{\ell=1}^n \sum_{j=0}^{m-1} a_\ell b_j S_\ell(0) e^{(r-\frac{1}{2}\sigma_\ell^2)(T-j)}}{\sum_{\ell=1}^n \sum_{j=0}^{m-1} a_\ell b_j \sqrt{\text{Var}(Y_\ell^{FA}(T-j))}}, \quad (30)$$

and with the first order approximations for  $S_\ell(T-j)$  given by

$$S_\ell^{FA}(T-j) = S_\ell(0) e^{(r-\frac{1}{2}\sigma_\ell^2)(T-j)} (1 + \sigma_\ell W_\ell(T-j)). \quad (31)$$

$$Y_\ell^{FA}(T-j) = S_\ell^{FA}(T-j) - \bar{\sigma} K \left[ \sigma_\ell W_\ell(T-j) - \sum_{i=1}^n \sum_{k=0}^{m-1} a_i b_k \sigma_i W_i(T-k) \right] \quad (32)$$

is a first order approximation of

$$Y_\ell(T-j) = S_\ell(T-j) - \bar{\sigma} K \left[ \sigma_\ell W_\ell(T-j) - \sum_{i=1}^n \sum_{k=0}^{m-1} a_i b_k \sigma_i W_i(T-k) \right]. \quad (33)$$

In case of Lord's generalized results, a shifted lognormal approximation for  $Y_\ell(T-j)$  is of the form:

$$Y_\ell^{SLN}(T-j) = \alpha(\ell, T-j) + \exp[\theta(\ell, T-j) + \omega(\ell, T-j)Z_\ell], \quad (34)$$

where  $\alpha(\ell, T-j)$ ,  $\theta(\ell, T-j)$  and  $\omega(\ell, T-j)$  are the shift, mean and volatility functions and  $Z_\ell$  is a standard normal distribution. Further,

$$\mu_\ell^{SLN}(T-j) = \frac{1}{K} \left[ \alpha(\ell, T-j) + \exp[\theta(\ell, T-j) + \gamma^{SLN} \omega(\ell, T-j)] \right], \quad (35)$$

with the constant  $\gamma^{SLN}$  determined by the condition

$$\sum_{\ell=1}^n \sum_{j=0}^{m-1} a_\ell b_j \mu_\ell^{SLN}(T-j) = 1. \quad (36)$$

In this theorem,  $\bar{\sigma}$  is an arbitrary parameter. In [19], only the choice of  $\bar{\sigma} = 1$  is considered and this in the case of an Asian option. Numerically, we find that the upper bound is quadratic around the optimal value of  $\bar{\sigma}$  and therefore one can use an algorithm suggested by Lord [15] to determine the optimal upper bound.

#### Algorithm

- (i) Calculate the upper bound using  $\mu_\ell^{FA}(T-j)$  (resp.  $\mu_\ell^{SLN}(T-j)$ ) for three carefully chosen values of  $\bar{\sigma}$ ;
- (ii) Fit a quadratic function in  $\bar{\sigma}$  to these computed values;
- (iii) Determine the value of  $\bar{\sigma}$  in which the upper bound attains its minimum;
- (iv) Recalculate the upper bound in the approximately optimal  $\bar{\sigma}$ .

In the numerical section, we will compare the bound for  $\bar{\sigma} = 1$ , which will be called 'ThompUB', with (among others) the bound obtained by the optimized  $\bar{\sigma}$ , which will be called 'ThompUBquad'. Especially for long maturities, high volatilities and high strike values, the effects of optimizing  $\bar{\sigma}$  are considerable.

	initial	weight	volatility	dividend yield
stock	stock price	(in %)	(in %)	(in %)
BASF	42.55	25	33.34	2.59
Bayer	48.21	20	31.13	2.63
Degussa-Hüls	34.30	30	33.27	3.32
FMC	100.00	10	35.12	0.69
Schering	66.19	15	36.36	1.24

Table 1  
Stock characteristics

	BASF	Bayer	Degussa-Hüls	FMC	Schering
BASF	1.00	0.84	-0.07	0.45	0.43
Bayer	0.84	1.00	0.08	0.62	0.57
Degussa-Hüls	-0.07	0.08	1.00	-0.54	-0.59
FMC	0.45	0.62	-0.54	1.00	0.86
Schering	0.43	0.57	-0.59	0.86	1.00

Table 2  
Correlation structure

## 5. Numerical results

In this section we consider a numerical example for an Asian basket option in the Black & Scholes setting and compare the different lower and upper bounds. We recall the following notations where  $\Lambda$  can be FA1, FA2, FA3 or GA:  $L\Lambda$  for both the comonotonic lower bound (13) and the non-comonotonic lower bound (23) or (24),  $PECUB\Lambda$  for partially exact/comonotonic upper bound (19),  $UBRS\Lambda$  for upper bound (18) (with a comonotonic or non-comonotonic lower bound) based on the Rogers & Shi approach and CUB for comonotonic upper bound (7). We use the notation  $PECUB$  for the  $\min(PECUB_{FA1}, PECUB_{FA2}, PECUB_{FA3}, PECUB_{GA})$ ,  $UBRS$  for  $\min(UBRS_{FA1}, UBRS_{FA2}, UBRS_{FA3}, UBRS_{GA})$ ,  $LB$  for  $\max(LB_{FA1}, LB_{FA2}, LB_{FA3}, LB_{GA})$ ,  $ThompUB$  for upper bound based on Thompson's first order approximation with  $\bar{\sigma} = 1$ ,  $ThompUBquad$ , and  $SLNquad$  for upper bound based on the first order approximation and on the shift lognormal approximations, which use a numerical optimization algorithm to approximate the optimal scale  $\bar{\sigma}$ . The moneyness of the option is defined as

$$\frac{K}{\sum_{\ell=1}^n \sum_{j=0}^{m-1} a_{\ell} b_j \mathbb{E}^{\mathbb{Q}} [S_{\ell}(T-j)]} - 1. \quad (37)$$

Negative moneyness corresponds to in-the-money options, positive moneyness to out-of-the-money options. A moneyness of zero indicates that the option is at-the-money. In order to illustrate our bounds for an Asian basket options, we take a set of input data from [2] where the valuation results for Asian basket option with monthly averaging were written on a fictitious chemistry-pharma basket that consists of the five German DAX stocks listed in Tables 1 and 2.

The annual risk-free interest rate  $r$  is equal to 6% and we compute bounds for options with three different maturity dates (half a year, one year and five years). The exercise prices are chosen in such a way that Table 3 shows results for in-the-money, at-the-money and out-of-the money options. The averaging period of all options is five months and starts five months before maturity.

In Table 3, we compare the upper and lower bounds with Monte Carlo (MC) estimates. These

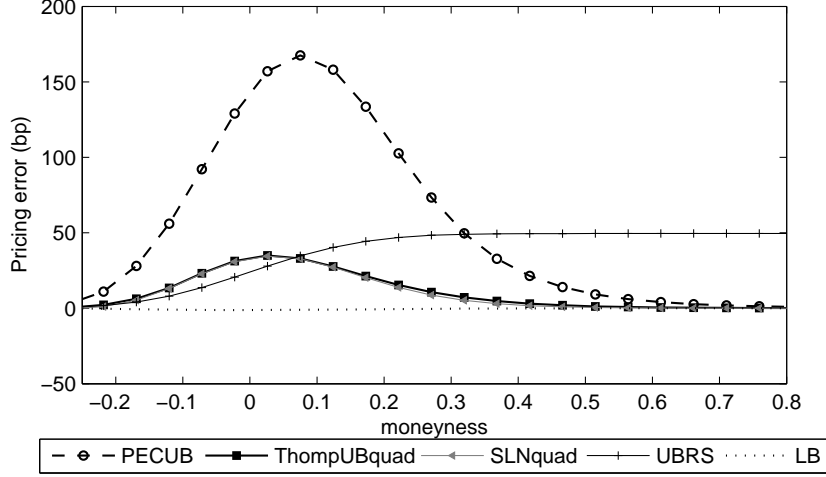


Fig. 1. Comparison of bounds for an Asian basket option with  $T = 0.5$

Monte Carlo estimates (and also the standard deviations (SE)) are obtained by generating 1 000 000 paths using antithetic variables, by following the algorithm of [6], [7] and of [12]. In Fig. 1 - Fig. 3 we plot the pricing error of a bound with respect to the moneyness (37) for different maturities  $T$ . This pricing error expressed in basis points (bp) is defined as

$$\frac{\text{bound} - \text{MC value}}{\sum_{\ell=1}^n a_{\ell} S_{\ell}(0)} 10\,000$$

where ‘bound’ takes the value of LB, UBRS, PECUB, ThompUBquad or SLNquad and the denominator equals 50.498 according to the data in Table 1.

For this data set,  $FA1$ ,  $FA2$  and  $FA3$  lead not to comonotonic lower bounds since the correlations  $r_{\ell,j}$  of (12) do not have the same sign for all  $\ell$  and  $j$ . However, in this case the method of Section 3 can be applied. The results from Table 3 reveal that the non-comonotonic lower bounds  $LBFA1$ ,  $LBFA2$  and  $LBFA3$  perform better than the comonotonic lower bound  $LBGA$ . The non-comonotonic lower bounds  $LBFA1$ ,  $LBFA2$  and  $LBFA3$  equal up to the 20-th decimal the sum of the last two terms in (24) as the sum of the first two terms is almost negligible.

From Table 3 and the figures Fig. 1 - Fig. 3, we notice that only for short maturities and in- and at-the-money, UBRS outperforms all the other upper bounds. In all other cases, ThompUBquad and SLNquad provide the best upper bounds, with SLNquad beating ThompUBquad most of the times. SLNquad is significantly sharper for long maturities and out-of-the money. We notice that the lower bound is very close to the Monte Carlo value but loses a bit of its sharpness for larger maturities. Also the precision of ThompUBquad and SLNquad decreases with the maturity  $T$ . PECUB is too high to be useful in comparison with ThompUBquad and SLNquad. Only (far) out of the money, PECUB becomes better than UBRS. It is easy to prove that UBRS converges to the constant  $\frac{1}{2}e^{-rT}\mathbb{E}^{\mathbb{Q}}[\text{Var}(S^{\ell})]^{1/2}$  for  $K$  tending to infinity, whereas both PECUB and CUB converge to zero for  $K$  tending to infinity. The CUB however, which can be seen as the price of a static hedging portfolio as mentioned before, leads to much higher upper bounds (see Table 3) since this bound does not take the correlations (2) into account. We further notice that UBRS $\Lambda$  obtains the best values for  $\Lambda = FA3$ , which by construction minimizes (a first order approximation

$T$	$K$ (moneyness)	To compare	Thomp UB	Thomp UBquad	SLN quad	LBA	UBRSA	PECUBA	$\Lambda$	
$\frac{1}{2}$	40 (-0.2181)	MC:10.8465	10.8582	10.8580	10.8520	10.8448	10.8556	10.9042	FA1	
		SE:0.0057				10.8448	10.8558	10.9054	FA2	
		CUB:11.1221				10.8448	10.8554	10.9023	FA3	
	50 (-0.0227)	MC:2.7860	2.9564	2.9442	2.9415	2.7801	2.8937	3.4912	FA1	
		SE:0.0040				2.7801	2.8930	3.4919	FA2	
		CUB:4.3465				2.7800	2.8903	3.4376	FA3	
						2.6705	3.2836	3.9378	GA	
						0.2299	0.4617	0.9288	FA1	
	60 (0.1728)	MC:0.2338	0.3591	0.3417	0.3361	0.2299	0.4613	0.9272	FA2	
SE:0.0012		0.2300				0.4573	0.9080	FA3		
CUB:1.1856		0.1742				1.1034	1.0407	GA		
1	40 (-0.2332)	MC:11.7157	11.7727	11.7718	11.7678	11.6984	11.8013	12.0623	FA1	
		SE:0.0097				11.6988	11.8025	12.0710	FA2	
		CUB:12.8736				11.6979	11.7986	12.0473	FA3	
	50 (-0.0415)	MC:4.7336	4.9739	4.9579	4.9505	4.7094	5.0155	5.7644	FA1	
		SE:0.0074				4.7095	5.0127	5.7664	FA2	
		CUB:6.9693				4.7092	5.0007	5.6535	FA3	
						4.5289	5.4830	6.2004	GA	
						1.3882	1.9106	2.7179	FA1	
	60 (0.1502)	MC:1.4083	1.7048	1.6562	1.6542	1.3875	1.9080	2.7092	FA2	
		SE:0.0043				1.3886	1.8903	2.6042	FA3	
		CUB:3.4347				1.1935	2.9853	2.9852	GA	
	5	40 (-0.3346)	MC:17.3030	17.6536	17.6361	17.6159	16.9863	18.1608	18.5893	FA1
			SE:0.1319				17.0030	18.1698	18.6527	FA2
			CUB: 20.2517				16.9727	18.1300	18.5112	FA3
		50 (-0.1807)	MC:12.5916	13.2374	13.1674	13.1334	12.2352	13.9249	14.5541	FA1
SE:0.0295			12.2421				13.8929	14.5678	FA2	
CUB:16.4350			12.2282				13.7679	14.1973	FA3	
			11.9023				14.6519	14.9816	GA	
			8.7834				11.0153	11.6879	FA1	
60 (-0.0168)		MC:9.1299	10.0549	9.8545	9.8331	8.7774	10.9485	11.6439	FA2	
		SE:0.0268				8.7853	10.7215	11.0728	FA3	
		CUB:13.4094				8.2379	12.0024	12.0553	GA	
70 (0.1470)		MC: 6.6520	7.8539	7.4618	7.4451	6.3285	9.0801	9.5618	FA1	
		SE: 0.0241				6.3127	9.0051	9.5048	FA2	
		CUB: 11.0082				6.3376	8.6661	8.8910	FA3	
					5.6654	10.2258	9.7925	GA		

Table 3  
Valuation results for Asian basket call option

of  $\mathbb{E}^{\mathbb{Q}}[\text{Var}(\mathbb{S}^{\ell})]$ , a term which is related to  $\mathbb{E}^{\mathbb{Q}}[\text{Var}(\mathbb{S}^{\ell})1_{\{\Lambda < d_{\Lambda}\}}]$  showing up in the expression of UBRSA.

We also compare ThompUB, the Asian basket option bound obtained by following Thompson's approach for  $\bar{\sigma} = 1$ , with ThompUBquad, the bound obtained by the optimized  $\bar{\sigma}$ . Especially for long maturities and high strike values, the effects of optimizing  $\bar{\sigma}$  are considerable.

**Conclusion** The lower bound LB, which can be calculated in both comonotonic and non-

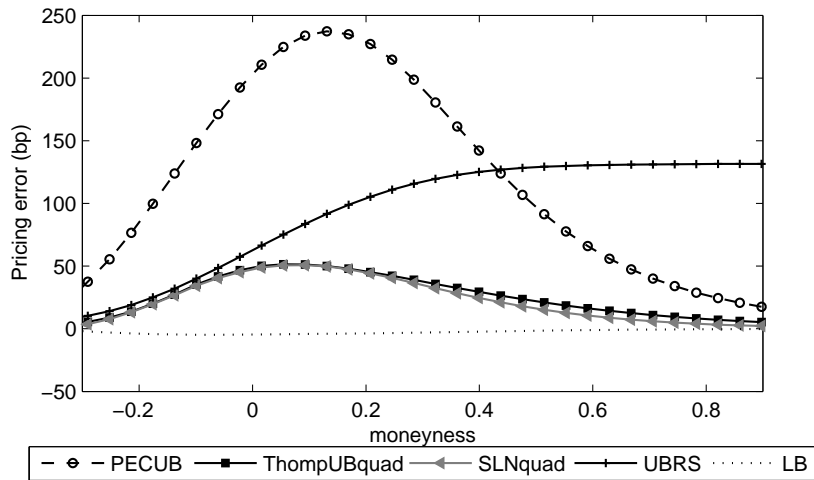


Fig. 2. Comparison of bounds for an Asian basket option value with  $T = 1$

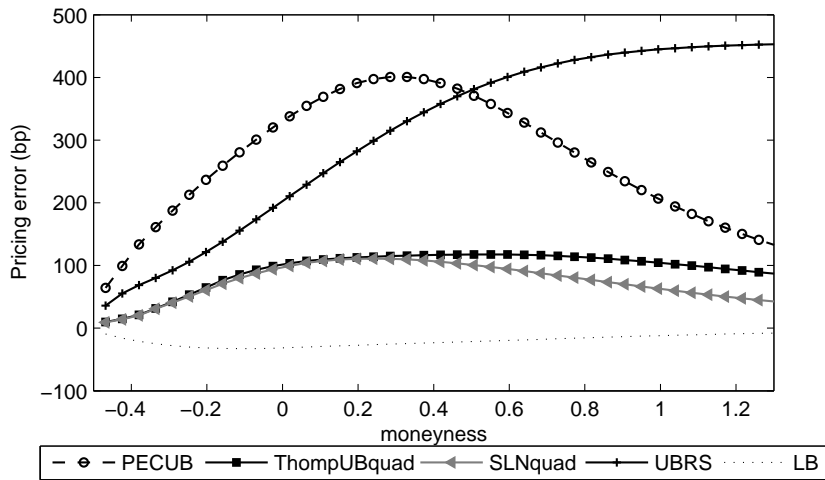


Fig. 3. Comparison of bounds for an Asian basket option with  $T = 5$

comonotonic situations, leads to very precise lower bounds. Based on our numerical tests, we recommend the reader to use for short maturities and in- and at-the money the upper bound UBRS, which also can be derived in both comonotonic and non-comonotonic situations. In the other cases, SLNquad seems to be the best upper bound.

*Acknowledgements* The authors would like to thank Roger Lord for helpful discussions. They also would like to thank an anonymous referee for useful comments. Michèle Vanmaele would like to acknowledge the financial support by the BOF-project 011/155/04 of the Ghent University.

## References

- [1] Albrecher H., Dhaene J., Goovaerts M.J. and Schoutens W. (2005). "Static hedging of Asian options under Lévy models: the comonotonicity approach", *The Journal of Derivatives*, **12**(3), 63-72.

- [2] Beisser J. (2001). *Topics in Finance - A conditional expectation approach to value Asian, Basket and Spread Options*. Ph.D. Thesis, Johannes Gutenberg University Mainz.
- [3] Black F. and Scholes M. (1973). "The price of options and corporate liabilities", *Journal of Political Economy*, **81**, 637-659.
- [4] Chen X., Deelstra G., Dhaene J. and Vanmaele M. (2006). "Static Super-Replicating Strategies for Exotic Options" Working Paper, Katholieke Universiteit Leuven.
- [5] Curran M. (1994). "Valuing Asian and portfolio by conditioning on the geometric mean price", *Management science*, **40**, 1705-1711.
- [6] Dahl L.O. and Benth F.E. (2001a). "Valuing Asian basket options with quasi-Monte Carlo techniques and singular value decomposition", *Pure Mathematics*, **2** February, 1-21.
- [7] Dahl L.O. and Benth F.E. (2001b). "Fast evaluation of the Asian basket option by singular value decomposition", *Pure Mathematics*, **8** March, 1-14.
- [8] Deelstra G., Liinev J. and Vanmaele M. (2004). "Pricing of arithmetic basket options by conditioning", *Insurance: Mathematics & Economics*, **34**, 55-57.
- [9] Dhaene J., Wang S., Young V.R. and Goovaerts M.J. (2002). "Comonotonicity and maximal stop-loss premiums", *Bulletin of the Swiss Association of Actuaries*, **2000**(2), 99-113.
- [10] Dhaene J., Denuit M., Goovaerts M.J., Kaas R. and Vyncke D. (2002a). "The concept of comonotonicity in actuarial science and finance: theory", *Insurance: Mathematics & Economics*, **31**(1), 3-33.
- [11] Dhaene J., Denuit M., Goovaerts M.J., Kaas R. and Vyncke D. (2002b). "The concept of comonotonicity in actuarial science and finance: Applications", *Insurance: Mathematics & Economics*, **31**(2), 133-161.
- [12] Glasserman P. (2003). *Monte Carlo Methods in Financial Engineering*. Springer.
- [13] Hobson D., Laurence P. and Wang T. (2005). "Static-Arbitrage Upper Bounds for the Prices of Basket Options", *Quantitative finance*, **5**(4), 329-342.
- [14] Kaas R., Dhaene J. and Goovaerts M.J. (2000). "Upper and lower bounds for sums of random variables", *Insurance: Mathematics & Economics*, **27**, 151-168.
- [15] Lord R. (2006). "Partially exact and bounded approximations for arithmetic Asian options", forthcoming in *Journal of Computational Finance*.
- [16] Nielsen J.A. and Sandmann K. (2003). "Pricing Bounds on Asian options", *Journal of Financial and Quantitative Analysis*, **38**(2), 449-473.
- [17] Rogers L.C.G. and Shi Z. (1995). "The value of an Asian option", *Journal of Applied Probability*, **32**, 1077-1088.
- [18] Simon S., Goovaerts M.J. and Dhaene J. (2000). "An easy computable upper bound for the price of an arithmetic Asian option", *Insurance: Mathematics & Economics*, **26**, 175-183.
- [19] Thompson G.W.P. (1999a). "Fast narrow bounds on the value of Asian options", Working paper, University of Cambridge.
- [20] Thompson G.W.P. (1999b). "Topics in mathematical Finance", Ph.D. Thesis, University of Cambridge.
- [21] Vanduffel S., Hoedemakers T. and Dhaene J. (2005). "Comparing Approximations for Risk Measures of Sums of Non-independent lognormal Random Variables" *North American Actuarial Journal*, **9**(4), 71-82.
- [22] Vanduffel S., Dhaene J. and Goovaerts M.J. (2005). "On the evaluation of 'saving-consumption' plans", *Journal of Pension Economics and Finance*, **4**(1), 17-30.
- [23] Vanduffel S., Chen X., Dhaene J., Goovaerts M.J., Henrard L. and Kaas R. (2006). "A note on optimal lower bound approximations for risk measures of sums of lognormals", Working paper KULeuven.
- [24] Vanmaele M., Deelstra G., Liinev J., Dhaene J. and Goovaerts M. J. (2006). "Bounds for the price of discrete sampled arithmetic Asian options", *Journal of Computational and Applied Mathematics*, **185**(1), 51-90.