REMARKS ON THE METHODOLOGY INTRODUCED BY GOOVAERTS ET AL.

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Abstract: In "A stochastic approach to insurance cycles", M.J. Goovaerts, R. Kaas and E. De Vylder use path integrals. Our paper presents some remarks on this methodology. We will show how the path integral models are related to stochastic differential equations in order to outline a method to evaluate the distribution of a random variable $Z\left(n, (x_t)_{o \le t \le n}\right)$ for a fixed time n, where $(x_t)_{t \varepsilon o}$ denotes a Gaussian stochastic process. An explicit expression for the distribution of the discount function $\exp -\int_0^n dt(\delta + x_t)$ follows immediately from the theory of Gaussian processes and its relation with stochastic differential equations.

Keywords: Stochastic processes, Lagrange function, path integrals, stochastic differential equations, infinitesimal generators and semi-groups.

Section 1 Introduction.

We would like to outline a method to evaluate the distribution of a random variable $Z\left(n, (x_t)_{o \le t \le n}\right)$ for a fixed time n, where $(x_t)_{t \varepsilon o}$ denotes a Gaussian stochastic process. The most general form of a Lagrange function which gives rise to a Gaussian path integral is a polynomial of at most second degree in x and $\dot{x}$ or

$$L_G = a\,\dot{x}^2 + 2b\,x\,\dot{x} + c\,x^2 + 2d\,\dot{x} + 2e\,x + f.$$

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In "A stochastic approach to insurance cycles" ([1]), M.J. Goovaerts, R. Kaas and E. De Vylder use path integrals. Following the underlying idea of that paper, we could examine the following functional integral :

$$\int_{-\infty}^{\infty} f(x_n)dx_n \int_{(0,0)}^{(x_n,n)} \exp\left(- \int_0^n L_G(x, \dot{x}, t)dt - \gamma\, Z\left(n, (x_t)_{o\le t\le n}\right)\right) Dx(t)$$

with f the density function of x_n.

This expression describes the generating function of the random variable $Z\left(n, (x_t)_{o\le t\le n}\right)$. This functional integral is not always easy to compute because it is possible that $E\left[e^{-\gamma Z}\right]$ is not analytic around $\gamma = 0$. This is the case for the discount function

$$Z_1\left(n, (x_t)_{o\le t\le n}\right) = \exp - \int_0^n dt(\delta + x_t)$$

and for the present value of future payments

$$Z_2\left(n, (B_t)_{o\le t\le n}\right) = \int_0^n dt \exp(at + bB_t)$$

where $(B_s, s \varepsilon 0)$ denotes a real valued Brownian motion starting from 0.

Because of the computational difficulties, we will avoid path integrals. In this paper, we will show how the path integral models are related to stochastic differential equations. Stochastic differential equations have the advantage that they show how individual paths behave. The distribution of the discount function

$$Z_1\left(n, (x_t)_{o\le t\le n}\right) = \exp - \int_0^n dt(\delta + x_t)$$

follows immediately from the theory of Gaussian processes and its relation with stochastic differential equations.

The distribution of the present value

$$Z_2\left(n, (B_t)_{o\le t\le n}\right) = \int_0^n dt \exp(at + bB_t)$$

is computed by Yor in "On some exponential functionals of Brownian motion". With the help of computations for Bessel processes, he obtained an explicit expression for the density, which is nonetheless complicated.

In the second section, we start with the analysis of the definition of a path integral. From the Lagrange function, we will derive the stochastic differential equation and potential. We refer to [1] for the details. Some applications of the derivation in this section will be given in section three and four.

In the third section, we will give the distribution of the discounting function $Z_1\left(n, (x_t)_{o\leq t\leq n}\right) = \exp - \int_0^n dt(\delta + x_t)$.

In the fourth section we intend to evaluate the distribution of a function $Z\left(n, (x_t)_{o\leq t\leq n}\right) = \int_0^n A\left(x_s\right)ds$, with $A \in C^2$. Knowing the stochastic differential equation, we find the infinitesimal operator and the "Carré du Champ" operator. We recall that a change of drift is related with the Feynman-Kac formula. We notice that the definition of a path integral is linked with the definition of a semi-group via the Trotter-Kato formula. So one can work with semi-groups instead of path integrals.
We make use of the idea of Dynkin's theorem to outline a method to evaluate the distribution of the function

$$Z\left(n, (x_t)_{o\leq t\leq n}\right) = \int_0^n A\left(x_s\right)ds .$$

Section 2 Path integrals and stochastic differential equations.

If x + y denotes the place at time t + ε, then x could be replaced by x+y/2 and $\dot{x}$ by y/ε. Since the path integral uses normalised forms, we multiply with ε. This represents integration over a time interval of length ε (see [1]).

$$\varepsilon\left(f+\frac{2d\,y}{\varepsilon}+\frac{a\,y^2}{\varepsilon^2}+e(2\,x+y)+\frac{b\,y(2\,x+y)}{\varepsilon}+\frac{c\,(2\,x+y)^2}{4}\right)$$

To find the drift, we interpret, as in the theory of path integrals, the above expression as the exponent in the density function of a normal variable. The mean will be found as the expression μ, where μ is chosen in such a way that after the substitution of y by z + μ, the coefficient of z is zero.

$$\mu = -\frac{4d\varepsilon + 2e\varepsilon^2 + 4b\varepsilon\,x + 2c\varepsilon^2\,x}{4a + 4b\varepsilon + c\varepsilon^2}$$

The drift of the motion is obtained by dividing μ by ε and by taking the limit of ε to 0. This gives

$$\text{drift} = -\frac{d}{a} - \frac{b\,x}{a}$$

After the substitution z = y - μ, we found an expression with one term in z^2 and terms of degree zero in z :

$$\left(b+\frac{a}{\varepsilon}+\frac{c\varepsilon}{4}\right)z^2 + \text{ terms of degree zero in z}$$

The coefficient of z^2 is related to the diffusion coefficient. In a time interval of length ε the particle moves as a Brownian motion multiplied by the diffusion coefficient. We find as diffusion coefficient

$$\sigma = \frac{1}{\sqrt{a}}$$

The stochastic differential equation is therefore known :

$$dx_t = -\left(\frac{d + bx_t}{a}\right) dt + \frac{1}{\sqrt{a}} \; dB_t$$

This defines a Markov process.
The potential is found as the term of degree zero in z. However we have to divide by ε because it will afterwards appear as a time integral. We then take the limit for ε to zero.

$$-\frac{d^2}{a} + f + \left(\frac{-2bd}{a} + 2e\right) x + \left(\frac{-b^2}{a} + c\right) x^2$$

The potential is a quadratic function of x alone. It turns out that the Lagrange function can be decomposed in a pure square containing the values $\dot{x}$ and the potential.

$$\frac{(d + b\,x + a\,\dot{x})^2}{a} + \left(-\frac{d^2}{a} + f + \left(\frac{-2bd}{a} + 2e\right) x + \left(\frac{-b^2}{a} + c\right) x^2\right)$$

Section 3 The distribution of a discount function.

Let x_t be determined by its stochastic differential equation

$$dx_t = -\left(\frac{d + bx_t}{a}\right) dt + \frac{1}{\sqrt{a}} \; dB_t$$

or, in order to simplify the notation, by

$$dx_t = (\alpha + \beta x_t)\, dt + \sigma\, dB_t$$

with $\alpha = -d/a$, $\beta = -\, b/a$ and $\sigma = 1/\sqrt{a}$
Then, the discounting function

$$\exp - \int_0^n ds(\delta + x_s)$$

has a lognormal distribution since

$$- \int_0^n ds(\delta + x_s) \overset{d}{=} N(-m\text{-}\delta n, \text{Var})$$

The mean -m-δn and the variation Var follow from the solution of the stochastic differential equation

$$x_t = x_o \exp\left(\int_0^t \beta(v)dv\right) + \int_0^t \alpha(u) \exp\left(\int_u^t \beta(v)dv\right) du$$

$$+ \int_0^t \sigma(u) \exp\left(\int_u^t \beta(v)dv\right) dB_u$$

Namely :

$$-m - \delta n = - \int_0^n \left(x_o \exp\left(\int_0^s \beta(v)dv \right) + \int_0^s \alpha(u) \exp\left(\int_u^s \beta(v)dv \right) du \right) ds - \delta n$$

$$\mathrm{Var} = \int_0^n \int_0^n \exp\left(\int_0^t \beta(v)dv \right) \exp\left(\int_0^s \beta(v)dv \right) \int_0^{t \Box s} \sigma^2(u) \exp\left(-2 \int_0^u \beta(v)dv \right) du \; ds \, dt$$

Section 4 General function $Z\left(n, (x_t)_{o\le t\le n}\right) = \int_0^n A\left(x_s\right)ds.$

From the general stochastic differential equation

$$dx_t = (\alpha + \beta x_t)\, dt + \sigma\, dB_t$$

it follows that the infinitesimal operator of the process is given by

$$L\phi = \frac{1}{2}\ \sigma\, \phi\,'' \ + \ (\alpha + \beta x)\, \phi\,'$$

and the "Carré du Champ" operator by $\Gamma\,(\phi,\phi) = \sigma^2\,(\phi')^2$.

The formula of Feynman-Kac says that

$$E_{x_o}\left[f(x_t)\ \exp\left(-\int_0^t A(x_u)du \right)\right]$$

defines a process with generator $\widetilde{L}\, f = Lf - Af$, for A □ C^2.

The theorem of Trotter-Kato gives a relation between infinitesimal generators and semi-groups. If a sequence of infinitesimal generators converge, the corresponding semi-groups converge to the semi-group which corresponds with the limit of the infinitesimal operators. Let us assume in general that $(x_s,\ s\ \varepsilon\ 0)$ is a Gaussian process with drift μ and diffusion coefficient σ and that A □ C^2. If we make the discrete approximation of the integral

$$\int_0^t A(x_u)du$$

by dividing the interval [0, t] in n equal parts, we obtain

$$\sum_{k=0}^{n-1} A(x_{kt/n})\,\frac{t}{n}$$

Then, if we take the limit of n to infinity, we find that

$$E_{x_o}\left[f(x_t)\, \exp\left(-\int_0^t A(x_u)du \right) \right] =$$

$$= \lim_{n \to \infty} \frac{1}{(\sqrt{2\pi\varepsilon}\,\sigma)^n} \int_{-\infty}^{\infty} dx_1 \exp\left(-\frac{(x_1 - x_o - \mu\varepsilon)^2}{2\sigma^2\varepsilon} \right) e^{-A(x)\varepsilon}$$

$$\cdot \int_{-\infty}^{\infty} dx_2 \exp\left(-\frac{(x_2 - x_1 - \mu\varepsilon)^2}{2\sigma^2\varepsilon} \right) e^{-A(x_1)\varepsilon}$$

$$\cdot \ldots \cdot \int_{-\infty}^{\infty} dx_n \exp\left(-\frac{(x_n - x_{n-1} - \mu\varepsilon)^2}{2\sigma^2\varepsilon} \right) e^{-A(x_{n-1})\varepsilon} f(x_n)$$

with $\varepsilon = \frac{t}{n}$

Because the infinitesimal generators (for each n) of the discrete approximation process converge to the infinitesimal generator
$Lf - Af$, the semi-groups (for each n) of the discrete approximation process converge to the expression $E_{x_o}\left[f(x_t)\, \exp\left(-\int_0^t A(x_u)du \right) \right]$.
Thus, one can use semi-groups instead of path integrals.

We recall that by the theorem of Dynkin,

$$\exp\left(\phi(x_t) - \phi(x_0) - \int_0^t \left[L\phi(x_s) + \frac{1}{2}\Gamma(\phi,\phi)(x_s) \right] ds \right)$$

is a martingale. We now would like to express $A(x_t)$ in terms of functions $\phi(x_t)$ so that

$$\exp\left(\phi(x_t) - \phi(x_o) - \int_0^t A(x_u)\, du\right)$$

has expectation one, conditional to x_t.
First we need the distribution of $(x_s)_{0\le s\le t}$ conditional to x_t.
We find as the stochastic differential equation for $(x_s)_{0\le s\le t}$ conditional to x_t

$$dx_s = (\bar{\alpha} + \bar{\beta}\; x_s)\, ds + \sigma\, d\gamma_s$$

with

$$\bar{\alpha} = \alpha + \frac{\sigma^2 \exp\left(\int_s^t \beta(v)dv\right)}{\int_s^t \sigma^2 \exp\left(2\int_v^t \beta(q)dq\right) dv}\left(x_t - \mu(t) + \mu(s)\exp\left(\int_s^t \beta(q)dq\right)\right)$$

$$\bar{\beta} = \beta - \frac{\sigma^2 \exp\left(\int_s^t \beta(v)dv\right)}{\int_s^t \sigma^2 \exp\left(2\int_v^t \beta(q)dq\right) dv}$$

$$\gamma_s = B_s - \int_0^s \frac{\sigma(u)\exp\left(\int_u^t \beta(v)dv\right) du}{\int_u^t \sigma^2(v)\exp\left(2\int_v^t \beta(q)dq\right) dv}\left(\int_u^t \sigma(v)\exp\left(\int_v^t \beta(q)dq\right) dB_v\right)$$

Unfortunately it is not always easy to express $A(x_t)$ in terms of functions $\phi(x_t)$.

REFERENCES

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[2] Yor, M. (1992). On some exponential functionals of Brownian motion. *Advances in Applied Probability* 24, no. 3, 509-532.