# Long-term returns in stochastic interest rate models

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**Abstract**

In this paper, we observe the convergence of the long-term return, using an extension of the Cox-Ingersoll-Ross (1985) stochastic model of the short interest rate $r$. Using the theory of Bessel processes, we are able to prove the convergence almost everywhere of $\frac{1}{t}\int_0^t X_s ds$ with $X$ a generalized Besselsquare process with drift with stochastic reversion level.

**Key words**

Long-term return, stochastic processes, generalized Besselsquare processes, convergence almost everywhere.

Abbreviated title: Long-term returns

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# 1 Introduction.

In this paper, we are interested in the long-term return $\frac{1}{t}\int_0^t r_u du$ where $(r_u)_{u\geq 0}$ denotes the instantaneous interest rate. Insurance companies promise a certain fixed percentage of interest on their insurance products such as bonds, life-insurances, etcetera. We wonder how this percentage should be determined. It has to be lower than the estimated random return in order to cover expenses and to hold some reserves so that the company can get through difficult periods of economic crisis without bankruptcy. On the other hand, the competitive struggle, strengthened by the open market of the European Community, forces rival companies to take the wishes of the clientele into considerations. Naturally, the costumer wants a return as high as possible.

In this light, we think it is interesting to study and to model the long-term return in a mathematical way. We analyse the convergence of the long-term return, using an extension of the Cox, Ingersoll and Ross (1985) stochastic model of the short interest rate $r$. Cox, Ingersoll and Ross express the short interest rate dynamics as

$$dr_t = \kappa(\gamma - r_t)dt + \sigma\sqrt{r_t}dB_t.$$

with $(B_t)_{t\geq 0}$ a Brownian motion and $\kappa, \gamma$ and $\sigma$ positive constants. It is a well-known fact that this model has some empirically relevant properties. In this model, $r$ never becomes negative and for $2\kappa\gamma \geq \sigma^2$, $r$ does not reach zero. For $\kappa > 0$ and $\gamma \geq 0$, the randomly moving interest rate is elastically pulled towards the long-term constant value $\gamma$.

However, it is reasonable to conjecture that the market will constantly change this level $\gamma$ and the volatility $\sigma$. In the footsteps of Schaefer and Schwartz (1984), Hull and White (1990) and Longstaff and Schwartz (1992), we extend the CIR model in order to reflect the time-dependence caused by the cyclical nature of the economy or by expectations concerning the future impacts of monetary policies. We assume the reversion level to be stochastic and we also generalize the volatility. In this situation, we examine the convergence of the long-term return $\frac{1}{t}\int_0^t r_u du$ and we propose some applications.

We consider a family of stochastic processes X, which contains the Bessel-square processes with drift. The many results known about these processes, e.g. Pitman-Yor (1982), Revuz-Yor (1991), convinced us that these processes are very tractable. Using the theory of Bessel processes, we found the following theorem, which is very useful for deducing the convergence almost everywhere of the long-term return in quite general situations:

**Theorem 1**

*Suppose that a probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, I\!\!P)$ is given and that a Brownian* *motion $(B_t)_{t\geq 0}$ is defined on it. A stochastic process $X : \Omega \times I\!R^+ \to I\!R^+$ is assumed to satisfy the stochastic differential equation*

$$dX_s = (2\beta X_s + \delta_s)ds + g(X_s)dB_s \quad \forall s \in I\!R^+$$

*with $\beta < 0$ and $g : I\!R \to I\!R^+$ a function, vanishing at zero and such that there is a constant $b$ with $|g(x) - g(y)| \leq b\sqrt{|x-y|}$.*
*The measurable and adapted process $\delta : \Omega \times I\!R^+ \to I\!R^+$ is assumed to satisfy:*

$$\frac{1}{s}\int_0^s \delta_u du \overset{a.e.}{\to} \overline{\delta} \text{ with } \overline{\delta} : \Omega \to I\!R^+.$$

*Under these conditions, the following convergence almost everywhere holds*

$$\frac{1}{s}\int_0^s X_u du \xrightarrow{a.e.} \frac{-\overline{\delta}}{2\beta}.$$

We will give a proof of this theorem in section 2.

In section 3, we show an immediate application of theorem 1. We consider the long-term return in the two-factormodel:

$$\begin{aligned} dr_t &= \kappa(\gamma_t - r_t)dt + \sigma\sqrt{r_t}dB_t \\ d\gamma_t &= \tilde{\kappa}(\gamma^* - \gamma_t)dt + \tilde{\sigma}\sqrt{\gamma_t}d\tilde{B}_t \end{aligned}$$

with $(B_t)_{t\geq 0}$ and $(\tilde{B}_t)_{t\geq 0}$ two Brownian Motions and with $\kappa$, $\tilde{\kappa}$, $\sigma$, $\tilde{\sigma}$ and $\gamma^*$ positive constants. The short interest rate process has a reversion level which is a stochastic process itself. We do not need any assumptions about the correlation between the Brownian motions of the instantaneous interest rate and of the stochastic reversion level process. We stress this fact because it is not trivial. Most authors of two-factormodels require, for technical reasons, that the Wiener processes are uncorrelated or have a deterministic and fixed correlation.

Without further notice we assume that the filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfies the usual assumptions with respect to $I\!P$, a fixed probability on the sigma-algebra $\mathcal{F}_\infty = \bigvee_{t\geq 0} \mathcal{F}_t$. Also $B$ is a continuous process that is a Brownian motion with respect to $(\mathcal{F}_t)_{t\geq 0}$.

## 2 Convergence a.e. of the long-term return.

Using the theory of Bessel processes, see Pitman-Yor (1982), we found a theorem which is very useful for deducing the convergence almost everywhere of the long-term return in quite general situations. In this section, we give a proof of the convergence result, which relies on the theory of stochastic differential equations and on Kronecker's lemma.

First, we recall Kronecker's lemma which is a standard lemma, even for stochastic integrals [see Revuz-Yor (1991, p.175 exercise 1.16)]. For completeness, we give a proof.

**Kronecker's lemma**

*Let us assume a continuous semimartingale $Y$ and a strictly positive increasing function $f$ which tends to infinity.*
*If $\int_0^\infty \frac{dY_u}{f(u)}$ exists a.e., then $\frac{Y_t}{f(t)} \longrightarrow 0$ a.e.*
Proof

If we denote $dZ_t = \frac{dY_t}{f(t)}$, then $Y_t = Y_0 + \int_0^t f(u)dZ_u$. By partial integration, recall that the semimartingale Z is continuous, we find that:

$$\begin{aligned} Y_t &= Y_0 + f(t)Z_t - f(0)Z_0 - \int_0^t Z_u df(u) \\ &= Y_0 + f(0)Z_t + \int_0^t (Z_t - Z_u)df(u) - f(0)Z_0. \end{aligned}$$

Consequently,

$$\frac{Y_t}{f(t)} = \frac{Y_0 + f(0)Z_t - f(0)Z_0}{f(t)} + \frac{1}{f(t)}\int_0^t (Z_t - Z_u)df(u).$$

Since $Z_\infty = \int_0^\infty \frac{dY_u}{f(u)}$ exists a.e., $(Z_t)_{t\geq 0}$ converges to $Z_\infty$ a.e. and therefore $\sup Z_t < \infty$ a.e.. Hence, the first term converges to zero a.e..
Let us look at the second term:

$$\frac{1}{f(t)}\int_0^t (Z_t - Z_u)df(u) = \frac{1}{f(t)}\int_0^s (Z_t - Z_u)df(u) + \frac{1}{f(t)}\int_s^t (Z_t - Z_u)df(u).$$

Since $(Z_t)_{t\geq 0}$ converges to $Z_\infty$ a.e., we can choose for a given $\varepsilon > 0$, a number $s$ large enough such that $|Z_t - Z_u| < \varepsilon$ for all $t, u \geq s$. For this fixed $s$, we can choose $t$ such that $2\max_{u\geq 0}|Z_u|\frac{f(s)}{f(t)} < \varepsilon$.

q.e.d.

To study the convergence a.e. of the long-term return, we consider a family of stochastic processes X, which contains the Besselsquare processes with drift. We define the (continuous) adapted process $X$ by the stochastic differential equation

$$dX_s = (2\beta X_s + \delta_s)ds + g(X_s)dB_s \quad \forall s \in \mathbb{R}^+$$

with $\beta$ strictly negative and $g$ a function, vanishing at zero and satisfying a Hölder condition of order one half. In order to have a unique solution of this stochastic differential equation, we suppose that $\int_0^t \delta_s ds < \infty$ for all $t$. The unique solution $X$ is non-negative and satisfies

$$X_s = e^{2\beta s}\left(X_0 + \int_0^s \delta_u e^{-2\beta u}\, du + \int_0^s e^{-2\beta u}\, g(X_u)\, dB_u\right). \tag{1}$$

In view of the applications, we think that the problem of existence and uniqueness goes beyond the scope of this paper. The interested reader is referred to Deelstra-Delbaen (1994).

For this family of stochastic processes, we first prove a technical lemma needed in the proof of the main theorem.

**Lemma 1**

*Suppose that a probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is given and that a stochastic process $X : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ is defined by the stochastic differential equation*

$$dX_s = (2\beta X_s + \delta_s)ds + g(X_s)dB_s \quad \forall s \in \mathbb{R}^+$$

*with*

- $\beta \leq 0$,
- $g : \mathbb{R} \to \mathbb{R}^+$ *is a function, vanishing at zero and such that there is a constant $b$ with $|g(x) - g(y)| \leq b\sqrt{|x-y|}$,*
- $\delta : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ *is an adapted and measurable process,*
- $\int_0^t \delta_u du < \infty$ *a.s. for all $t$.*

*Then for all $t \geq 0$ we have $\sup_{0\leq u\leq t} X_u$ is finite and if $\int_0^t \mathbb{E}[\delta_u]du < \infty$ we have for all $s \leq t$:*

$$\mathbb{E}[X_s] = e^{2\beta s}X_0 + \int_0^s e^{2\beta(s-u)}\mathbb{E}[\delta_u]du\,.$$

Proof

Let us first consider the case $\int_0^t E[\delta_u]du < \infty$.
We define the sequence $(T_n)_{n\geq 0}$ of stopping times by

$$T_n = \inf\{u \mid X_u \geq n\}.$$

From (1), we have that for all $s \leq t$

$$e^{-2\beta(s\wedge T_n)} X_{s\wedge T_n} = X_0 + \int_0^{s\wedge T_n} \delta_u e^{-2\beta u} du + \int_0^{s\wedge T_n} g(X_u) e^{-2\beta u}\, dB_u. \tag{2}$$

Since $X$ is bounded on the interval $[\![0, T_n]\!]$ and since $|g(x)|$ is bounded by $b\sqrt{x}$, we obtain that $\left(\int_0^{s\wedge T_n} g(X_u)e^{-2\beta u}\, dB_u\right)_{0\leq s\leq t}$ is a martingale, bounded in $L^2$. Indeed, let us calculate the square of the $L^2$ norm:

$$\begin{aligned} & I\!\!E\left[\int_0^{t\wedge T_n} g^2(X_u)e^{-4\beta u}\, du\right] \\ \leq \;& e^{-4\beta t} I\!\!E\left[\int_0^{t\wedge T_n} b^2 X_u\, du\right] \\ \leq \;& b^2 e^{-4\beta t} I\!\!E\left[\int_0^{t\wedge T_n} X_u\, du\right] \\ \leq \;& b^2 e^{-4\beta t} nt < \infty\,. \end{aligned}$$

Therefore, the expected value of $\int_0^{s\wedge T_n} e^{-2\beta u} g(X_u)\, dB_u$ is zero and equation (2) reduces to

$$I\!\!E\left[e^{-2\beta(s\wedge T_n)} X_{s\wedge T_n}\right] = X_0 + I\!\!E\left[\int_0^{s\wedge T_n} \delta_u e^{-2\beta u}\, du\right].$$

Taking the limit for $n$ going to infinity and applying Fatou's lemma, we obtain for all $s \leq t$:

$$\begin{aligned} I\!\!E[X_s] &\leq e^{2\beta s} X_0 + e^{2\beta s} I\!\!E\left[\int_0^s \delta_u e^{-2\beta u} du\right] \\ &\leq e^{2\beta s} X_0 + I\!\!E\left[\int_0^s \delta_u du\right] \\ &< \infty. \end{aligned}$$

We now show that $\sup_{t\geq s\geq 0} X_s$ is integrable. From the solution of the stochastic differential equation, it is known that

$$\begin{aligned} \sup_{s\leq t} X_s &\leq \sup_{s\leq t}\left(e^{2\beta s}\left(X_0 + \int_0^s \delta_u e^{-2\beta u} du + \int_0^s g(X_u)e^{-2\beta u} dB_u\right)\right) \\ &\leq X_0 + \int_0^t \delta_u\, du + \sup_{s\leq t}\left|\int_0^s g(X_u)e^{-2\beta u}\, dB_u\right|. \end{aligned}$$

Consequently,

$$\mathbb{E}\left[\sup_{s\le t} X_s\right] \le X_0 + \int_0^t \mathbb{E}[\delta_u]\,du + \left\|\sup_{s\le t}\int_0^s g(X_u)e^{-2\beta u}\,dB_u\right\|_2.$$

Using Doob's inequality, we remark that the last term is bounded.

$$\begin{aligned}
&\left\|\sup_{s\le t}\int_0^s g(X_u)e^{-2\beta u}dB_u\right\|_2^2 \\
&\quad\le\quad 4\left\|\int_0^t g(X_u)e^{-2\beta u}dB_u\right\|_2^2 \\
&\quad\le\quad 4e^{-4\beta t}\mathbb{E}\left[\int_0^t g(X_u)^2du\right] \\
&\quad\le\quad 4e^{-4\beta t}\mathbb{E}\left[\int_0^t b^2X_u\,du\right] \\
&\quad\le\quad 4e^{-4\beta t}b^2\int_0^t \mathbb{E}[X_u]\,du \\
&\quad<\quad \infty.
\end{aligned}$$

This shows that $(X_{s\wedge T_n})_{0\le s\le t, n\ge 1}$ is a uniformly integrable family and that we are allowed to interchange limits and expectations in the expression

$$\lim_{n\to\infty}\mathbb{E}\left[e^{-2\beta(s\wedge T_n)}X_{s\wedge T_n}\right] = \lim_{n\to\infty}\left(X_0 + \mathbb{E}\left[\int_0^{s\wedge T_n} e^{-2\beta u}\delta_u\right]\right).$$

We conclude that in the case of $\int_0^t \mathbb{E}[\delta_u]du < \infty$, the result is obtained:

$$\mathbb{E}[X_s] = e^{2\beta s}X_0 + \int_0^s e^{2\beta(s-u)}\mathbb{E}[\delta_u]du.$$

Let us now look at the general case with the local assumption $\int_0^t \delta_u du < \infty$ a.e. for all $t$. We define the sequence $(\sigma_n)_{n\ge1}$ by $\sigma_n = \inf\{t \mid \int_0^t \delta_u \ge n\}$ and we denote $\delta_u 1_{[\![0,\sigma_n]\!]}$ by $\delta_u^{(n)}$. The stochastic differential equation

$$dX_s^{(n)} = \left(2\beta X_s^{(n)} + \delta_s^{(n)}\right)ds + g(X_s^{(n)})dB_s$$

has a unique solution and by the definition of $\sigma_n$, we have $\int_0^t \mathbb{E}[\delta_u^{(n)}]du \le n$. Applying the first part of the proof, we obtain $\sup_{0\le s\le t} X_s^{(n)} < \infty$ a.e.. On $[\![0,\sigma_n]\!]$, all $X^{(k)}, k\ge n$ are equal by the uniqueness of the solution of the stochastic differential equation. Since $\bigcup[\![0,\sigma_n]\!] \supset [\![0,t]\!]$, the result holds under the local assumption $\int_0^t \delta_u du < \infty$ a.e. for all $t$.

q.e.d.

Now, we are ready for the convergence theorem itself.

**Theorem 1**

*Suppose that a probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, I\!P)$ is given and that a Brownian motion $(B_t)_{t\geq 0}$ is defined on it. A stochastic process $X : \Omega \times I\!R^+ \to I\!R^+$ is assumed to satisfy the stochastic differential equation*

$$dX_s = (2\beta X_s + \delta_s)ds + g(X_s)dB_s \quad \forall s \in I\!R^+$$

*with $\beta < 0$ and $g : I\!R \to I\!R^+$ a function, vanishing at zero and such that there is a constant $b$ with $|g(x) - g(y)| \leq b\sqrt{|x-y|}$.*
*The measurable and adapted process $\delta : \Omega \times I\!R^+ \to I\!R^+$ is assumed to satisfy:*

$$\frac{1}{s}\int_0^s \delta_u du \overset{a.e.}{\to} \overline{\delta} \text{ with } \overline{\delta} : \Omega \to I\!R^+.$$

*Under these conditions, the following convergence almost everywhere holds*

$$\frac{1}{s}\int_0^s X_u du \overset{a.e.}{\longrightarrow} \frac{-\overline{\delta}}{2\beta}.$$

Proof
Integrating the stochastic differential equation

$$dX_s = (2\beta X_s + \delta_s)ds + g(X_s)dB_s \quad \forall s \in I\!R^+$$

over the time-interval [0,t] and dividing this integral by $2\beta(t+1)$, gives us the equality:

$$\frac{1}{t+1}\int_0^t (X_s + \frac{\delta_s}{2\beta})ds = -\int_0^t \frac{g(X_s)}{2\beta(t+1)}dB_s + \frac{X_t - X_0}{2\beta(t+1)}. \tag{3}$$

It remains to prove that both terms on the right hand side converge to zero almost everywhere.

In order to show that $\int_0^t \frac{g(X_s)}{2\beta(t+1)}dB_s$ converges to zero a.e., we use Kronecker's lemma and check the existence a.e. of $\int_0^\infty \frac{g(X_u)}{u+1}dB_u$.
Let us introduce the sequence $(T_n)_{n\geq 1}$ of stopping times:

$$T_n = \inf\left\{t \mid \int_0^t \frac{\delta_u}{(u+1)^2}du \geq n\right\}.$$

Since by hypothesis $\frac{1}{s+1}\int_0^s \delta_u du \overset{a.e.}{\to} \overline{\delta}$, we obtain that $\int_0^u \delta_s ds \leq K(u+1)$ a.e. for some constant $K$, depending on $\omega$. Straightforward calculations show that

$\int_0^\infty \frac{\delta_u}{(u+1)^2} du < \infty$ a.e.:

$$\begin{aligned}
\int_0^\infty \frac{\delta_u}{(u+1)^2} du &= \left.\frac{\int_0^u \delta_s ds}{(u+1)^2}\right|_0^\infty + 2\int_0^\infty \left(\int_0^u \delta_s ds\right) \frac{du}{(u+1)^3} \\
&\leq \lim_{u\to\infty} \frac{\int_0^u \delta_s ds}{(u+1)^2} - 0 + 2\int_0^\infty \frac{K(u+1)}{(u+1)^3} du \\
&< \infty.
\end{aligned}$$

Hence, $\{T_n = \infty\} \uparrow \Omega$ and consequently, we only need to prove the existence a.e. of $\int_0^\infty \frac{g(X_u^{T_n})}{u+1} dB_u$ on $\{T_n = \infty\}$.

Moreover, since $\int_0^\infty \frac{g(X_u^{T_n})}{u+1} dB_u$ is a local martingale, it suffices to remark that $\int_0^t \frac{g(X_u^{T_n})}{u+1} dB_u$ is a $L^2$-bounded martingale:

$$\begin{aligned}
&\left\| \int_0^t \frac{g(X_u^{T_n})}{u+1} dB_u \right\|_2^2 \\
&= \int_0^t I\!E\left[g^2(X_u^{T_n})\right] \frac{1}{(u+1)^2} du \\
&\leq \int_0^t I\!E\left[X_u^{T_n}\right] \frac{b^2}{(u+1)^2} du.
\end{aligned}$$

In order to evaluate this last integral, we remark that

$$\begin{aligned}
I\!E\left[X_u^{T_n}\right] &= I\!E\left[X_u 1_{(u\leq T_n)}\right] \\
&\leq e^{2\beta u} I\!E\left[e^{-2\beta u} X_u 1_{(u\leq T_n)}\right] \\
&\leq e^{2\beta u} I\!E\left[e^{-2\beta(u\wedge T_n)} X_{u\wedge T_n}\right].
\end{aligned}$$

In lemma 1, we obtained the equality

$$I\!E[e^{-2\beta(s\wedge T_n)} X_{s\wedge T_n}] = X_0 + I\!E\left[\int_0^{s\wedge T_n} e^{-2\beta u}\delta_u du\right].$$

Consequently:

$$\begin{aligned}
I\!E\left[X_u^{T_n}\right] &\leq e^{2\beta u}\left(X_0 + I\!E\left[\int_0^{u\wedge T_n} e^{-2\beta s}\delta_s ds\right]\right) \\
&\leq e^{2\beta u} X_0 + e^{2\beta u}\int_0^u e^{-2\beta s} I\!E\left[\delta_s 1_{(s\leq T_n)}\right] ds.
\end{aligned}$$

Using this result, we obtain

$$\int_0^t I\!E\left[X_u^{T_n}\right] \frac{1}{(u+1)^2} du$$

$$
\begin{aligned}
&\leq \int_0^t X_0 \frac{e^{2\beta u}}{(u+1)^2} du \\
&\quad + \int_0^t \frac{e^{2\beta u}}{(u+1)^2} du \int_0^u e^{-2\beta s} \mathbb{E}\left[\delta_s 1_{(s \leq T_n)}\right] ds.
\end{aligned}
$$

Obviously, the first term is uniformly bounded in $t$.
It remains to look at the second term. We apply Fubini's theorem to find a bound which is not depending on $t$:

$$
\begin{aligned}
&\int_0^t \frac{e^{2\beta u}}{(u+1)^2} du \int_0^u e^{-2\beta s} \mathbb{E}\left[\delta_s 1_{(s \leq T_n)}\right] ds \\
&\quad = \int_0^t e^{-2\beta s} \mathbb{E}\left[\delta_s 1_{(s \leq T_n)}\right] ds \int_s^t \frac{e^{2\beta u}}{(u+1)^2} du \\
&\quad \leq \int_0^t \mathbb{E}\left[\delta_s 1_{(s \leq T_n)}\right] \frac{1}{(s+1)^2} \left(\frac{-1}{2\beta}\right) ds \\
&\quad \leq \frac{-1}{2\beta} \mathbb{E}\left[\int_0^{T_n} \frac{\delta_s}{(s+1)^2} ds\right] \\
&\quad \leq \frac{-n}{2\beta}.
\end{aligned}
$$

In order to show that the second term of (3), namely $\frac{X_t - X_0}{2\beta(t+1)}$, converges to zero a.e., we divide the solution of the stochastic differential equation (1) by $t+1$:

$$
\frac{X_t}{t+1} = \frac{e^{2\beta t} X_0}{t+1} + \int_0^t \frac{e^{2\beta(t-u)}}{t+1} \delta_u du + \int_0^t \frac{e^{2\beta(t-u)}}{t+1} g(X_u) dB_u. \tag{4}
$$

Under the hypothesis $\frac{1}{s+1}\int_0^s \delta_u du \overset{a.e.}{\to} \overline{\delta}$, the second term in (4) can be made arbitrarily small, since for all $\omega \in \Omega$ and for all $\varepsilon > 0$ :

$$
\begin{aligned}
&\frac{1}{t+1}\int_0^t e^{2\beta(t-u)} \delta(\omega, u) du \\
&\quad \leq \frac{1}{t+1}\int_0^{t-\sqrt{t}} e^{2\beta\sqrt{t}} \delta(\omega, u) du + \frac{1}{t+1}\int_{t-\sqrt{t}}^t \delta(\omega, u) du.
\end{aligned}
$$

Given $\varepsilon > 0$, the first term can be rewritten

$$
\begin{aligned}
&\frac{1}{t+1}\int_0^{t-\sqrt{t}} e^{2\beta\sqrt{t}} \delta(\omega, u) du \\
&\quad \leq \frac{1}{t+1-\sqrt{t}}\int_0^{t-\sqrt{t}} e^{2\beta\sqrt{t}} \delta(\omega, u) du \\
&\quad \leq \overline{\delta}(1+\varepsilon) e^{2\beta\sqrt{t}}.
\end{aligned}
$$

Consequently, the first term converges to zero for $t$ going to infinity.
The second term also tends to zero since

$$\begin{aligned}
&\lim_{t\to\infty} \frac{1}{t+1} \int_{t-\sqrt{t}}^{t} \delta(\omega,u)du \\
&= \lim_{t\to\infty} \left( \frac{1}{t+1} \int_0^t \delta(\omega,u)du - \frac{1}{t+1-\sqrt{t}} \int_0^{t-\sqrt{t}} \delta(\omega,u)du \right) \\
&\quad + \lim_{t\to\infty} \left( \frac{-1}{t+1} + \frac{1}{t+1-\sqrt{t}} \right) \int_0^{t-\sqrt{t}} \delta(\omega,u)du \\
&= \overline{\delta} - \overline{\delta} + \lim_{t\to\infty} \frac{\sqrt{t}}{t+1} \frac{1}{t+1-\sqrt{t}} \int_0^{t-\sqrt{t}} \delta(\omega,u)du \\
&= 0.
\end{aligned}$$

In order to check the convergence a.e. of $\frac{e^{2\beta t}}{t+1} \int_0^t e^{-2\beta u} g(X_u)dB_u$ to zero, we again use Kronecker's lemma and we look at the existence of $\int_0^\infty \frac{e^{-2\beta t} g(X_t)}{(t+1)e^{-2\beta t}} dB_t$. However, since this integral is equal to $\int_0^\infty \frac{g(X_t)}{t+1} dB_t$, the result follows from the calculations above.

q.e.d.

This theorem can be generalized to stochastic processes $X$ with a time-dependent strictly negative drift rate $\beta$. Let us define the process $X$ by the stochastic differential equation

$$dX_s = (2\beta_s X_s + \delta_s)ds + g(X_s)dB_s \quad \text{for all} \quad s \in I\!R^+$$

where the function $g$ and the process $\delta$ satisfy the hypothesis of theorem 1; and where $\sup_s \beta_s < 0$ and $\int_0^\infty dv I\!E[\delta_v] e^{-\int_0^v 2\beta_s ds} \int_v^\infty \frac{e^{\int_0^u 2\beta_s ds}}{(u+1)^2} du < \infty$. Then, the following convergence almost everywhere holds:

$$\frac{1}{s} \int_0^s \left( X_u + \frac{\delta_u}{2\beta_u} \right) du \xrightarrow{a.e.} 0.$$

## 3 A two-factor CIR model.

In this section, we give an example of theorem 1. We study the two-factormodel

$$dr_t = \kappa(\gamma_t - r_t)dt + \sigma\sqrt{r_t}dB_t$$
$$d\gamma_t = \tilde{\kappa}(\gamma^* - \gamma_t)dt + \tilde{\sigma}\sqrt{\gamma_t}d\tilde{B}_t$$

with $\kappa, \tilde{\kappa} > 0$; $\gamma^*, \sigma$ and $\tilde{\sigma}$ positive constants and $(B_t)_{t\geq 0}$ and $(\tilde{B}_t)_{t\geq 0}$ two Brownian motions. The short interest rate $r$ follows an extended Cox, Ingersoll and Ross square root process with reversion level $(\gamma_t)_{t\geq 0}$, which follows a square root process itself. We know that the time-dependent reversion level $(\gamma_t)_{t\geq 0}$ is itself elastically pulled towards the long-term constant value $\gamma^*$. We are interested in the convergence of the long term return $\frac{1}{t}\int_0^t r_u du$.

Remark that we do not make any assumption about the way the two Wiener processes are correlated. In contrast with most authors, we do not demand the Brownian motions to be independent, they may have an arbitrary random correlation.

Before looking at the convergence almost everywhere of the long-term return $\frac{1}{s}\int_0^s r_u du$ itself, we first use theorem 1 to check that indeed

$$\frac{1}{s}\int_0^s \gamma_u du \overset{a.e.}{\to} \gamma^*.$$

If we define $Y_u = \frac{4}{\tilde{\sigma}^2}\gamma_u$, then $Y_u$ satisfies the stochastic differential equation :

$$dY_u = \left(\frac{4\tilde{\kappa}\gamma^*}{\tilde{\sigma}^2} + 2\left(\frac{-\tilde{\kappa}}{2}\right)Y_u\right)du + 2\sqrt{Y_u}d\tilde{B}_u.$$

Thus, $(Y_u)_{u\geq 0}$ is a Besselsquare process with drift, namely in the notation of Pitman-Yor :

$$Y \sim {}^{-\tilde{\kappa}/2}\mathbb{Q}^{\frac{4\tilde{\kappa}\gamma^*}{\tilde{\sigma}^2}}_{\frac{4\gamma_0}{\tilde{\sigma}^2}}.$$

In general, $dY_u = \left(\delta_u + 2\tilde{\beta}Y_u\right)du + \tilde{g}\left(Y_u\right)d\tilde{B}_u$
with

- $\tilde{\beta} = \frac{-\tilde{\kappa}}{2} < 0$
- $\delta_u = \frac{4\tilde{\kappa}\gamma^*}{\tilde{\sigma}^2} \quad \forall u \in \mathbb{R}_+$
- $\tilde{g}\left(Y_u\right) = 2\sqrt{Y_u}$.

Trivially, the conditions of theorem 1 are satisfied and it follows that

$$\frac{1}{s}\int_0^s Y_u du \overset{a.e.}{\to} \frac{4\gamma^*}{\tilde{\sigma}^2}$$

and consequently:

$$\frac{1}{s}\int_0^s \gamma_u du \overset{a.e.}{\to} \gamma^*.$$

Analogously, we consider the instantaneous interest rate $r_u$ itself. The transformation $X_u = \frac{4}{\sigma^2} r_u$ satisfies the stochastic differential equation

$$dX_u = \left(\frac{4\kappa\gamma_u}{\sigma^2} + 2\left(\frac{-\kappa}{2}\right) X_u\right) du + 2\sqrt{X_u} dB_u.$$

In terms of theorem 1:

$$dX_u = (\delta_u + 2\beta X_u)\, du + g\,(X_u)\, dB_u$$

with

- $\beta = \frac{-\kappa}{2} < 0$
- $\delta_u = \frac{4\kappa\gamma_u}{\sigma^2} \qquad \forall u \in I\!R_+$
- $g\,(X_u) = 2\sqrt{X_u}$.

Since $\delta$ is a transformation of the continuous, adapted process $\gamma$, $\delta$ itself is measurable and adapted. Because $\frac{1}{s}\int_0^s \gamma_u du \overset{a.e.}{\to} \gamma^*$, we know that

$$\frac{1}{s}\int_0^s \delta_u du = \left(\frac{1}{s}\int_0^s \gamma_u du\right)\frac{4\kappa}{\sigma^2} \quad \xrightarrow{a.e.} \quad \gamma^* \frac{4\kappa}{\sigma^2}.$$

Therefore, the conditions of theorem 1 are fulfilled and we find that

$$\frac{1}{s}\int_0^s X_u du \overset{a.e.}{\to} \frac{4\gamma^*}{\sigma^2}$$

and finally that

$$\frac{1}{s}\int_0^s r_u du \overset{a.e.}{\to} \gamma^*.$$

We conclude that the long-term return converges a.e. to $\gamma^*$, the long-term constant value towards which the drift rate is pulled in this two-factormodel.

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