Static Super-Replicating Strategies for a Class of Exotic Options

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Abstract

In this paper, we investigate static super-replicating strategies for European-type call options written on a weighted sum of asset prices. This class of exotic options includes Asian options and basket options among others. We assume that there exists a market where the plain vanilla options on the different assets are traded and hence their prices can be observed in the market. Both the infinite market case (where prices of the plain vanilla options are available for all strikes) and the finite market case (where only a finite number of plain vanilla option prices are observed) are considered. We prove that the finite market case converges to the infinite market case when the number of observed plain vanilla option prices tends to infinity.

We show how to construct a portfolio consisting of the plain vanilla options on the different assets, whose pay-off super-replicates the pay-off of the exotic option. As a consequence, the price of the super-replicating portfolio is an upper bound for the price of the exotic option. The super-hedging strategy is model-free in the sense that it is expressed in terms of the *observed* option prices on the individual assets, which can be e.g. dividend paying stocks with no explicit dividend process known. This paper is a generalization of the work of Simon et al. (2000) who considered this problem for Asian options in the infinite market case. Laurence and Wang (2004) and Hobson et al. (2005) considered this problem for basket options, in the infinite as well as in the finite market case.

As opposed to Hobson et al. (2005) who use Lagrange optimization techniques, the proofs in this paper are based on the theory of integral stochastic orders and on the theory of comonotonic risks.

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1 Introduction

In this paper, we investigate static super-replicating strategies for European-type call options with a pay-off at future expiration date T given by $(w_1X_1 + \cdots + w_nX_n - K)_+$, where for each i in $\{1, \ldots, n\}$, the notation X_i is used for the positive price at a future time T_i, $0 \leq T_i \leq T$, of some underlying i and where w_i is the corresponding positive weight factor. Further, K is the exercise price of the exotic option at the maturity date and $(x)_+$ is a notation for $\max\{x, 0\}$. The notation $\mathbb{S}$ is used to indicate the weighted sum of asset prices:

$$\mathbb{S} = w_1X_1 + \cdots + w_nX_n. \tag{1}$$

We assume that there exists a market where European call options on the different assets are traded. To be more specific, we assume that for each i, $i = 1, \ldots, n$, the current time-0 prices $C_i[K]$ of the options with pay-off $(X_i - K)_+$ at expiration date T_i are known for a (finite or infinite) number of K-values.

The only assumptions that we make about the pricing process is that there are no arbitrage opportunities and that the market prices of all vanilla options involved are given by discounted expectations under some (unknown) probability measure Q. And under that Q, all discounted gain processes are martingales, with a gain process being the sum of the processes of the discounted prices and the accumulated discounted dividends. We concentrate upon the gain process since the underlyings will usually have dividends to be taken into account in the case of some rather long-term exotic options, like e.g. Asian options (see below at (6)). The current time-0 prices of the options available in the market are given by

$$C_i[K] = e^{-\delta T_i}\mathrm{E}\left[(X_i - K)_+\right], \qquad i = 1, \ldots, n, \tag{2}$$

with δ the risk free interest rate, which is supposed to be constant.
In the remainder of this paper, expectations with respect to X_i as in (2) have to be understood as expectations under the Q-measure. Also, statements about the distribution of X_i have to be understood in terms of the Q-measure. We will not explicitly mention the Q in the notations. We will use the notations F_{X_i}, $i = 1, \ldots, n$, and $F_{\mathbb{S}}$ to denote the cumulative distribution functions (cdf) of X_i and $\mathbb{S}$ in the Q-world.

In Section 3, we assume that for each underlying X_i, the price $C_i[K]$ of the vanilla call option is known for any exercise price $K \geq 0$. We will call this case the *infinite market* case. Either all vanilla call option prices may be known because we assume a specific Q-measure, and we call this approach the 'model-based' approach. Or these prices may be known because there exists a market where all the prices $C_i[K]$ can be observed. In the latter case, the approach is called 'model-free' as it is based on the observed option prices.

As noticed by Breeden and Litzenberger (1978), knowing the call prices $C_i[K]$ for all strikes K is equivalent to knowing the full pricing distribution of the asset prices X_i at times T_i.

As opposed to the infinite market case, in Section 4 we will consider the more realistic situation of a *finite market*. Here we will assume that for each underlying X_i only a finite number of possible option prices are available.

Consider the European-type exotic option with pay-off at expiration time T given by

$$(\mathbb{S} - K)_+ . \tag{3}$$

Assume that this option is not available in the market so that no price for it can be observed. Let $C[K]$ be a 'fair price' a rational decision maker is willing to pay for this exotic option. With a 'fair price' we mean that this price does not exceed the price of any investment strategy consisting of buying a portfolio of available plain vanilla options with a pay-off that super-replicates the pay-off of the exotic option. It is our goal to derive the 'largest possible fair price' for this exotic option, given the available information from the market. This largest fair price is equal to the price of the cheapest super-replicating strategy for the exotic option which consists of buying a linear combination of the available plain vanilla options. In this sense, the largest fair price can be considered as a 'least upper bound' for all possible 'fair' prices, given the observed plain vanilla option prices $C_i[K]$.

The upper bound for $C[K]$ may also be useful in a model-based approach where the price of the exotic option is given by

$$C[K] = e^{-\delta T} \mathrm{E}\left[(\mathbb{S} - K)_+\right] \tag{4}$$

for some given Q-measure. Indeed, even in a Black & Scholes setting, this price is difficult to evaluate. In this case, the upper bound may serve as an approximation for the real price.

Another possible application is the case where the exotic option is available in the market so that its price can be observed. In that case, the upper bound can be derived in a model-free framework and the observed price can be compared to it in order to detect eventual arbitrage opportunities or model-error.

Examples of options with a pay-off of the form (3) are basket options and Asian options. For basket options, the X_i have to be interpreted as the prices $S_i(T)$ of n different assets i at the exercise date T:

$$X_i = S_i(T), \qquad i = 1, \ldots, n. \tag{5}$$

The weights w_i are the weights in the basket. So our goal in this case is to super-replicate the pay-off $\left(\sum_{i=1}^n w_i S_i(T) - K\right)_+$ of the basket option by a linear combination of the pay-offs $(S_i(T) - K_{i,j})_+$ of the available plain vanilla options with strikes $K_{i,j}$.

In case of Asian options, all assets i are identical. The X_i represent the prices $S(T - i + 1)$ of a fixed asset at different times $T_i \equiv (T - i + 1)$:

$$X_i = S(T - i + 1), \qquad i = 1, \ldots, n. \tag{6}$$

The weights w_i typically equal $\frac{1}{n}$ such that $\mathbb{S}$ is the average price of the asset over the last n periods prior to expiration. In this case, our goal is to super-replicate the pay-off $\left(\frac{1}{n}\sum_{i=1}^n S(T - i + 1) - K\right)_+$ of the Asian option by linear combination of the pay-offs $(S(T - i + 1) - K_{i,j})_+$.

Another application of our results concerns a pure unit-linked contract of duration $n = T$, where at each time $n - i$ a fixed amount P is used to buy units of the underlying asset. In this case, the value at time n of the investment of P at time $n - i$ is given by

$$X_i = P \frac{S(n)}{S(n-i)}, \qquad i = 1, \ldots, n, \tag{7}$$

where $S(n-i)$ represents the price of the underlying asset at time $n-i$. At time n, the value of the invested sum will equal $P \sum_{i=1}^{n} \frac{S(n)}{S(n-i)}$. Assume that this contract is sold with a maturity guarantee at time n which equals $\alpha n P$ with $0 < \alpha \leq 1$. This means that the pay-off at contract termination n is given by

$$\alpha n P + \left(\sum_{i=1}^{n} X_i - \alpha n P \right)_+, \tag{8}$$

which can also be written as

$$\sum_{i=1}^{n} X_i + \left(\alpha n P - \sum_{i=1}^{n} X_i \right)_+. \tag{9}$$

Upper bounds for the call option in (8) or the put option in (9) can be determined from the general results that we will derive in this paper. The problem of evaluating these types of options is considered in a forthcoming paper.

Concerning super-replicating strategies for exotic options, to the best of our knowledge, the optimal super-replicating strategy for the infinite market case applied to Asian options, was first presented in Simon et al. (2000), see also Dhaene et al. (2002b). The case of basket options, both in the infinite and the finite market case has been considered in Hobson et al. (2005). These authors use Lagrange optimization to characterize the optimal strikes in the upper bound that consists of a linear combination of the vanilla call prices. In a second step they show that the optimal strategy is attained for a special model where $\mathbb{S}$ is a comonotonic sum and where call prices are given by the discounted expected pay-offs. In this paper, we extend the results of Simon et al. (2000) and Hobson et al. (2005) to general exotic options with pay-off of the form (3), both in the infinite and the finite market case, regardless of dividend paying underlying assets or non-dividend paying underlying assets of the exotic options.

As compared to the methodology of Hobson et al. (2005) using Lagrange multipliers, our approach is more straightforward, the proofs and characterization of the optimal strikes being directly based on the properties of comonotonicity, a simple concept that describes an extreme form of dependency between the components of a random vector. This concept has received a lot of attention in the actuarial literature since the paper of Wang et al. (1998).

Moreover, our approach doesn't require a distinction between the behaviours — strictly increasing/non-decreasing, continuous/discontinuous — of the marginals; all cases are dealt with in a same way. We also prove directly that the upper bound represents an optimal super-replicating strategy without usage of a primal-dual formulation.

Hobson et al. (2005) also show how the results for the finite market case follow from the corresponding results of the infinite market case. At first sight, one may end up with the case where the upper bound consists of European call prices which are not observable in the market. However, these so-called unreachable European call prices can be expressed in terms of a convex combination which consists of proportions α and $(1-\alpha)$ of their neighbouring reachable call prices. We prove that these optimal proportions α and $(1-\alpha)$ are identical for all underlying assets. Additionally, we prove that also in the finite market case the upper bound represents an optimal super-replicating strategy in a much broader class of admissible strategies. We also prove the convergence of the upper bound in the finite market case to the one in the infinite market case when the number of strikes and hence the number of observed plain vanilla option prices for each underlying X_i tends to infinity.

The structure of the paper is as follows: In Section 2, we recall definitions and results on integral stochastic orders, inverse distributions and the concept of comonotonicity, which are of importance for the derivation of the results in this paper. In Section 3, we consider the infinite market case. We first prove an upper bound for any fair price of the exotic option with a pay-off of the form $\left(\sum_{i=1}^{n} w_i X_i - K\right)_+$. We show (directly) that this upper bound can be interpreted as a linear combination of European call option prices which corresponds to an optimal super-replicating strategy. Further, we prove that this upper bound can also be interpreted as a worst-case expectation of the pay-off of the exotic option in a certain Fréchet class of all multivariate pricing distributions with fixed marginal distributions. This section is finished by discussing some computational aspects. Section 4 deals with the finite market case. We explore a model-free approach to derive an upper bound for any fair price of the exotic option in a market where only finitely many strikes for the European call options are traded. Hereto we introduce random variables with a discrete distribution based on the traded European call options. Following the structure of Section 3 a first theorem contains the upper bound in terms of the traded European call options. Then we give a direct proof that this upper bound can be interpreted as the price of the cheapest super-replicating strategy as well as a worst case expectation. We discuss some important computational issues. Finally, we prove the convergence of the upper bound in the finite market case to the one in the infinite market. Section 5 concludes the paper. In the appendix, an algorithm concerning some computation in the finite market case is proposed.

2 Some definitions and results concerning comonotonicity

In this section, we recall some definitions and results concerning stochastic orders, inverse distributions and the concept of comonotonicity. These results will turn out to be essential for deriving the optimal super-replicating strategy for exotic options, both in the infinite and the finite market case.

First, we introduce the concepts of convex order and increasing convex order between (distributions of) random variables. When using these ordering concepts, it is always silently assumed that the random variables involved have finite means.

Definition 1 *A random variable X is said to precede a random variable Y in the increasing convex order sense, notation $X \leq_{\text{icx}} Y$, in case*

$$E\left[(X-d)_+\right] \leq E\left[(Y-d)_+\right], \quad \textit{for all } d. \tag{10}$$

In an actuarial context, the increasing convex order is called 'stop-loss order' because of its straightforward relation with stop-loss reinsurance, see e.g. Kaas et al. (2001).

Definition 2 *A random variable X is said to precede a random variable Y in the convex order sense, notation $X \leq_{\text{cx}} Y$, if the following conditions hold:*

$$\begin{aligned} E[X] &= E[Y], \\ E\left[(X-d)_+\right] &\leq E\left[(Y-d)_+\right], \quad \textit{for all } d. \end{aligned} \tag{11}$$

Other characterizations of these orders can be found e.g. in Shaked and Shanthikumar (1994) in a general context, or in Denuit et al. (2005) in an actuarial context.

The inverse of a cumulative distribution function is usually defined as follows:

Definition 3 *The inverse of the cumulative distribution function F_X of a random variable X is given by*

$$F_X^{-1}(p) = \inf\left\{x \in \mathbb{R} \mid F_X(x) \geq p\right\}, \quad p \in [0,1]. \tag{12}$$

However, for $p \in [0,1]$, a possible choice for the inverse of F_X in p is any point in the interval

$$\left[\inf\left\{x \in \mathbb{R} \mid F_X(x) \geq p\right\};\ \sup\left\{x \in \mathbb{R} \mid F_X(x) \leq p\right\}\right]. \tag{13}$$

Here we take $\inf \emptyset = +\infty$ and $\sup \emptyset = -\infty$. Taking the left hand border of this interval to be the value of the inverse cdf at p, leads to the inverse as defined in (12). Similarly, we define $F_X^{-1+}(p)$ as the right hand border of the interval:

Definition 4

$$F_X^{-1+}(p) = \sup\left\{x \in \mathbb{R} \mid F_X(x) \leq p\right\}, \quad p \in [0,1]. \tag{14}$$

Note that $F_X^{-1}(0) = -\infty$ and $F_X^{-1+}(1) = +\infty$, while $F_X^{-1}(p)$ and $F_X^{-1+}(p)$ are finite for all $p \in (0,1)$. We also have that $x \in \left(F_X^{-1+}(0), F_X^{-1}(1)\right)$ implies that $F_X(x) \in (0,1)$. Finally, we have that $\Pr\left[X \in \left[F_X^{-1+}(0), F_X^{-1}(1)\right]\right] = 1$.

Following Kaas et al. (2000), for any α in $[0,1]$, we define the α-inverse of F_X as follows:

Definition 5 *The α-inverse of the cumulative distribution function F_X of a random variable X is defined as the following convex combination of the inverses F_X^{-1} and F_X^{-1+} of F_X:*

$$F_X^{-1(\alpha)}(p) = \alpha F_X^{-1}(p) + (1-\alpha) F_X^{-1+}(p), \qquad p \in (0,1)\,,\ \alpha \in [0,1]. \tag{15}$$

Next, we define comonotonicity of a random vector.

Definition 6 *A random vector $(Y_1, \ldots, Y_n)$ with marginal cdf's $F_{Y_i}(x) = \Pr[Y_i \leq x]$ is said to be comonotonic if it has the same distribution as $\left(F_{Y_1}^{-1}(U), F_{Y_2}^{-1}(U), \ldots, F_{Y_n}^{-1}(U)\right)$, with U a random variable which is uniformly distributed on the unit interval $(0,1)$.*

In the sequel, the notation U will uniquely be used to denote a random variable which is uniformly distributed on the unit interval $(0,1)$. The components of the comonotonic random vector $\left(F_{Y_1}^{-1}(U), F_{Y_2}^{-1}(U), \ldots, F_{Y_n}^{-1}(U)\right)$ are maximally dependent in the sense that all of them are non-decreasing functions of the same random variable.

Consider a random vector $(Y_1, \ldots, Y_n)$. Its comonotonic counterpart $(Y_1^c, \ldots, Y_n^c)$ is a comonotonic random vector with the same marginal distributions:

$$(Y_1^c, \ldots, Y_n^c) \overset{d}{=} \left(F_{Y_1}^{-1}(U), F_{Y_2}^{-1}(U), \ldots, F_{Y_n}^{-1}(U)\right), \tag{16}$$

where the notation $\overset{d}{=}$ stands for 'equality in distribution'. The sum of the components of $(Y_1^c, \ldots, Y_n^c)$ is denoted by S^c,

$$S^c = Y_1^c + \cdots + Y_n^c. \tag{17}$$

The distribution function of S^c is completely specified when the marginals F_{Y_i} are given. The probabilities $F_{S^c}(x)$ follow from

$$F_{S^c}(x) = \sup\left\{p \in [0,1] \mid \sum_{i=1}^{n} F_{Y_i}^{-1}(p) \leq x\right\}. \tag{18}$$

Notice that

$$F_{S^c}^{-1+}(0) = \sum_{i=1}^{n} F_{Y_i}^{-1+}(0) \qquad \text{and} \qquad F_{S^c}^{-1}(1) = \sum_{i=1}^{n} F_{Y_i}^{-1}(1)\,. \tag{19}$$

If $x \in \left(F_{S^c}^{-1+}(0), F_{S^c}^{-1}(1)\right)$, then we have that $0 < F_{S^c}(x) < 1$. In case all Y_i are lognormal e.g., the condition $x \in \left(F_{S^c}^{-1+}(0), F_{S^c}^{-1}(1)\right)$ reduces to $x \in (0, +\infty)$.

The α-inverses of S^c, with $\alpha \in [0,1]$, can easily be obtained from the inverses of the marginals involved, as they fulfil the following additivity property:

$$F_{S^c}^{-1(\alpha)}(p) = \sum_{i=1}^{n} F_{Y_i}^{-1(\alpha)}(p), \qquad p \in (0,1)\,. \tag{20}$$

One can prove that for any $K \in \left(F_{S^c}^{-1+}(0), F_{S^c}^{-1}(1)\right)$, the following relation holds:

$$\mathrm{E}\left[(S^c - K)_+\right] = \sum_{i=1}^{n} \mathrm{E}\left[\left(Y_i - F_{Y_i}^{-1(\alpha)}\left(F_{S^c}(K)\right)\right)_+\right], \tag{21}$$

with $\alpha \in [0,1]$ such that

$$F_{S^c}^{-1(\alpha)}\left(F_{S^c}(K)\right) = K. \tag{22}$$

Making use of Definition 5 for the α-inverse, equation (22) can easily be solved for α when $F_{S^c}^{-1+}\left(F_{S^c}(K)\right) \neq F_{S^c}^{-1}\left(F_{S^c}(K)\right)$, namely:

$$\alpha = \frac{F_{S^c}^{-1+}\left(F_{S^c}(K)\right) - K}{F_{S^c}^{-1+}\left(F_{S^c}(K)\right) - F_{S^c}^{-1}\left(F_{S^c}(K)\right)}. \tag{23}$$

On the other hand, when $F_{S^c}^{-1+}\left(F_{S^c}(K)\right) = F_{S^c}^{-1}\left(F_{S^c}(K)\right)$, it is easily seen that (21) and (22) hold for any α in $[0,1]$. In the remainder, we will use for simplicity $\alpha = 1$ for this case.
For proofs and more details about (18), (20) and (21), we refer to the overview paper on comonotonicity by Dhaene et al. (2002a).

The expression (21) can also be written in terms of the usual inverse cdf's $F_{Y_i}^{-1}$. Indeed, for any $K \in \left(F_{S^c}^{-1+}(0), F_{S^c}^{-1}(1)\right)$, one has that

$$\mathrm{E}\left[(S^c - K)_+\right] = \sum_{i=1}^{n} \mathrm{E}\left[\left(Y_i - F_{Y_i}^{-1}\left(F_{S^c}(K)\right)\right)_+\right] - \left(K - F_{S^c}^{-1}\left(F_{S^c}(K)\right)\right)\left(1 - F_{S^c}(K)\right). \tag{24}$$

This expression was derived in Dhaene et al. (2000) and follows from using an integration by parts to rewrite each term of the form $\mathrm{E}\left[(Y_i - K_i)_+\right]$ in (21) as

$$\mathrm{E}\left[(Y_i - K_i)_+\right] = \int_{K_i}^{+\infty} \left(1 - F_{Y_i}(x)\right) dx, \tag{25}$$

and hence, by noticing that $\mathrm{E}\left[(Y_i - K_i)_+\right]$ can be interpreted as the surface above F_{Y_i}, from K_i until $+\infty$.

The following convex ordering relation holds for any random vector $(Y_1, \ldots, Y_n)$:

$$Y_1 + \cdots + Y_n \leq_{\mathrm{cx}} Y_1^c + \cdots + Y_n^c. \tag{26}$$

This ordering relation can already be found in Rüschendorf (1983). A proof for this inequality in the bivariate case can also be found in Wang et al. (1998), while a proof in terms of 'supermodular ordering' is given in Müller (1997) and a simple geometric proof is in Kaas et al. (2002). Related ordering results for the convex increasing ordering of sums have been stated in Meilijson & Nadas (1979).
The ordering relation (26) can be generalized as follows. Consider the random vectors $(X_1, \ldots, X_n)$ and $(Y_1, \ldots, Y_n)$. Then we have that

$$X_i \leq_{\mathrm{icx}} Y_i \text{ for } i = 1, \ldots, n \quad \Rightarrow \quad X_1 + \cdots + X_n \leq_{\mathrm{icx}} Y_1^c + \cdots + Y_n^c. \tag{27}$$

A proof for this result can be found in Dhaene et al. (2000).

3 A least upper bound for the price of the exotic option in case of full marginal information

3.1 Deriving the upper bound

In the remainder of the paper we will use the notations and conventions of Sections 1 and 2. We introduce the random variable $\mathbb{S}^c$ to indicate the 'comonotonic counterpart' of the random variable $\mathbb{S}$ which was defined in (1):

$$\mathbb{S}^c = w_1 F_{X_1}^{-1}(U) + w_2 F_{X_2}^{-1}(U) + \cdots + w_n F_{X_n}^{-1}(U). \tag{28}$$

Notice that (the distribution of) $\mathbb{S}^c$ is obtained from (the distribution of) $\mathbb{S}$ by keeping the marginals of the terms in the sum $\mathbb{S}$ but replacing the dependency structure between these terms by the comonotonic dependency structure.

The 'extreme' outcomes of $\mathbb{S}^c$ are, according to (19), given by

$$F_{\mathbb{S}^c}^{-1+}(0) = \sum_{i=1}^{n} w_i F_{X_i}^{-1+}(0) \quad \text{and} \quad F_{\mathbb{S}^c}^{-1}(1) = \sum_{i=1}^{n} w_i F_{X_i}^{-1}(1). \tag{29}$$

In this section, we consider the infinite market case, where it is assumed that for each i, $i = 1, \ldots, n$ and for each $K \geq 0$, the price of the option with pay-off $(X_i - K)_+$ at expiration date T_i is known. The current time-0 price of this option is denoted by $C_i[K]$ and is given by (2).
Knowledge of the prices $C_i[K]$ for all $K \geq 0$ is equivalent to knowledge of $\mathrm{E}\left[(X_i - K)_+\right]$ for all $K \geq 0$, which in turn is equivalent to knowing the cdf $F_{X_i}(x)$ for all x. Indeed, as the call prices are decreasing convex functions in K, the distribution function F_{X_i} of X_i is given by

$$F_{X_i}(x) = \Pr[X_i \leq x] = 1 + e^{\delta T_i} C_i'[x+], \tag{30}$$

where $C_i'[x+]$ is the right derivative in x. Notice that F_{X_i} is the cdf of X_i used for pricing purposes. This information enables us to determine the distribution function of $\mathbb{S}^c$ as well, as can be seen from (18). Note that the observable plain vanilla call prices $C_i[K]$ do not allow us to specify the multivariate pricing distribution $F_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n)$.

It is our goal to find an upper bound for any fair price $C[K]$ of the exotic option with pay-off $\left(\sum_{i=1}^n w_i X_i - K\right)_+$ which can be expressed in terms of the marginal information contained in the observed plain vanilla option prices.

In the following theorem, we derive such an upper bound in terms of the plain vanilla option prices $C_i[K]$ of the underlyings X_i, $i = 1, \ldots, n$. In the next section we will discuss the optimality of this upper bound which can be interpreted as the price of a super-replicating strategy.

Theorem 1 *Let us assume the infinite market as described above.*

(i) *For any* $K \in \left(F_{\mathbb{S}^c}^{-1+}(0), F_{\mathbb{S}^c}^{-1}(1)\right)$, *any fair price* $C[K]$ *of the exotic option with pay-off* $(\mathbb{S}-K)_+$ *at time* T *is constrained from above as follows:*

$$C[K] \leq e^{-\delta T} E\left[(\mathbb{S}^c - K)_+\right] \tag{31}$$

$$= \sum_{i=1}^{n} w_i e^{-\delta(T-T_i)} C_i \left[F_{X_i}^{-1(\alpha)}\left(F_{\mathbb{S}^c}(K)\right)\right] \tag{32}$$

with α *given by*

$$\alpha = \frac{F_{\mathbb{S}^c}^{-1+}\left(F_{\mathbb{S}^c}(K)\right) - K}{F_{\mathbb{S}^c}^{-1+}\left(F_{\mathbb{S}^c}(K)\right) - F_{\mathbb{S}^c}^{-1}\left(F_{\mathbb{S}^c}(K)\right)} \tag{33}$$

in case $F_{\mathbb{S}^c}^{-1+}\left(F_{\mathbb{S}^c}(K)\right) \neq F_{\mathbb{S}^c}^{-1}\left(F_{\mathbb{S}^c}(K)\right)$ *and* $\alpha = 1$ *otherwise.*

(ii) *For* $K \notin \left(F_{\mathbb{S}^c}^{-1+}(0), F_{\mathbb{S}^c}^{-1}(1)\right)$, *the exact exotic option price* $C[K]$ *is given by*

$$C[K] = \begin{cases} \sum_{i=1}^{n} w_i e^{-\delta(T-T_i)} C_i[0] - e^{-\delta T} K & \text{if } K \leq F_{\mathbb{S}^c}^{-1+}(0) \quad (34a) \\ 0 & \text{if } K \geq F_{\mathbb{S}^c}^{-1}(1). \quad (34b) \end{cases}$$

Proof.

(i) Applying (21) and (23), we can decompose $\mathrm{E}\left[(\mathbb{S}^c - K)_+\right]$ into

$$\mathrm{E}\left[(\mathbb{S}^c - K)_+\right] = \sum_{i=1}^{n} w_i \mathrm{E}\left[\left(X_i - F_{X_i}^{-1(\alpha)}\left(F_{\mathbb{S}^c}(K)\right)\right)_+\right] \tag{35}$$

with $\alpha \in [0,1]$ given by (33) in case $F_{\mathbb{S}^c}^{-1+}\left(F_{\mathbb{S}^c}(K)\right) \neq F_{\mathbb{S}^c}^{-1}\left(F_{\mathbb{S}^c}(K)\right)$ and $\alpha = 1$ otherwise. According to (22), α can also be determined from

$$F_{\mathbb{S}^c}^{-1(\alpha)}\left(F_{\mathbb{S}^c}(K)\right) = K, \tag{36}$$

or in view of (20), equivalently from

$$\sum_{i=1}^{n} w_i F_{X_i}^{-1(\alpha)}\left(F_{\mathbb{S}^c}(K)\right) = K. \tag{37}$$

Taking into account expression (2) for the European option prices $C_i[K]$, we further transform (35) into

$$e^{-\delta T}\mathrm{E}\left[(\mathbb{S}^c - K)_+\right] = \sum_{i=1}^{n} w_i e^{-\delta(T-T_i)} C_i \left[F_{X_i}^{-1(\alpha)}\left(F_{\mathbb{S}^c}(K)\right)\right]. \tag{38}$$

Hence, we have proven that the right-hand side of (32) is equal to (31).
Next, we use a no-arbitrage argument to prove that $e^{-\delta T}\mathrm{E}\left[(\mathbb{S}^c - K)_+\right]$ is an upper

bound for any fair price $C[K]$ of the exotic option with pay-off $(\mathbb{S}-K)_+$ at time T. For the $(\cdot)_+$-function it obviously holds that

$$\left(\sum_{i=1}^{n} w_i X_i - K\right)_+ \leq \sum_{i=1}^{n} w_i (X_i - K_i)_+ \tag{39}$$

for all decompositions $(K_1, K_2, \ldots, K_n)$ of K satisfying $\sum_{i=1}^{n} w_i K_i \leq K$ and $K_i \geq 0$, $i = 1, \ldots, n$. Here and in the sequel, stochastic inequalities such as the one in (39) have to be understood as holding for all outcomes $\omega \in \Omega$ of the measurable space $(\Omega, \mathcal{F})$ on which the random variables are defined.

In particular, relation (37) guarantees that the inequality (39) holds for the decomposition $\left(F_{X_1}^{-1(\alpha)}(F_{\mathbb{S}^c}(K)), F_{X_2}^{-1(\alpha)}(F_{\mathbb{S}^c}(K)), \ldots, F_{X_n}^{-1(\alpha)}(F_{\mathbb{S}^c}(K))\right)$ with α defined as above:

$$\left(\sum_{i=1}^{n} w_i X_i - K\right)_+ \leq \sum_{i=1}^{n} w_i \left(X_i - F_{X_i}^{-1(\alpha)}(F_{\mathbb{S}^c}(K))\right)_+ . \tag{40}$$

The right-hand side of this inequality can be interpreted as the pay-off at time T of a strategy consisting of buying at time zero $w_i e^{-\delta(T-T_i)}$ European options with pay-off $(X_i - F_{X_i}^{-1(\alpha)}(F_{\mathbb{S}^c}(K)))_+$ at time T_i, holding these options until they expire at time T_i and investing their eventual pay-offs at that time in the risk free account until time T.

In order to avoid arbitrage opportunities, any fair price $C[K]$ of the exotic option with pay-off $(\mathbb{S}-K)_+$ should not exceed the price of the strategy corresponding to the right-hand side of the inequality (40), hence

$$C[K] \leq \sum_{i=1}^{n} w_i e^{-\delta(T-T_i)} C_i \left[F_{X_i}^{-1(\alpha)}\left(F_{\mathbb{S}^c}(K)\right)\right]. \tag{41}$$

Combining (38) and (41) leads to (31) and (32).

(ii) In case $K \leq F_{\mathbb{S}^c}^{-1+}(0)$, we know with certainty that $\mathbb{S} \geq K$. For the pay-off, this implies that

$$(\mathbb{S}-K)_+ = \mathbb{S} - K = \sum_{i=1}^{n} w_i X_i - K,$$

and by a no-arbitrage argument that the option price $C[K]$ is given by (34a). Thus $C[K]$ directly follows from the observed asset prices $C_i[0]$.

When $K \geq F_{\mathbb{S}^c}^{-1}(1)$, we know with certainty that $\mathbb{S} \leq K$, which implies that the pay-off of the option will be zero and hence this option has no value.

■

In the infinite market case considered in this section, the information that is available concerning the pricing distribution of the vector $(X_1, X_2, \ldots, X_n)$ consists of the distributions of the marginals X_i. The upper bound $e^{-\delta T}\mathrm{E}\left[(\mathbb{S}^c - K)_+\right]$ for the exotic option price $C[K]$ holds for all pricing vectors with given marginal distributions.

The essential part of the proof of Theorem 1 consists of a no-arbitrage argument based on the relations (37) and (40). Furthermore, from this proof we can easily conclude that the upper bound (41) for any fair exotic option price $C[K]$ remains to hold without assuming that the plain vanilla option prices are given by discounted expectations under some measure Q. On the other hand, in order to derive the expression $e^{-\delta T}\mathrm{E}\left[(\mathbb{S}^c - K)_+\right]$ for this upper bound, we have to make this assumption and as explained before, we assume that Q is a martingale measure for the discounted gain process since the underlying(s) of the exotic option can pay out dividends.

Notice that in case the exotic option is priced as a discounted expected pay-off under the Q-measure, the upper bound (31) is a direct consequence of the convex order relation

$$\mathbb{S} \leq_{\mathrm{cx}} \mathbb{S}^c \tag{42}$$

which follows from (26) and Definition 2.

To the best of our knowledge, the upper bound of the exotic option price, restricted to the Asian option case, and using the stochastic order relation (42) was first derived in Simon et al. (2000). Dhaene et al. (2002b) compare the upper bound (32) for an Asian option with the exact (simulated) price in case of a Black & Scholes market, whereas Albrecher et al. (2005) consider the same problem in the case of asset prices modeled by Lévy processes. Nielsen et al. (2003) apply Lagrange optimization to derive the upper bound for the Asian option as a portfolio of European call option prices, but only in the special case of a Black & Scholes setting. The infinite market case, applied to an Asian option in a discrete-time binary tree model, was considered in Reynaerts et al. (2006). A comparison of different approximations for the price of Asian options is given in Vanmaele et al. (2006) and bounds for Asian basket options are dealt with in Deelstra et al. (2006).

3.2 The upper bound as the price of the cheapest super-replicating strategy

The upper bound for $C[K]$ presented in (32) is a linear combination of n observable option prices. To be more precise, the linear combination contains $w_i e^{-\delta(T-T_i)}$ options on the underlying X_i with exercise price $F_{X_i}^{-1(\alpha)}\left(F_{\mathbb{S}^c}(K)\right)$.

As noticed in Simon et al. (2000), the proposed exercise prices $F_{X_i}^{-1(\alpha)}\left(F_{\mathbb{S}^c}(K)\right)$ are optimal, in the sense that any other linear combination $\sum_{i=1}^n w_i e^{-\delta(T-T_i)} C_i\left[K_i\right]$ with $\sum_{i=1}^n w_i K_i \leq K$ will lead to a higher upper bound. This statement can easily be verified from the fact that

$$\mathrm{E}\left[(\mathbb{S}^c - K)_+\right] \leq \sum_{i=1}^n w_i \mathrm{E}\left[(X_i - K_i)_+\right] \tag{43}$$

holds for any K_i such that $\sum_{i=1}^n w_i K_i \leq K$, whereas equality holds when $K_i = F_{X_i}^{-1(\alpha)}\left(F_{\mathbb{S}^c}(K)\right)$.

Hence,

$$\begin{aligned}
\mathrm{E}\left[(\mathbb{S}^c - K)_+\right] &= \sum_{i=1}^{n} w_i \mathrm{E}\left[\left(X_i - F_{X_i}^{-1(\alpha)}\left(F_{\mathbb{S}^c}(K)\right)\right)_+\right] \\
&= \min_{K_i \geq 0, \sum w_i K_i \leq K} \sum_{i=1}^{n} w_i \mathrm{E}\left[(X_i - K_i)_+\right]. \qquad (44)
\end{aligned}$$

Translating this result into option prices leads to

$$\begin{aligned}
e^{-\delta T}\mathrm{E}\left[(\mathbb{S}^c - K)_+\right] &= \sum_{i=1}^{n} w_i e^{-\delta(T-T_i)} C_i\left[F_{X_i}^{-1(\alpha)}\left(F_{\mathbb{S}^c}(K)\right)\right] \qquad (45) \\
&= \min_{K_i \geq 0, \sum w_i K_i \leq K} \sum_{i=1}^{n} w_i e^{-\delta(T-T_i)} C_i[K_i]. \qquad (46)
\end{aligned}$$

In a setting with continuous cdf's F_{X_i} and only considering Asian options, Albrecher et al. (2005) notice that the upper bound (46) for the exotic option $C[K]$ can be interpreted in terms of an optimal static super-hedging strategy.
Their interpretation can easily be extended to our general setting. Indeed, consider the strategy where at time 0, for each i, one buys $w_i e^{-\delta(T-T_i)}$ European calls $C_i[K_i]$, where the exercise prices K_i are such that $\sum_{i=1}^{n} w_i K_i \leq K$. Further, hold each of these calls until its expiration time T_i and when $T_i < T$ invest the pay-off in the risk free account from time T_i until time T.
At time T, the pay-off of this strategy is given by $\sum_{i=1}^{n} w_i(X_i - K_i)_+$ and it is easy to verify that it super-replicates the pay-off of the exotic option:

$$(\mathbb{S} - K)_+ \leq \sum_{i=1}^{n} w_i(X_i - K_i)_+. \qquad (47)$$

The time-0 price of this super-replicating strategy is given by $\sum_{i=1}^{n} w_i e^{-\delta(T-T_i)} C_i[K_i]$.
From (45) and(46) it follows that the particular choice $K_i = F_{X_i}^{-1(\alpha)}\left(F_{\mathbb{S}^c}(K)\right)$ for the exercise prices leads to the cheapest super-replicating strategy in the class of strategies as described above.

Next, we will show that the optimal strategy $K_i = F_{X_i}^{-1(\alpha)}\left(F_{\mathbb{S}^c}(K)\right)$ is also optimal in a much broader class of admissible strategies. In the sequel of this section, we consider the class of investment strategies where for each X_i at current time 0, call options can be bought at any exercise price and where at exercise date $T_i < T$, the pay-off is invested in the risk free account until time T. The pay-off at time T of any such strategy is given by

$$\sum_{i=1}^{n} \int_0^{+\infty} e^{\delta(T-T_i)}(X_i - k)_+ \, \mathrm{d}\nu_i(k), \qquad (48)$$

where the real functions ν_i are used to describe the number of investments in the respective options on the different X_i. The investment strategy with pay-off given by (48) will be denoted by $\underline{\nu} \equiv (\nu_1, \nu_2, \ldots, \nu_n)$. Assuming that we can apply Fubini's theorem for interchanging the integration order, the price of this investment strategy is given by

$$\sum_{i=1}^{n} \int_{0}^{+\infty} C_i\left[k\right] \mathrm{d}\nu_i(k). \tag{49}$$

We are only interested in investment strategies that super-replicate the pay-off $(\mathbb{S} - K)_+$ of the exotic option. Therefore we will only consider investment strategies $\underline{\nu}$ belonging to the set $\mathcal{A}_K$ which is defined by

$$\mathcal{A}_K = \left\{ \underline{\nu} \mid \left(\sum_{i=1}^{n} w_i X_i - K \right)_+ \leq \sum_{i=1}^{n} \int_{0}^{+\infty} e^{\delta(T-T_i)} (X_i - k)_+ \, \mathrm{d}\nu_i(k) \right\}. \tag{50}$$

It is easy to verify that the investment strategies considered in the optimisation problem (46) are a subclass of the class of investment strategies $\mathcal{A}_K$. Indeed, the solution of optimisation problem (46) is contained in $\mathcal{A}_K$ as it can be represented as

$$\nu_i(k) = \left\{ \begin{array}{cl} 0 & \text{if } k < F_{X_i}^{-1(\alpha)}\left(F_{\mathbb{S}^c}(K)\right) \\ w_i e^{-\delta(T-T_i)} & \text{if } k \geq F_{X_i}^{-1(\alpha)}\left(F_{\mathbb{S}^c}(K)\right) \end{array} \right. \tag{51}$$

with α defined by (33) in case $F_{\mathbb{S}^c}^{-1+}\left(F_{\mathbb{S}^c}(K)\right) \neq F_{\mathbb{S}^c}^{-1}\left(F_{\mathbb{S}^c}(K)\right)$ and $\alpha = 1$ otherwise.

In the following theorem, we look for the cheapest super-replicating investment strategy $\underline{\nu} \in \mathcal{A}_K$.

Theorem 2 *Consider the infinite market case. For any* $K \in \left(F_{\mathbb{S}^c}^{-1+}(0), F_{\mathbb{S}^c}^{-1}(1)\right)$ *it holds that*

$$e^{-\delta T} E\left[(\mathbb{S}^c - K)_+\right] = \min_{\underline{\nu} \in \mathcal{A}_K} \sum_{i=1}^{n} \int_{0}^{+\infty} C_i\left[k\right] d\nu_i(k). \tag{52}$$

Proof. For any $\underline{\nu} \in \mathcal{A}_K$ the pay-off inequality (50) is independent of the underlying multivariate distribution of $(X_1, X_2, \ldots, X_n)$. In particular, it has to hold for the comonotonic case where $(X_1, X_2, \ldots, X_n)$ has the same distribution as $(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), \ldots, F_{X_n}^{-1}(U))$. Let us concentrate on this case. Taking expectations of both sides of the inequality (50) and taking into account that

$$e^{-\delta T_i} \mathrm{E}\left[\left(F_{X_i}^{-1}(U) - k\right)_+\right] = C_i\left[k\right], \qquad i = 1, \ldots, n;\ k \geq 0$$

in the infinite market case, we find

$$e^{-\delta T} \mathrm{E}\left[(\mathbb{S}^c - K)_+\right] \leq \sum_{i=1}^{n} \int_{0}^{+\infty} C_i\left[k\right] \mathrm{d}\nu_i(k), \qquad \underline{\nu} \in \mathcal{A}_K.$$

Hence, we can conclude that

$$e^{-\delta T} \mathrm{E}\left[(\mathbb{S}^c - K)_+\right] \leq \inf_{\underline{\nu} \in \mathcal{A}_K} \sum_{i=1}^{n} \int_{0}^{+\infty} C_i\left[k\right] \mathrm{d}\nu_i(k). \tag{53}$$

Taking into account (45), which states that $e^{-\delta T}\mathrm{E}\left[(\mathbb{S}^c - K)_+\right]$ is equal to the price of the investment strategy $\underline{\nu} \in \mathcal{A}_K$ defined in (51), we find that the infimum is reached and (53) holds with equality.

■

From the proof of Theorem 2, we can conclude that the cheapest super-replicating strategy $\underline{\nu}$ contained in $\mathcal{A}_K$ is given by (51), with α defined by (33) in case $F_{\mathbb{S}^c}^{-1+}\left(F_{\mathbb{S}^c}(K)\right) \neq F_{\mathbb{S}^c}^{-1}\left(F_{\mathbb{S}^c}(K)\right)$ and $\alpha = 1$ otherwise. The price of this cheapest super-replicating strategy is precisely $e^{-\delta T}\mathrm{E}\left[(\mathbb{S}^c - K)_+\right]$.

Notice that our results concerning the optimal super-replicating strategy are related with but slightly different from the results contained in Hobson et al. (2005) and Laurence et al. (2004), who derive their results in a setting of primal and dual problems.

We have that $e^{-\delta T}\mathrm{E}\left[(\mathbb{S}^c - K)_+\right]$ corresponds to a reasonable offer price for the seller of the exotic option. Indeed, if he sells at this price and acquires the optimal super-replication portfolio $\underline{\nu}$ defined in (51), he will incur no losses. On the other hand, $e^{-\delta T}\mathrm{E}\left[(\mathbb{S}^c - K)_+\right]$ corresponds to a maximum on the price that the buyer of the exotic option is willing to pay. Indeed, if the exotic option has a higher price, the buyer better purchases the optimal combination of plain vanilla options that super-replicates the pay-off of the exotic option.

Theorem 2 can be generalized in a straightforward way to the broader class of super-replicating strategies which also contain investments in cash, in calls on assets different from the one used to define the exotic option and also investments in puts on all the above mentioned underlyings. In this case, we simply have to redefine $\mathcal{A}_K$ in terms of the available investment instruments, whereas the proof of the result proceeds in the same way as considered above.

3.3 The upper bound as a worst-case expectation

In this subsection, we will show that the upper bound $e^{-\delta T}\mathrm{E}\left[(\mathbb{S}^c - K)_+\right]$ for any fair price $C[K]$ of the exotic option can be interpreted as a worst-case expectation of its pay-off, in the Fréchet class $\mathcal{R}_n$ of all multivariate pricing distributions with fixed marginals.

Definition 7 *The Fréchet class $\mathcal{R}_n$ of all n-dimensional random vectors with marginals equal to the respective pricing distributions F_{X_i} of the asset prices X_i is given by*

$$\mathcal{R}_n = \{(Y_1, Y_2, \ldots, Y_n) \mid F_{Y_i}(x) = F_{X_i}(x);\, x \geq 0,\, i = 1, \ldots, n\}. \tag{54}$$

As we are working with a model under which the prices of vanilla call prices are discounted expected pay-offs, we can equivalently define $\mathcal{R}_n$ as follows:

$$\mathcal{R}_n = \left\{(Y_1, Y_2, \ldots, Y_n) \mid e^{-\delta T_i}\mathrm{E}\left[(Y_i - K)_+\right] = C_i[K];\, K \geq 0,\, i = 1, \ldots, n\right\}, \tag{55}$$

which means that $\mathcal{R}_n$ can be interpreted as the set of all n-dimensional random vectors for which the discounted expectations $e^{-\delta T_i}\mathrm{E}\left[(Y_i - K)_+\right]$ coincide with the respective observed option prices. We immediately find that $(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), \ldots, F_{X_n}^{-1}(U)) \in \mathcal{R}_n$.

Theorem 3 *In the infinite market case, we have that*

$$e^{-\delta T} E\left[(\mathbb{S}^c - K)_+\right] = \max_{(Y_1,Y_2,\ldots,Y_n)\in\mathcal{R}_n} e^{-\delta T} E\left[\left(\sum_{i=1}^{n} w_i Y_i - K\right)_+\right]. \tag{56}$$

Proof. Relation (26) implies that for any $(Y_1, Y_2, \ldots, Y_n)$ in $\mathcal{R}_n$ it holds that

$$\sum_{i=1}^{n} w_i Y_i \leq_{\text{cx}} \mathbb{S}^c,$$

and by Definition 2 also that

$$e^{-\delta T}\mathrm{E}\left[\left(\sum_{i=1}^{n} w_i Y_i - K\right)_+\right] \leq e^{-\delta T}\mathrm{E}\left[(\mathbb{S}^c - K)_+\right].$$

The stated result (56) follows then from the fact that $(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), \ldots, F_{X_n}^{-1}(U))$ belongs to $\mathcal{R}_n$. ■

Theorem 3 states that the upper bound of Theorem 1 can be interpreted as a worst-case expectation of the pay-off of the exotic option, in the sense that it corresponds to the largest possible expectation, given the marginal pricing distributions of the underlying assets. Hence, in case we can only observe the option prices of the underlying plain vanilla options, we can find an upper bound for the non-observed exotic option price by considering the worst case possible, given the partial information available concerning the pricing distribution of $(X_1, X_2, \ldots, X_n)$. From Theorem 3 we can conclude that this worst case corresponds to the comonotonic case.

From a mathematical point of view, the results for the infinite market case described in Theorem 1 and Theorem 3 are very similar to finding a best upper bound for a stop-loss premium of a sum of non-independent random variables in terms of stop-loss premia of the marginals involved, as described in Goovaerts et al. (2000) and Kaas et al. (2000). Early references to solutions for this problem are Meilijson & Nadas (1979) and Rüschendorf (1983).

Furthermore, similar results to the one presented in Theorem 3 have been presented in the finance literature. For example in the simple arbitrage-free market of only one risk-free asset and one risky asset with price S at the initial date and price S_1 at the final date in an interval $[S_l, S_h]$, the static selling price of a derivative with pay-off $g(S_1)$ with g a convex function, is given by $\sup_{P\epsilon\mathcal{P}} \mathrm{E}_P[g(S_1)]$ with $\mathcal{P}$ the set of risk-neutral probability measures, a result which holds in much more general situations, see El Karoui and Quenez (1995). This upper bound in the simple market mentioned above is obtained for the probability measure such that S_1 can only take the values S_l and S_h, which is the worst case scenario under the given information, leading to the maximal variance of S_1, see e.g. Dana & Jeanblanc-Picqué (1998).

3.4 Computational aspects

The coefficient α in (32) is independent of i and is determined by relation (33) when $F_{\mathbb{S}^c}^{-1+}(F_{\mathbb{S}^c}(K)) \neq F_{\mathbb{S}^c}^{-1}(F_{\mathbb{S}^c}(K))$. In order to be able to calculate α defined in (33), one has to determine $F_{\mathbb{S}^c}(K)$, $F_{\mathbb{S}^c}^{-1+}(F_{\mathbb{S}^c}(K))$ and $F_{\mathbb{S}^c}^{-1}(F_{\mathbb{S}^c}(K))$. According to (18), $F_{\mathbb{S}^c}(K)$ can be determined from

$$F_{\mathbb{S}^c}(K) = \sup \left\{ p \in [0,1] \mid \sum_{i=1}^{n} w_i F_{X_i}^{-1}(p) \leq K \right\}, \tag{57}$$

while from (20), we find that $F_{\mathbb{S}^c}^{-1+}(F_{\mathbb{S}^c}(K))$ and $F_{\mathbb{S}^c}^{-1}(F_{\mathbb{S}^c}(K))$ are given by

$$F_{\mathbb{S}^c}^{-1+}(F_{\mathbb{S}^c}(K)) = \sum_{i=1}^{n} w_i F_{X_i}^{-1+}(F_{\mathbb{S}^c}(K)) \tag{58}$$

and

$$F_{\mathbb{S}^c}^{-1}(F_{\mathbb{S}^c}(K)) = \sum_{i=1}^{n} w_i F_{X_i}^{-1}(F_{\mathbb{S}^c}(K)), \tag{59}$$

respectively.

In case all marginals F_{X_i} are strictly increasing on $\left(F_{X_i}^{-1+}(0), F_{X_i}^{-1}(1)\right)$ and at least one is continuous on $\mathbb{R}$, one has that $F_{\mathbb{S}^c}$ is strictly increasing on $\left(F_{\mathbb{S}^c}^{-1+}(0), F_{\mathbb{S}^c}^{-1}(1)\right)$ and continuous on $\mathbb{R}$. Hence, in this case the value $F_{\mathbb{S}^c}(K)$ can unambiguously be obtained for any $K \in \left(F_{\mathbb{S}^c}^{-1+}(0), F_{\mathbb{S}^c}^{-1}(1)\right)$ from (37) with $\alpha = 1$:

$$\sum_{i=1}^{n} w_i F_{X_i}^{-1}(F_{\mathbb{S}^c}(K)) = K, \tag{60}$$

see Dhaene et al. (2002a). A particular case where these conditions are fulfilled is the Black & Scholes model.

The upper bound (32) can also be written in terms of the inverses $F_{X_i}^{-1}$, as is shown in the following corollary.

Corollary 1 *Consider the infinite market case. For any* $K \in \left(F_{\mathbb{S}^c}^{-1+}(0), F_{\mathbb{S}^c}^{-1}(1)\right)$ *it holds that*

$$\begin{aligned} e^{-\delta T} E\left[(\mathbb{S}^c - K)_+\right] = & \sum_{i=1}^{n} w_i e^{-\delta(T-T_i)} C_i \left[F_{X_i}^{-1}(F_{\mathbb{S}^c}(K))\right] \\ & - e^{-\delta T}\left(K - F_{\mathbb{S}^c}^{-1}(F_{\mathbb{S}^c}(K))\right)(1 - F_{\mathbb{S}^c}(K)). \end{aligned} \tag{61}$$

Proof. The proof follows immediately from (2), (24) and (31). ∎

By (36) and Definition 5, it obviously holds that

$$F_{\mathbb{S}^c}^{-1}(F_{\mathbb{S}^c}(K)) \leq K = F_{\mathbb{S}^c}^{-1(\alpha)}(F_{\mathbb{S}^c}(K)).$$

Taking into account that the $C_i[\cdot]$ are decreasing functions of the exercise price, we find from (32) that $\sum_{i=1}^{n} w_i e^{-\delta(T-T_i)} C_i \left[F_{X_i}^{-1}\left(F_{\mathbb{S}^c}(K)\right)\right]$ is also an upper bound for the exotic option price $C[K]$, but it is not necessarily the optimal one in the sense that the time-0 price of this portfolio of plain vanilla options $C_i[F_{X_i}^{-1}(F_{\mathbb{S}^c}(K))]$ may not be the cheapest one.

When $\alpha = 1$, the α-inverses $F_{X_i}^{-1(\alpha)}$ coincide with the usual inverses $F_{X_i}^{-1}$, see Definition 5, implying that according to (36) $F_{\mathbb{S}^c}^{-1}\left(F_{\mathbb{S}^c}(K)\right) = K$. Hence, the upper bounds (32) and (61) also coincide and reduce to

$$e^{-\delta T}\mathrm{E}\left[\left(\mathbb{S}^c - K\right)_+\right] = \sum_{i=1}^{n} w_i e^{-\delta(T-T_i)} C_i\left[F_{X_i}^{-1}\left(F_{\mathbb{S}^c}(K)\right)\right]. \tag{62}$$

This situation will occur in the case described above leading to relation (60).

An expression very similar to (62) can already be found in Jamshidian (1989) who proves that in the Vasicek (1977) model, an option on a portfolio of pure discount bonds (in particular, an option on a coupon paying bond) decomposes into a portfolio of options on the individual discount bonds in the portfolio. This holds true because in the Vasicek model, the prices of all pure discount bonds at some future time T are decreasing functions of a single random variable, namely the spot rate at that time. The same holds true in the class of affine one-factor term structure models studied by Duffie and Kan (1996). This implies that the price at time T of the portfolio of pure discount bonds in these models is a comonotonic sum, for which a decomposition similar to the one in (62) holds.

Expression (21) of Kaas et al. (2000) can be considered as an extension of the result of Jamshidian to the case where the cdf's of the F_{X_i} are not necessarily strictly increasing. As we will see in the next section, this situation will naturally appear when generalizing the derived upper bounds for the price of exotic options to the case of a market where the European options involved are only traded for a limited number of exercise prices.

4 A least upper bound in the finite market case

4.1 Deriving the upper bound

In the preceding section we assumed that the prices of the European calls $C_i[K]$ were given for all $K \geq 0$. In this section we will explore a model-free approach in a market where only finitely many strikes are traded.

To be more precise, we assume that for each i, only the prices of the European call options with strikes $K_{i,j}$, $j = 0, 1, \ldots, m_i$, are available. These options have respective pay-offs $\left(X_i - K_{i,j}\right)_+$ at time $T_i \leq T$ and we denote their observed prices by $C_i[K_{i,j}]$. As before, we assume that these prices can be expressed as

$$C_i[K_{i,j}] = e^{-\delta T_i}\mathrm{E}\left[\left(X_i - K_{i,j}\right)_+\right], \qquad j = 0, 1, \ldots, m_i,\ i = 1, \ldots, n, \tag{63}$$

where for each i, the expectations are taken with respect to the unknown pricing distribution F_{X_i}. The only information we have about F_{X_i} is contained in the observed option prices $C_i\left[K_{i,j}\right]$.

The pay-off of the call option with strike $K_{i,0} = 0$ coincides with receiving X_i at time T_i. From (63), we find that $C_i\left[0\right] = e^{-\delta T_i}\mathrm{E}\left[X_i\right]$. When no dividends are paid until time T_i, a no-arbitrage argument leads to the conclusion that $C_i\left[0\right]$ equals the time-0 price of the underlying X_i. When dividends are paid, $C_i\left[0\right]$ will be a lower bound for the time-0 price of the underlying asset.
For any underlying X_i, we introduce the notation $C_i\left[K\right]$ to denote the following continuous functions of K:

$$C_i\left[K\right] = e^{-\delta T_i}\mathrm{E}\left[\left(X_i - K\right)_+\right]. \tag{64}$$

This function is decreasing and convex and has a derivative with respect to the strike K equal to $-e^{-\delta T_i}$ for $K < 0$.

For any underlying X_i, the available strikes are assumed to fulfill the following chain of inequalities:

$$0 = K_{i,0} < K_{i,1} < K_{i,2} < \cdots < K_{i,m_i} < K_{i,m_i+1}, \tag{65}$$

where the K_{i,m_i+1} are defined by

$$K_{i,m_i+1} = \sup\left\{K \geq 0 \mid C_i\left[K\right] > 0\right\}. \tag{66}$$

Hence, K_{i,m_i+1} is equal to the supremum of the support of X_i under the pricing distribution F_{X_i}. From the definition of K_{i,m_i+1}, we immediately find that $C_i\left[K\right] > 0$ for all $K < K_{i,m_i+1}$, whereas $C_i\left[K\right] = 0$ for all $K \geq K_{i,m_i+1}$. The dashed line in Figure 2 corresponds to an example of how the function $C_i\left[K\right]$ could look like.

In the model-free approach of this section, the K_{i,m_i+1} will in general not be known and theoretically might be equal to infinity. In the sequel however, we will take a practical approach and assume that all K_{i,m_i+1} have a finite value, however large enough — this will be specified later.

It is our goal to derive an upper bound for any fair price $C\left[K\right]$ of the exotic option with pay-off $\left(\mathbb{S} - K\right)_+$ at time T, in terms of the observed plain vanilla call prices $C_i\left[K_{i,j}\right]$. We will show that this upper bound corresponds to the price of a super-replicating strategy that involves only investments in the traded calls.

The results presented in this section are a generalization of the work of Hobson et al. (2005), who consider this problem for the case of a basket option. They construct a convex approximation $\overline{C}_i[\cdot]$ to each function $C_i[\cdot]$ via a linear interpolation such that $\overline{C}_i[K_{i,j}] = C_i[K_{i,j}]$ for the observed call prices. As in the infinite market case, they use Lagrange optimization to derive their results. We will extend the results of Hobson et al. (2005) to the more general exotic option as introduced in Section 1 and derive these results in a more unified framework. Our proofs are based on some basic results from the theory of stochastic orders and comonotonic risks, as presented in Section 2.

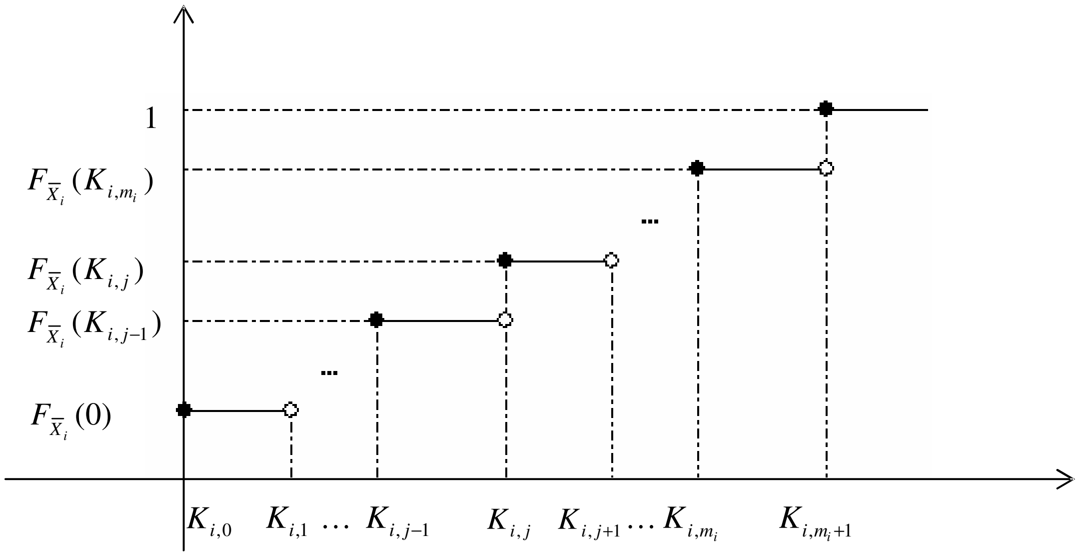

Figure 1: The cdf of $\overline{X}_i$.

We start by introducing random variables $\overline{X}_i$, $i = 1, \ldots, n$ with cdf's $F_{\overline{X}_i}$ defined by

$$F_{\overline{X}_i}(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 + e^{\delta T_i} \dfrac{C_i\left[K_{i,j+1}\right] - C_i\left[K_{i,j}\right]}{K_{i,j+1} - K_{i,j}} & \text{if } K_{i,j} \leq x < K_{i,j+1}, \quad j = 0, 1, \ldots, m_i \\ 1 & \text{if } x \geq K_{i,m_i+1} \end{cases} \tag{67}$$

The functions $F_{\overline{X}_i}(x)$ are well-defined cdf's. This follows from the mentioned properties of the functions $C_i\left[K\right]$.

The random variables $\overline{X}_i$ have a discrete distribution, with possible outcomes given by the $K_{i,j}$, see Figure 1.

In the following lemma, we show that the function $\overline{C}_i\left[K\right]$ defined by

$$\overline{C}_i\left[K\right] = e^{-\delta T_i} \mathrm{E}\left[\left(\overline{X}_i - K\right)_+\right], \tag{68}$$

coincides with a linear interpolation of the function $C_i[K]$. We further derive expressions for the quantile function $F_{\overline{X}_i}^{-1(\alpha)}(p)$. To simplify the description of those quantiles we define an artificial strike $K_{i,-1} = -1$, for which one immediately finds that $F_{\overline{X}_i}(K_{i,-1}) = 0$.

Lemma 1 *Consider the random variable $\overline{X}_i$ with cdf defined in (67). Let $\overline{C}_i\left[K\right]$ be defined by (68), then we have that $\overline{C}_i\left[K\right] = 0$ for $K \geq K_{i,m_i+1}$, whereas*

$$\overline{C}_i\left[K\right] = \frac{C_i\left[K_{i,j+1}\right] - C_i\left[K_{i,j}\right]}{K_{i,j+1} - K_{i,j}}\left(K - K_{i,j}\right) + C_i\left[K_{i,j}\right], \tag{69}$$

$$K_{i,j} \leq K < K_{i,j+1},\ j = 0, 1, \ldots, m_i.$$

Furthermore, the α-quantile $F_{\overline{X}_i}^{-1(\alpha)}(p)$, $0<p<1$, $0\le\alpha\le1$, is given by:

$$F_{\overline{X}_i}^{-1(\alpha)}(p)=\begin{cases} K_{i,j} & \text{if } F_{\overline{X}_i}(K_{i,j-1})<p<F_{\overline{X}_i}(K_{i,j}),\ j=0,1,\ldots,m_i+1,\\ \alpha K_{i,j}+(1-\alpha)K_{i,j+1} & \text{if } p=F_{\overline{X}_i}(K_{i,j}),\ j=0\ldots,m_i. \end{cases} \tag{70}$$

Proof. First, from (25), (67) and (68) it follows immediately that $\overline{C}_i[K]=0$ for $K\ge K_{i,m_i+1}$.
For $K_{i,j}\le K<K_{i,j+1}$, $j=0,\ldots,m_i$, we invoke (25) and use the fact that $F_{\overline{X}_i}$ is piecewise constant to arrive at

$$\begin{aligned}\overline{C}_i[K]&=e^{-\delta T_i}\int_K^{+\infty}\left(1-F_{\overline{X}_i}(x)\right)dx=e^{-\delta T_i}\Big(\int_{K_{i,j}}^{+\infty}\left(1-F_{\overline{X}_i}(x)\right)dx-\int_{K_{i,j}}^{K}\left(1-F_{\overline{X}_i}(x)\right)dx\Big)\\ &=e^{-\delta T_i}\sum_{\ell=j}^{m_i}\int_{K_{i,\ell}}^{K_{i,\ell+1}}\left(1-F_{\overline{X}_i}(x)\right)dx-e^{-\delta T_i}\int_{K_{i,j}}^{K}\left(1-F_{\overline{X}_i}(x)\right)dx.\end{aligned}$$

Inserting the constant values of $F_{\overline{X}_i}$ in the respective intervals, this expression could be easily transformed into (69). The quantiles $F_{\overline{X}_i}^{-1(\alpha)}(p)$ follow from (67), see also Figure 1. ■

From this lemma it follows that

$$\overline{C}_i[K_{i,j}]=C_i[K_{i,j}],\qquad j=0,1,\ldots,m_i+1, \tag{71}$$

while the function $\overline{C}_i[K]$ is piecewise linear. We can conclude that the function $\overline{C}_i[K]$ coincides with the linear interpolation of the function $C_i[K]$ with common values in $K_{i,j}$, $j=0,\ldots,m_i+1$. In Figure 2, we illustrate the function $\overline{C}_i[K]$, as well as the function $C_i[K]$. From the convexity property of $C_i[K]$, we can conclude that $\overline{C}_i[K]\ge C_i[K]$ holds for all K. Taking into account Definition 1, this can also be stated as

$$X_i\le_{\text{icx}}\overline{X}_i,\qquad i=1,\ldots,n. \tag{72}$$

In the following theorem, we derive an upper bound for any fair exotic option price $C[K]$ in terms of the observed plain vanilla option prices $C_i[K_{i,j}]$. We first define the comonotonic sum $\bar{\mathbb{S}}^c$ by:

$$\bar{\mathbb{S}}^c=w_1F_{\overline{X}_1}^{-1}(U)+w_2F_{\overline{X}_2}^{-1}(U)+\cdots+w_nF_{\overline{X}_n}^{-1}(U). \tag{73}$$

From (19) and (67), we find that the extreme outcomes for $\bar{\mathbb{S}}^c$ are given by

$$F_{\bar{\mathbb{S}}^c}^{-1+}(0)=\sum_{i=1}^n w_iF_{\overline{X}_i}^{-1+}(0)=0, \tag{74}$$

$$F_{\bar{\mathbb{S}}^c}^{-1}(1)=\sum_{i=1}^n w_iF_{\overline{X}_i}^{-1}(1)=\sum_{i=1}^n w_iK_{i,m_i+1}. \tag{75}$$

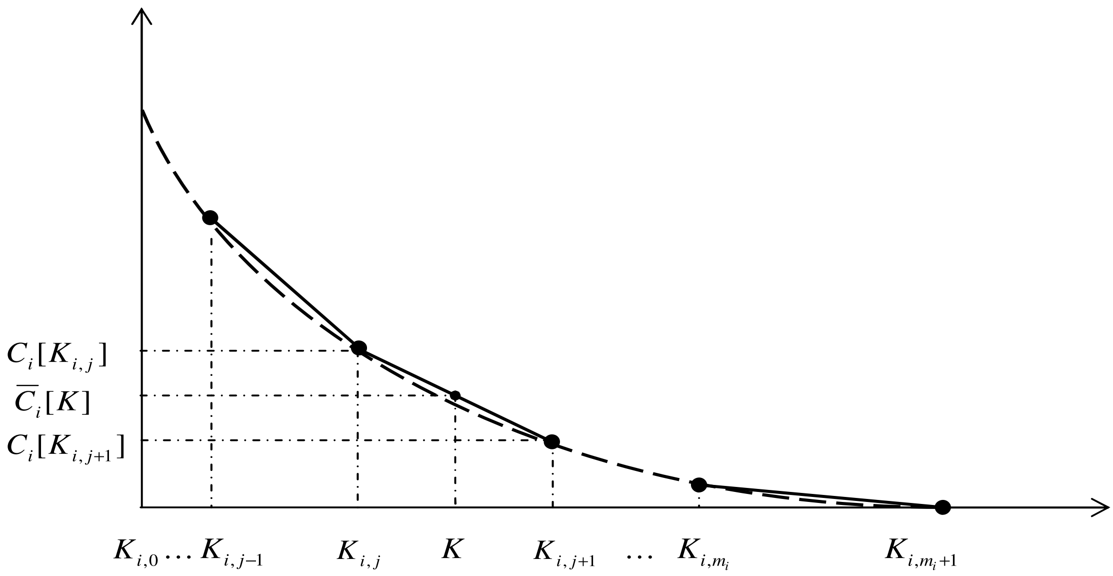

Figure 2: The functions $\overline{C}_i[K]$ and $C_i[K]$.

Let $K \in (0, \sum_{i=1}^n w_i K_{i,m_i+1})$, then we have that $F_{\bar{\mathbb{S}}^c}(K) \in (0,1)$. For any such K and underlying X_i, we define $j_i(K)$, in the sequel often abbreviated as j_i, as the unique index contained in the set $\{0,1,\ldots,m_i+1\}$ that satisfies

$$F_{\overline{X}_i}(K_{i,j_i-1}) < F_{\bar{\mathbb{S}}^c}(K) \leq F_{\overline{X}_i}(K_{i,j_i}). \tag{76}$$

Further, for any $K \in (0, \sum_{i=1}^n w_i K_{i,m_i+1})$ we define the set N_K as follows:

$$N_K = \left\{ i \in \{1,2,\ldots,n\} \mid F_{\overline{X}_i}(K_{i,j_i-1}) < F_{\bar{\mathbb{S}}^c}(K) < F_{\overline{X}_i}(K_{i,j_i}) \right\}. \tag{77}$$

Its complement $\overline{N}_K = \{1,2,\ldots,n\} \setminus N_K$ can be defined as

$$\overline{N}_K = \left\{ i \in \{1,2,\ldots,n\} \mid F_{\bar{\mathbb{S}}^c}(K) = F_{\overline{X}_i}(K_{i,j_i}) \right\}. \tag{78}$$

Notice that $i \in \overline{N}_K$ implies that $j_i \in \{0,1,\ldots,m_i\}$. The indices $j_i(K)$ and the set N_K play a crucial role in describing the upper bound for the exotic option price $C[K]$.

Theorem 4 *Let us assume the finite market as described above.*

(i) For any $K \in (0, \sum_{i=1}^n w_i K_{i,m_i+1})$ we have that any fair price $C[K]$ of the exotic option with pay-off $(\mathbb{S}-K)_+$ at time T is constrained from above as follows:

$$C[K] \leq e^{-\delta T} E\left[\left(\bar{\mathbb{S}}^c - K\right)_+\right] \tag{79}$$

$$= \sum_{i \in N_K} w_i e^{-\delta(T-T_i)} C_i[K_{i,j_i}] + \sum_{i \in \overline{N}_K} w_i e^{-\delta(T-T_i)} \left(\alpha C_i[K_{i,j_i}] + (1-\alpha) C_i[K_{i,j_i+1}]\right) \tag{80}$$

with α given by

$$\alpha = \frac{\sum_{i \in N_K} w_i K_{i,j_i} + \sum_{i \in \overline{N}_K} w_i K_{i,j_i+1} - K}{\sum_{i \in \overline{N}_K} w_i \left(K_{i,j_i+1} - K_{i,j_i} \right)} \tag{81}$$

in case $N_K \neq \{1, 2, \ldots, n\}$ and $\alpha = 1$ otherwise, and with the j_i defined by (76).

(ii) For any $K \notin (0, \sum_{i=1}^{n} w_i K_{i,m_i+1})$, the exotic option price $C[K]$ is given by:

$$C[K] = \begin{cases} \sum_{i=1}^{n} w_i e^{-\delta(T-T_i)} C_i[0] - e^{-\delta T} K & \textit{if } K \leq 0 \\ 0 & \textit{if } K \geq \sum_{i=1}^{n} w_i K_{i,m_i+1}. \end{cases}$$

Proof.

(i) Let $K \in (0, \sum_{i=1}^{n} w_i K_{i,m_i+1})$. From (70) and the definitions of the indices j_i and the sets N_K and $\overline{N}_K$ in (76), (77) and (78), we find that

$$F_{\overline{X}_i}^{-1(\alpha)}(F_{\bar{\mathbb{S}}^c}(K)) = \begin{cases} K_{i,j_i} & \text{if } i \in N_K \\ \alpha K_{i,j_i} + (1-\alpha) K_{i,j_i+1} & \text{if } i \in \overline{N}_K \end{cases} \tag{82}$$

holds for any $\alpha \in [0, 1]$.
Combining (82) with the expression (69) or (71) for the $\overline{C}_i\,[K]$, we arrive at:

$$\begin{aligned} \overline{C}_i \left[F_{\overline{X}_i}^{-1(\alpha)} \left(F_{\bar{\mathbb{S}}^c}(K) \right) \right] &= \begin{cases} \overline{C}_i \left[K_{i,j_i} \right] & \text{if } i \in N_K \\ \overline{C}_i \left[\alpha K_{i,j_i} + (1-\alpha) K_{i,j_i+1} \right] & \text{if } i \in \overline{N}_K \end{cases} \\ &= \begin{cases} C_i \left[K_{i,j_i} \right] & \text{if } i \in N_K \\ \alpha C_i \left[K_{i,j_i} \right] + (1-\alpha) C_i \left[K_{i,j_i+1} \right] & \text{if } i \in \overline{N}_K. \end{cases} \end{aligned} \tag{83}$$

Let us now consider $\mathrm{E}\left[\left(\bar{\mathbb{S}}^c - K \right)_+ \right]$. Relations (21) and (22) enable us to decompose this expectation into

$$\mathrm{E}\left[\left(\bar{\mathbb{S}}^c - K \right)_+ \right] = \sum_{i=1}^{n} w_i \mathrm{E}\left[\left(\overline{X}_i - F_{\overline{X}_i}^{-1(\alpha)} \left(F_{\bar{\mathbb{S}}^c}(K) \right) \right)_+ \right] \tag{84}$$

with $\alpha \in [0, 1]$ determined from

$$F_{\bar{\mathbb{S}}^c}^{-1(\alpha)} \left(F_{\bar{\mathbb{S}}^c}(K) \right) = K, \tag{85}$$

or, equivalently by (20), from

$$\sum_{i=1}^{n} w_i F_{\overline{X}_i}^{-1(\alpha)} (F_{\bar{\mathbb{S}}^c}(K)) = K. \tag{86}$$

Relation (82) ensures that α defined in (86) is given by (81) in case $N_K \neq \{1, 2, \ldots, n\}$, while $\alpha = 1$ (or any other value in $[0, 1]$) otherwise. Recall that we assumed that

all K_{i,m_i+1} are finite, which guarantees that α is well-defined. In the sequel of this proof, we will continue to work with the α defined in (86).
We now rewrite (84) in terms of the expressions (68) for the European call option prices:

$$e^{-\delta T}\mathrm{E}\left[\left(\bar{\mathbb{S}}^c - K\right)_+\right] = \sum_{i=1}^{n} w_i e^{-\delta(T-T_i)}\overline{C}_i\left[F_{\overline{X}_i}^{-1(\alpha)}\left(F_{\bar{\mathbb{S}}^c}(K)\right)\right]. \tag{87}$$

Inserting (83) in (87) then proves that $e^{-\delta T}\mathrm{E}\left[\left(\bar{\mathbb{S}}^c - K\right)_+\right]$ can be expressed by (80).
It remains to prove that $e^{-\delta T}\mathrm{E}\left[\left(\bar{\mathbb{S}}^c - K\right)_+\right]$ is an upper bound for any fair price of the exotic option with pay-off $(\mathbb{S}-K)_+$ at time T. In view of (86), we find that this pay-off is constrained from above by

$$(\mathbb{S}-K)_+ \leq \sum_{i=1}^{n} w_i\left(X_i - F_{\overline{X}_i}^{-1(\alpha)}\left(F_{\bar{\mathbb{S}}^c}(K)\right)\right)_+ \tag{88}$$

$$= \sum_{i\in N_K} w_i\left(X_i - K_{i,j_i}\right)_+ + \sum_{i\in \overline{N}_K} w_i\left(X_i - \alpha K_{i,j_i} - (1-\alpha)K_{i,j_i+1}\right)_+, \tag{89}$$

where in the last step we used (82). From the inequality above, we can derive the following inequality:

$$(\mathbb{S}-K)_+ \leq \sum_{i\in N_K} w_i\left(X_i - K_{i,j_i}\right)_+ + \sum_{i\in \overline{N}_K} w_i\left(\alpha\left(X_i - K_{i,j_i}\right)_+ + (1-\alpha)\left(X_i - K_{i,j_i+1}\right)_+\right). \tag{90}$$

The right-hand side of this inequality can be interpreted as the pay-off at time T of a strategy consisting of buying a number of the available European options, holding these options until they expire and investing their pay-offs at expiration in the risk free account until time T. To be more precise, for any $i \in N_K$, one buys $w_i e^{-\delta(T-T_i)}$ options with pay-off $(X_i - K_{i,j_i})_+$ at time T_i, whereas for any $i \in \overline{N}_K$, one buys $\alpha w_i e^{-\delta(T-T_i)}$ options with pay-off $(X_i - K_{i,j_i})_+$ and $(1-\alpha)w_i e^{-\delta(T-T_i)}$ options with pay-off $(X_i - K_{i,j_i+1})_+$.
By a no-arbitrage argument, any fair price $C[K]$ of the exotic option with pay-off $(\mathbb{S}-K)_+$ should be smaller than the price of the strategy on the right-hand side of the inequality (90):

$$C[K] \leq \sum_{i\in N_K} w_i e^{-\delta(T-T_i)} C_i\left[K_{i,j_i}\right] + \sum_{i\in \overline{N}_K} w_i e^{-\delta(T-T_i)}\left(\alpha C_i\left[K_{i,j_i}\right] + (1-\alpha)C_i\left[K_{i,j_i+1}\right]\right). \tag{91}$$

Combining the proven equality (80) and (91), we have shown part (i) of the theorem.

(ii) The case $K \leq F_{\bar{\mathbb{S}}^c}^{-1+}(0) = 0$ is proven in an analogous way to the corresponding case in Theorem 1. When $K \geq F_{\bar{\mathbb{S}}^c}^{-1}(1) = \sum_{i=1}^{n} w_i K_{i,m_i+1}$ we have that $\mathbb{S} \leq K$ holds with probability 1. Hence we have that $C[K] = e^{-\delta T}\mathrm{E}[(\mathbb{S}-K)_+] = 0$ in this case.■

Remark that in practice we assume that the K_{i,m_i+1} are finite and therefore N_K will be a strict subset of the set $\{1, 2, \ldots, n\}$, leading always to an α of the form (81). However in section 4.4.1, we will study also the case that the K_{i,m_i+1} tend to infinity leading to the situation where $\alpha = 1$.

The essential part of the proof of Theorem 4 consists of a no-arbitrage argument based on the relation (90). From this proof we can easily conclude that the upper bound (80) for any fair exotic option price $C[K]$ remains to hold without assuming that the plain vanilla option prices are given by discounted expectations under some measure Q. On the other hand, in order to derive the expression $e^{-\delta T}\mathrm{E}\left[\left(\bar{\mathbb{S}}^c - K\right)_+\right]$ for this upper bound, we have to make this assumption.

Based on relation (79) we can conclude that in the finite market case $C\left[K\right]$ is constrained from above by $e^{-\delta T}\mathrm{E}\left[\left(\bar{\mathbb{S}}^c - K\right)_+\right]$. In the limiting case where full information of the marginals is available, we have from (31) that $C\left[K\right]$ is constrained from above by $e^{-\delta T}\mathrm{E}\left[\left(\mathbb{S}^c - K\right)_+\right]$.
Taking into account the increasing convex order relations (72) and the definitions (28) and (73) of $\mathbb{S}^c$ and $\bar{\mathbb{S}}^c$, respectively, we find from (27) that $\mathbb{S}^c$ precedes $\bar{\mathbb{S}}^c$ in increasing convex order sense:

$$\mathbb{S}^c \leq_{\text{icx}} \bar{\mathbb{S}}^c. \tag{92}$$

From Definition 2, we can conclude that

$$e^{-\delta T}\mathrm{E}\left[\left(\mathbb{S}^c - K\right)_+\right] \leq e^{-\delta T}\mathrm{E}\left[\left(\bar{\mathbb{S}}^c - K\right)_+\right]. \tag{93}$$

Hence, we find the intuitive result that the upper bound in the finite market case exceeds the upper bound in the infinite market case.

Notice that in case the exotic option is priced as a discounted expected pay-off under the Q-measure, the upper bound (79) is a direct consequence of the increasing convex order relation

$$\mathbb{S} \leq_{\text{icx}} \bar{\mathbb{S}}^c \tag{94}$$

which follows from combining (42) and (92).

The super-replicating strategy of Theorem 4 might have investments in European options with the exercise price $K_{i,0} = 0$ when $K_{i,j_i} = K_{i,0} = 0$. This will be the case for those i for which $0 < F_{\bar{\mathbb{S}}^c}(K) \leq F_{\overline{X}_i}(0)$. From (67) and the properties of the function $\overline{C}_i\left[K\right]$, we find that $F_{\overline{X}_i}(0)$ is a non-decreasing function of $K_{i,1}$. This implies that the higher the value of the first available strictly positive strike $K_{i,1}$ is, the more values of K will lead to an optimal super-replicating strategy with investments in the option with the exercise price $K_{i,0}$.

Until here, we assumed that for each underlying X_i European call option with exercise price zero is available in the market. In practice, these options will often not be traded, except in the case of a non-dividend paying asset where these options can be identified with their underlying asset, and their time-0 price equals the time-0 asset price. However, we can easily adapt the upper bound (80) and the related super-replicating strategy in Theorem 4 to the case that European options with exercise price zero are not traded. This can be performed by buying the underlying asset associated with X_i instead of the corresponding European option with exercise price zero and by replacing each $C_i\left[0\right]$ in formula (80) by the higher current time-0 price of the underlying X_i.

4.2 The upper bound as the price of the cheapest super-replicating strategy

The upper bound (80) for $C[K]$ is a linear combination of the prices of observable options on the underlyings X_i and can be interpreted as the price of a static super-replicating strategy. Indeed, the right-hand side of (90) describes the pay-off at time T of a strategy consisting of buying a number of the available European options with pay-off $(X_i - K_{i,j_i})_+$ or $(X_i - K_{i,j_i+1})_+$ at time T_i and investing these pay-offs in the risk free account from time T_i until time T, $i = 1, \ldots, n$.

We will show that the super-replicating strategy corresponding to the upper bound in Theorem 4 is optimal in a broad class of admissible strategies. In the sequel of this section, we consider the class of investment strategies where for each X_i at current time 0, European call options can be bought at any available exercise price and where at exercise date $T_i < T$, the pay-off is invested in the risk free account until time T. The pay-off at time T of any such strategy is given by

$$\sum_{i=1}^{n} \sum_{j=0}^{m_i} e^{\delta(T-T_i)} \nu_{i,j} (X_i - K_{i,j})_+, \tag{95}$$

where $\nu_{i,j}$ is the number of options with pay-off $(X_i - K_{i,j})_+$ at time T_i. The investment strategy with pay-off given by (95) will be denoted by $\underline{\nu}$. The price of this investment strategy is given by

$$\sum_{i=1}^{n} \sum_{j=0}^{m_i} \nu_{i,j} C_i \left[K_{i,j}\right]. \tag{96}$$

As we are only interested in investment strategies that super-replicate the pay-off $(\mathbb{S} - K)_+$ of the exotic option, we will consider investment strategies $\underline{\nu}$ belonging to the following set

$$\bar{\mathcal{A}}_K = \left\{ \underline{\nu} \mid \left(\sum_{i=1}^{n} w_i X_i - K \right)_+ \leq \sum_{i=1}^{n} \sum_{j=0}^{m_i} e^{\delta(T-T_i)} \nu_{i,j} (X_i - K_{i,j})_+ \right\}. \tag{97}$$

Taking into account (90), we find that the super-replicating investment strategy corresponding to the upper bound $e^{-\delta T} \mathrm{E}\left[\left(\bar{\mathbb{S}}^c - K\right)_+\right]$ in Theorem 4 belongs to the set $\bar{\mathcal{A}}_K$. This strategy is given by

$$\nu_{i,j} = \begin{cases} w_i e^{-\delta(T-T_i)} & \text{if } i \in N_K \text{ and } j = j_i, \\ w_i e^{-\delta(T-T_i)} \alpha & \text{if } i \in \overline{N}_K \text{ and } j = j_i, \\ w_i e^{-\delta(T-T_i)} (1-\alpha) & \text{if } i \in \overline{N}_K \text{ and } j = j_i + 1, \end{cases} \tag{98}$$

whereas all other $\nu_{i,j}$ are equal to 0.

In the following theorem, we look for the cheapest super-replicating strategy $\underline{\nu} \in \bar{\mathcal{A}}_K$.

Theorem 5 *Consider the finite market case. For any $K \in (0, \sum_{i=1}^{n} w_i K_{i,m_i+1})$ we have that*

$$e^{-\delta T} E\left[\left(\bar{\mathbb{S}}^c - K\right)_+\right] = \min_{\underline{\nu} \in \bar{\mathcal{A}}_K} \sum_{i=1}^{n} \sum_{j=0}^{m_i} \nu_{i,j} C_i \left[K_{i,j}\right]. \tag{99}$$

Proof. For any $\underline{\nu} \in \bar{\mathcal{A}}_K$ the pay-off inequality (97) is independent of the underlying multivariate distribution function of $(X_1, X_2, \ldots, X_n)$. In particular, it has to hold for the case where $(X_1, X_2, \ldots, X_n)$ has the same distribution as $(F_{\overline{X}_1}^{-1}(U), F_{\overline{X}_2}^{-1}(U), \ldots, F_{\overline{X}_n}^{-1}(U))$. Let us concentrate on this case. Taking expectations of both sides of this inequality and taking into account that

$$e^{-\delta T_i}\mathrm{E}\left[\left(F_{\overline{X}_i}^{-1}(U) - K_{i,j}\right)_+\right] = C_i\left[K_{i,j}\right], \qquad j = 1, \ldots, m_i,\ i = 1, \ldots, n,$$

in the finite market case, we find

$$e^{-\delta T}\mathrm{E}\left[\left(\bar{\mathbb{S}}^c - K\right)_+\right] \leq \sum_{i=1}^{n}\sum_{j=0}^{m_i} \nu_{i,j} C_i\left[K_{i,j}\right], \qquad \underline{\nu} \in \bar{\mathcal{A}}_K.$$

Hence, we also have that

$$e^{-\delta T}\mathrm{E}\left[\left(\bar{\mathbb{S}}^c - K\right)_+\right] \leq \inf_{\underline{\nu} \in \bar{\mathcal{A}}_K} \sum_{i=1}^{n}\sum_{j=0}^{m_i} \nu_{i,j} C_i\left[K_{i,j}\right]. \tag{100}$$

Taking into account (79) and (80), which state that $e^{-\delta T}\mathrm{E}\left[\left(\bar{\mathbb{S}}^c - K\right)_+\right]$ is equal to the price of the investment strategy $\underline{\nu} \in \overline{\mathcal{A}}_K$ defined in (98), we find that the infimum is reached and that (100) holds with equality.

■

The upper bound $e^{-\delta T}\mathrm{E}\left[\left(\bar{\mathbb{S}}^c - K\right)_+\right]$ corresponds to a reasonable offer price for the seller of the exotic option. Indeed, selling the exotic option at this price allows the seller to super-replicate its pay-off. The buying price is defined as the supremum of the price of all strategies with a pay-off below the exotic option's pay-off (see El Karoui and Quenez (1995) or Dana and Jeanblanc-Picqué (1998)). In incomplete markets, the buying price will be in general strictly lower than the selling price and a fair price can be any price in between them. The selling price $e^{-\delta T}\mathrm{E}\left[\left(\bar{\mathbb{S}}^c - K\right)_+\right]$ is the upper bound of the pricing interval since it corresponds to a maximum on the price that the buyer of the exotic option is willing to pay. Indeed, if the exotic option has a higher price, the buyer better purchases the optimal combination of plain vanilla options that super-replicates the pay-off of the exotic option.

From (93) we see that the selling price and so the upper bound of the pricing interval will be lower in case of full marginal information. This result is in correspondence to intuition, as in the infinite market case, the class of admissible super-replicating strategies is larger.

4.3 The upper bound as a worst case expectation

We start this subsection by introducing the class $\overline{\mathcal{R}}_n$ of all (distributions of) n-dimensional random variables with a number of fixed stop-loss premia corresponding to the observable European option prices.

Definition 8 *The class* $\overline{\mathcal{R}}_n$ *of* n*-dimensional random vectors is defined as*

$$\overline{\mathcal{R}}_n = \left\{ (Y_1, Y_2, \dots, Y_n) \mid Y_i \geq 0 \text{ and } e^{-\delta T_i} E\left[(Y_i - K_{i,j})_+ \right] = C_i \left[K_{i,j} \right]; \quad j = 0, \dots, m_i + 1,\ i = 1, \dots, n \right\}. \tag{101}$$

Obviously, the comonotonic random vector $\left(F^{-1}_{\overline{X}_1}(U), F^{-1}_{\overline{X}_2}(U), \dots, F^{-1}_{\overline{X}_n}(U) \right)$ with marginal distributions defined in (67) is an element of $\overline{\mathcal{R}}_n$.

Theorem 6 *In the finite market case it holds that*

$$e^{-\delta T} E\left[\left(\bar{\mathbb{S}}^c - K \right)_+ \right] = \max_{(Y_1, \dots, Y_n) \in \overline{\mathcal{R}}_n} e^{-\delta T} E\left[\left(\sum_{i=1}^n w_i Y_i - K \right)_+ \right]. \tag{102}$$

Proof. For any $(Y_1, Y_2, \dots, Y_n) \in \overline{\mathcal{R}}_n$, the function $e^{-\delta T_i} \mathrm{E}\left[(Y_i - K)_+ \right]$ is a non-negative, decreasing and convex function of K which agrees with the given option prices $C_i \left[K_{i,j} \right]$, $j = 0, \dots, m_i + 1$. On the other hand, from (69), we find that any $\overline{C}_i \left[K \right]$ with $K_{i,j} \leq K \leq K_{i,j+1}$ is a convex linear combination of the option prices $C_i \left[K_{i,j} \right]$ and $C_i \left[K_{i,j+1} \right]$ in the endpoints. Hence, $\overline{C}_i \left[K \right]$ is the largest non-negative decreasing convex function which agrees with the observed option prices $C_i \left[K_{i,j} \right]$, see Figure 2. This implies by Definition 1 that for any $(Y_1, Y_2, \dots, Y_n) \in \overline{\mathcal{R}}_n$ the following increasing convex order relation holds:

$$Y_i \leq_{\mathrm{icx}} \overline{X}_i, \qquad i = 1, \dots, n. \tag{103}$$

From (27), we can conclude that these order relations imply that

$$\sum_{i=1}^n w_i Y_i \leq_{\mathrm{icx}} \bar{\mathbb{S}}^c.$$

Definition 2 then ensures that

$$e^{-\delta T} \mathrm{E}\left[\left(\sum_{i=1}^n w_i Y_i - K \right)_+ \right] \leq e^{-\delta T} \mathrm{E}\left[\left(\bar{\mathbb{S}}^c - K \right)_+ \right]$$

holds for all $(Y_1, Y_2, \dots, Y_n) \in \overline{\mathcal{R}}_n$.
The stated result then follows from the fact that $\left(F^{-1}_{\overline{X}_1}(U), F^{-1}_{\overline{X}_2}(U), \dots, F^{-1}_{\overline{X}_n}(U) \right) \in \overline{\mathcal{R}}_n$. ∎

Theorem 6 states that the upper bound for the price of the exotic option that we derived in Theorem 4 can be interpreted as a worst-case expectation, in the sense that it corresponds to the largest possible expectation of the pay-off of the exotic option, given the finite number of observable European option prices of the underlying plain vanilla options.

4.4 Computational aspects

4.4.1 The coefficient α

An equivalent upper bound as the one in expression (80) was derived in Hobson et al. (2005) for the particular case of basket options. However, they did not prove that for each $i \in \overline{N}_K$ the coefficient α that determines the proportions to be invested in the options with prices $C_i\left[K_{i,j_i}\right]$ and $C_i\left[K_{i,j_i+1}\right]$ does not depend on i. From our approach based on comonotonicity and generalized inverses it turns out naturally that the optimal upper bound can be described via a unique coefficient α, which is determined by (81). This theoretical remark is practical as it speeds up the numerical calculations needed (see also the description of an algorithm in the Appendix). However this interesting observation does not improve e.g. the numerical results for baskets obtained in Hobson et al. (2005) themselves since the sum over $i \in \overline{N}_K$ in (80) reduces to one term. Therefore, we do not include a numerical example in this paper.

We now discuss the artificial strikes K_{i,m_i+1} defined in (66) and their link with α, especially when $\alpha \neq 1$. As mentioned before, we assume all K_{i,m_i+1} to be finite, but 'large enough', see Section 4.1. In that sense, the choice of the K_{i,m_i+1} is somewhat arbitrary. Looking at the expression (81) for $\alpha \neq 1$, one might wonder whether α depends on this arbitrary choice of the strikes K_{i,m_i+1}. In order to answer this question we distinguish two different cases.

Case 1: When $K \leq \sum_{i=1}^{n} w_i K_{i,m_i}$ one has that α does not depend on any K_{i,m_i+1}, $i = 1, \ldots, n$, and the optimal super-replicating strategy does not require any investment in European call options with K_{i,m_i+1} as strike.

In order to prove this statement, notice that from the relations (76)-(78) and (82)-(83), it is clear that α will not depend on any K_{i,m_i+1} when the condition $F_{\overline{\mathbb{S}}^c}(K) < F_{\overline{X}_i}(K_{i,m_i})$ is fulfilled for all i. It remains to prove that $K \leq \sum_{i=1}^{n} w_i K_{i,m_i}$ implies all of these conditions.

Note however that it follows from (67) that the value $F_{\overline{X}_i}(K_{i,m_i})$ depends on K_{i,m_i+1}. In order to be able to give the proof we have to specify what we mean by 'large enough' for the finite value of K_{i,m_i+1}: the artificial strike K_{i,m_i+1} has to be chosen such that from (67) it follows that for all i and i':

$$F_{\overline{X}_i}(K_{i,m_i-1}) < F_{\overline{X}_{i'}}(K_{i',m_{i'}}). \tag{104}$$

This inequalitiy trivially holds when $i = i'$.

Proof. We follow a reasoning ex absurdo. We assume that there exists an i' such that

$$F_{\overline{\mathbb{S}}^c}(K) \geq F_{\overline{X}_{i'}}(K_{i',m_{i'}}), \tag{105}$$

while for all $i \neq i'$ in view of (104) one has that

$$F_{\overline{X}_i}(K_{i,m_i-1}) < F_{\overline{\mathbb{S}}^c}(K) < F_{\overline{X}_i}(K_{i,m_i}).$$

Combining (85)-(86) and (82) we find that

$$K = F_{\bar{\mathbb{S}}^c}^{-1(\alpha)}\left(F_{\bar{\mathbb{S}}^c}(K)\right) = \sum_{i=1}^{n} w_i F_{\overline{X}_i}^{-1(\alpha)}(F_{\bar{\mathbb{S}}^c}(K)) > \sum_{i=1}^{n} w_i K_{i,m_i},$$

where the last inequality is strict, due to the assumption (105).
This last inequality contradicts the initial assumption about K. Hence, we can conclude that the implication

$$K \leq \sum_{i=1}^{n} w_i K_{i,m_i} \quad \Rightarrow \quad F_{\bar{\mathbb{S}}^c}(K) < F_{\overline{X}_i}(K_{i,m_i}) \text{ for all } i$$

is true. ■

Case 2: When $\sum_{i=1}^{n} w_i K_{i,m_i} < K < \sum_{i=1}^{n} w_i K_{i,m_i+1}$ one has that α depends on some K_{i,m_i+1}.
In order to obtain a sharp upper bound, we should be able to observe the 'real' K_{i,m_i+1}, or at least to estimate these in a realistic way. However, in practice this will often not be possible. Therefore, we will investigate the behaviour of the upper bound when the K_{i,m_i+1} converge to infinity. Notice that we also assume that (104) holds as in case 1. Now we first prove the following implication:

$$\sum_{i=1}^{n} w_i K_{i,m_i} < K \quad \Rightarrow \quad \exists i' : F_{\overline{X}_{i'}}(K_{i',m_{i'}}) \leq F_{\bar{\mathbb{S}}^c}(K). \tag{106}$$

Proof. We give a proof ex absurdo. We assume that

$$\forall i : F_{\bar{\mathbb{S}}^c}(K) < F_{\overline{X}_i}(K_{i,m_i}).$$

In this case, we find from (76) and (82) that $F_{\overline{X}_i}^{-1(\alpha)}\left(F_{\bar{\mathbb{S}}^c}(K)\right) \leq K_{i,m_i}$. Hence, in view of (85)-(86), we find

$$K = \sum_{i=1}^{n} w_i F_{\overline{X}_i}^{-1(\alpha)}\left(F_{\bar{\mathbb{S}}^c}(K)\right) \leq \sum_{i=1}^{n} w_i K_{i,m_i},$$

which is in contradiction with the initial assumption on K. ■

We recall that we only consider the case (i) in Theorem 4 where

$$K < \sum_{i=1}^{n} w_i K_{i,m_i+1},$$

which corresponds to $F_{\bar{\mathbb{S}}^c}(K) < 1$, while relations (106) and (104) imply that for all i

$$F_{\overline{X}_i}(K_{i,m_i-1}) < F_{\bar{\mathbb{S}}^c}(K).$$

Hence $j_i(K) = m_i$ or $j_i(K) = m_i + 1$ holds in (82) for i in N_k, while $j_i(K) = m_i$ is valid in (82) for i in $\overline{N}_K$. Recalling that for all i we have that $C[K_{i,m_i+1}] = 0$, we can conclude from the reasoning above that the upper bound (80) equals

$$\begin{aligned}
&\sum_{i\in N_K, j_i=m_i} w_i e^{-\delta(T-T_i)} C_i\left[K_{i,m_i}\right] + \sum_{i\in \overline{N}_K, j_i=m_i} w_i e^{-\delta(T-T_i)} \left(\alpha C_i\left[K_{i,m_i}\right] + (1-\alpha) C_i\left[K_{i,m_i+1}\right]\right) \\
&+ \sum_{i\in N_K, j_i=m_i+1} w_i e^{-\delta(T-T_i)} C_i\left[K_{i,m_i+1}\right] \\
&= \sum_{i\in N_K, j_i=m_i} w_i e^{-\delta(T-T_i)} C_i\left[K_{i,m_i}\right] + \sum_{i\in \overline{N}_K, j_i=m_i} w_i e^{-\delta(T-T_i)} \alpha C_i\left[K_{i,m_i}\right],
\end{aligned} \tag{107}$$

with α given by

$$\frac{\sum_{i\in N_K, j_i=m_i} w_i K_{i,m_i} + \sum_{i\in \overline{N}_K, j_i=m_i} w_i K_{i,m_i+1} + \sum_{i\in N_K, j_i=m_i+1} w_i K_{i,m_i+1} - K}{\sum_{i\in \overline{N}_K, j_i=m_i} w_i \left(K_{i,m_i+1} - K_{i,m_i}\right)}.$$

When we choose all K_{i,m_i+1} to be large enough not only such that $K < \sum_{i=1}^n w_i K_{i,m_i+1}$ but also such that $F_{\overline{X}_i}(K_{i,m_i})$ is converging to one, the set $\overline{N}_K$ will be empty and the upper bound (107) reduces to

$$\sum_{i=1}^n w_i e^{-\delta(T-T_i)} C_i[K_{i,m_i}]. \tag{108}$$

Notice that in case $\sum_{i=1}^n w_i K_{i,m_i} < K$, the upper bound (108) also follows immediately from the inequality

$$\left(\sum_{i=1}^n w_i X_i - K\right)_+ \leq \sum_{i=1}^n w_i (X_i - K_{i,m_i})_+$$

by a no-arbitrage argument.

4.4.2 An algorithm to compute $F_{\bar{\mathbb{S}}^c}(K)$

In order to be able to calculate α defined in (81), one should determine N_K and $\overline{N}_K$ and therefore one should first be able to determine $F_{\bar{\mathbb{S}}^c}(K)$. An algorithm to compute $F_{\bar{\mathbb{S}}^c}(K)$ in the case that $K \leq \sum_{i=1}^n w_i K_{i,m_i}$ is proposed and explained in the Appendix.

4.4.3 The upper bound in terms of $F_{\overline{X}_i}^{-1}$

From (87) one sees that the upper bound $e^{-\delta T}\mathrm{E}\left[\left(\bar{\mathbb{S}}^c - K\right)_+\right]$ for any fair price $C[K]$ of the exotic option with pay-off $(\mathbb{S} - K)_+$ can be interpreted as an upper bound expressed in terms of the inverses $F_{\overline{X}_i}^{-1(\alpha)}$, with α defined by (81). However, the upper bound $e^{-\delta T}\mathrm{E}\left[\left(\bar{\mathbb{S}}^c - K\right)_+\right]$ can also be expressed in terms of the usual inverses $F_{\overline{X}_i}^{-1}$, as is shown in the following corollary.

Corollary 2 *Consider the finite market case. For any $K \in (0, \sum_{i=1}^{n} w_i K_{i,m_i+1})$, one has that*

$$e^{-\delta T} E\left[\left(\bar{\mathbb{S}}^c - K\right)_+\right] = \sum_{i=1}^{n} w_i e^{-\delta(T-T_i)} C_i\left[K_{i,j_i}\right] - e^{-\delta T}\left(K - F_{\bar{\mathbb{S}}^c}^{-1}\left(F_{\bar{\mathbb{S}}^c}(K)\right)\right)\left(1 - F_{\bar{\mathbb{S}}^c}(K)\right), \tag{109}$$

where the indices j_i are defined in (76).

Proof. From (24) we find that the upper bound $e^{-\delta T}\mathrm{E}\left[\left(\bar{\mathbb{S}}^c - K\right)_+\right]$ for $C[K]$ can be expressed as

$$\begin{aligned} & e^{-\delta T}\mathrm{E}\left[\left(\bar{\mathbb{S}}^c - K\right)_+\right] \\ & = \sum_{i=1}^{n} w_i e^{-\delta(T-T_i)} \overline{C}_i\left[F_{\overline{X}_i}^{-1}\left(F_{\bar{\mathbb{S}}^c}(K)\right)\right] - e^{-\delta T}\left(K - F_{\bar{\mathbb{S}}^c}^{-1}\left(F_{\bar{\mathbb{S}}^c}(K)\right)\right)\left(1 - F_{\bar{\mathbb{S}}^c}(K)\right). \end{aligned}$$

From (83) with $\alpha = 1$, we find that $\overline{C}_i\left[F_{\overline{X}_i}^{-1}\left(F_{\bar{\mathbb{S}}^c}(K)\right)\right] = C_i\left[K_{i,j_i}\right]$. Combining these observations leads to the desired result (109). ∎

From Theorem 4 and (109) we find the following upper bound for any fair price $C[K]$ of the exotic option with pay-off $(\mathbb{S} - K)_+$:

$$C[K] \leq \sum_{i=1}^{n} w_i e^{-\delta(T-T_i)} C_i\left[K_{i,j_i}\right]. \tag{110}$$

From (82) and (86) it follows that

$$\sum_{i=1}^{n} w_i K_{i,j_i} = \sum_{i=1}^{n} w_i F_{\overline{X}_i}^{-1}(F_{\bar{\mathbb{S}}^c}(K)) \leq \sum_{i=1}^{n} w_i F_{\overline{X}_i}^{-1(\alpha)}(F_{\bar{\mathbb{S}}^c}(K)) = K.$$

Hence,

$$(\mathbb{S} - K)_+ \leq \sum_{i=1} w_i \left(X_i - K_{i,j_i}\right)_+. \tag{111}$$

The right-hand side of this inequality can be interpreted as the pay-off at time T of a strategy where for each underlying X_i, one buys $w_i e^{-\delta(T-T_i)}$ European options with strike K_{i,j_i}, and holds these options until they expire and invests their pay-offs at expiration in the risk free account until time T. The price of this super-replicating strategy is equal to the upper bound in (110).

Hence, the first term on the right hand side in (109) can be interpreted as the price of a super-replicating strategy for the exotic option where only one plain vanilla European call option is bought for each underlying. The second term on the right hand side in (109) is the difference between the price of the super-replicating strategy corresponding to (110) and the price of the super-replicating strategy in Theorem 4, where for each asset options with different exercise prices may be bought. In this respect, it is worth mentioning the results of Laurence and Wang (2004) who have investigated the smallest super-replicating strategy for a basket option in case one only has knowledge of the prices of the underlying assets, the interest rate and of just one option price $C_i[K_i]$ per underlying asset.

4.5 Convergence proof

In this section we will prove the intuitive result that the finite market case converges to the infinite market case: When in the finite market case the number of plain vanilla option prices that can be observed for each underlying X_i increases, in other words when the number of available strikes for each underlying increases, then the finite market upper bound will converge to the one in the infinite market case.

Proposition 1 *The discrete random variable $\overline{X}_i$ with cdf $F_{\overline{X}_i}$ defined in* (67) *converges in distribution to the random variable X_i of* (1) *with cdf F_{X_i} when m_i tends to $+\infty$ while $h_i = \max_j |K_{ij} - K_{i,j-1}|$ tends to zero.*

Proof. We start from the definition (67) of $F_{\overline{X}_i}(x)$ for $x \in [K_{i,j-1}, K_{ij}[$ and rewrite it in terms of x, having in mind the piecewise linearity of $F_{\overline{X}_i}(x)$:

$$
\begin{aligned}
F_{\overline{X}_i}(x) &= 1 - e^{\delta T_i} \frac{C_i[K_{i,j-1}] - C_i[K_{ij}]}{K_{ij} - K_{i,j-1}} \\
&= 1 + e^{\delta T_i} \frac{C_i[K_{ij}] - C_i[x]}{K_{ij} - x}.
\end{aligned}
$$

When h_i tends to zero, the denominator $K_{ij} - x$ will also do, providing us with the right derivative of $C_i[x]$ which equals:

$$
\lim_{K_{ij} \to x} \frac{C_i[K_{ij}] - C_i[x]}{K_{ij} - x} = C_i'[x+].
$$

Recalling that

$$
F_{X_i}(x) = \Pr[X_i \leq x] = 1 + e^{\delta T_i} C_i'[x+]
$$

we may conclude

$$
\lim_{h_i \to 0} F_{\overline{X}_i}(x) = F_{X_i}(x).
$$

For $x > K_{i,m_i+1}$ we find that since $m_i \to +\infty$ also $x \to +\infty$ and thus that

$$
\lim_{m_i \to +\infty} F_{\overline{X}_i}(x) = \lim_{x \to +\infty} F_{\overline{X}_i}(x) = 1 = \lim_{x \to +\infty} F_{X_i}(x).
$$

Finally we note that for $x < 0$, $F_{\overline{X}_i}(x) = 0 = F_{X_i}(x)$ and certainly in the limit.
Hence we proved that

$$
\lim_{m_i \to +\infty, h_i \to 0} F_{\overline{X}_i}(x) = F_{X_i}(x), \quad \text{for all } x \in \mathbb{R}.
$$

■

Skorohod's theorem (see e.g. Billingsley (1995)) guarantees that there exist random variables $\overline{Y}_i$ and Y_i on a common probability space such that $\overline{Y}_i$ has distribution $F_{\overline{X}_i}$, Y_i has distribution F_{X_i}, and $\overline{Y}_i(\omega) \to Y_i(\omega)$ for each ω. Following the lines of the proof of Skorohod's theorem, we choose $\overline{Y}_i = F^{-1}_{\overline{X}_i}(U)$ within the set of random variables with $F_{\overline{X}_i}$

as distribution and similarly within the set of random variables with F_{X_i} as distribution we choose $Y_i = F_{X_i}^{-1}(U)$. Then $F_{\overline{X}_i}^{-1}(U)$ and $F_{X_i}^{-1}(U)$ are defined on the same probability space since they are driven by the same uniform $(0,1)$-random variable U and $F_{\overline{X}_i}^{-1}(U)$ converges almost surely to $F_{X_i}^{-1}(U)$, namely on the set of continuity points. Addition and multiplication preserve convergence with probability 1.
Denote $m = \min_i m_i$ and $h = \max_i h_i$. When m tends to infinity and h to zero, then all m_i tend to infinity while all h_i tend to zero and we can state the following result.

Proposition 2 *The comonotonic sum* $\bar{\mathbb{S}}^c$ (73) *converges almost surely to* $\mathbb{S}^c$ (28) *for* $m \to +\infty$ *and* $h \to 0$, *when* $\bar{\mathbb{S}}^c$ *and* $\mathbb{S}^c$ *are driven by the same uniform* $(0,1)$*-random variable.*

Now we come to the main result:

Theorem 7 *The upper bound* $e^{-\delta T}E[(\bar{\mathbb{S}}^c - K)_+]$ (79) *in the finite market case converges to the upper bound* $e^{-\delta T}E[(\mathbb{S}^c - K)_+]$ (31) *in the infinite market case when* $m \to +\infty$ *and* $h \to 0$.

Proof. Without loss of generality, we assume that $\bar{\mathbb{S}}^c$ and $\mathbb{S}^c$ are driven by the same uniform $(0,1)$-random variable.
Since convergence with probability 1 implies convergence in distribution and the function $(K - \cdot)_+$ is bounded and continuous, we obtain by the Helly-Bray theorem that in view of Proposition 2 it holds that

$$\lim_{m\to+\infty, h\to 0} E[(K - \bar{\mathbb{S}}^c)_+] = E[(K - \mathbb{S}^c)_+].$$

Next, we know that

$$E[(\bar{\mathbb{S}}^c - K)_+] = E[\bar{\mathbb{S}}^c] - K + E[(K - \bar{\mathbb{S}}^c)_+]$$

and find that

$$\begin{aligned} E[\bar{\mathbb{S}}^c] &= \sum_{i=1}^n w_i E[\overline{X}_i] \\ &= \sum_{i=1}^n w_i e^{\delta T_i} \overline{C}_i[0] \overset{(71)}{=} \sum_{i=1}^n w_i e^{\delta T_i} C_i[0] \\ &= \sum_{i=1}^n w_i E[X_i] = E[\mathbb{S}] = E[\mathbb{S}^c], \end{aligned}$$

implying that we do not only have stop-loss ordering but also convex ordering for $\bar{\mathbb{S}}^c$ and $\mathbb{S}^c$!
Thus we have

$$E[(\bar{\mathbb{S}}^c - K)_+] = E[\mathbb{S}^c] - K + E[(K - \bar{\mathbb{S}}^c)_+]$$

and

$$\begin{aligned}
\lim_{m\to+\infty,h\to0} E[(\bar{\mathbb{S}}^c - K)_+] &= E[\mathbb{S}^c] - K + \lim_{m\to+\infty,h\to0} E[(K - \bar{\mathbb{S}}^c)_+] \\
&= E[\mathbb{S}^c] - K + E[(K - \mathbb{S}^c)_+] \\
&= E[(\mathbb{S}^c - K)_+].
\end{aligned}$$

■

5 Conclusions

In this paper, we investigated super-replicating strategies for European-type call options written on a weighted sum $\mathbb{S} = w_1X_1 + \cdots + w_nX_n$. This class of exotic options includes Asian options and basket options among others.

Firstly, we assumed that for each underlying X_i the prices $C_i[K]$ of the European calls with pay-off $(X_i - K)_+$ are known for all $K \geq 0$ or, equivalently, full knowledge of the pricing distributions of the respective X_i is available. Using the theory on comonotonicity, we proved that in a very broad class of admissible investment strategies that super-replicate the pay-off of the exotic option with pay-off $(S - K)_+$, and the cheapest one is the one that consists of buying exactly one plain vanilla option per underlying X_i. The price of this optimal super-replicating portfolio is given by $e^{-\delta T}\mathrm{E}\left[(\mathbb{S}^c - K)_+\right]$.

A first situation where these results can be applied is the case where there exists a market where all option prices $C_i[K]$ can be observed for all exercise prices $K \geq 0$. In this case, we don't have to make an assumption about the underlying pricing process, and therefore, such an approach is called 'model-free'. It is clear that this is a purely theoretical situation, as in reality, there will only be a limited (finite) number of options traded on each underlying X_i. A second situation where we could apply these results arises when we make an assumption concerning the underlying pricing distributions F_{X_i}. Even in the case where the multivariate pricing distribution of the random vector $(X_1, X_2, \ldots, X_n)$ is known, the results may still be useful, as in this case determining the price $C[K]$ of the exotic option will often be not straightforward, mainly because of the dependency that exists among the X_i. In this case, the use of an easy computable upper bound in terms of the marginal distributions involved may be helpful.

Secondly, we explored a model-free approach and assumed that only finitely many strikes are traded per underlying X_i. Again using the theory on comonotonicity, we derived an upper bound for the price of the exotic option with pay-off $(\mathbb{S} - K)_+$. This bound can be interpreted as the cheapest super-replicating strategy in a broad class of super-replicating strategies consisting of buying the available European options. This optimal strategy consists of buying at most two plain vanilla options per underlying X_i. The price of this optimal super-replicating portfolio is given by $e^{-\delta T}\mathrm{E}\left[\left(\bar{\mathbb{S}}^c - K\right)_+\right]$. We proved that this price converges to the price $e^{-\delta T}\mathrm{E}\left[(\mathbb{S}^c - K)_+\right]$ of the optimal super-replicating portfolio in the infinite market case when the number of strikes, and hence the number of the observed vanilla call option prices, for each underlying X_i tends to infinity.

Many results presented in this paper are closely related to results in Hobson et al. (2005). We generalized and at the same time simplified their approach by deriving the bounds from basic results available in the theory on comonotonic risks and the theory on integral stochastic orderings. A related problem to the one we considered in this paper is to determine static hedging strategies for exotic options in case one can only buy European plain vanilla options on a subset of all underlying X_i. For the case of basket options, this problem is considered in Su (2005).

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References

[1] Albrecher, H., Dhaene, J., Goovaerts, M.J. and Schoutens, W. (2005). Static hedging of Asian options under Lévy models: the comonotonicity approach. *The Journal of Derivatives*, vol. 12(3), 63-72.

[2] Billingsley, P. (1995). *Probability and Measure*. New York : Wiley (3rd ed.), pp. 593.

[3] Breeden, D.T. and Litzenberger, R.H. (1978). Prices of state-contingent claims implicit in option prices. *Journal of Business*, vol. 51, 621-651.

[4] Dana, R.-A. and Jeanblanc-Picqué, M. (1998). *Marchés Financiers en Temps Continu*. Economica, p. 330.

[5] Deelstra, G., Diallo, I. and Vanmaele, M. (2006). Bounds for Asian basket options. *Journal of Computational and Applied Mathematics*, doi:10.1016/j.cam.2006.12.017.

[6] Denuit, M., Dhaene, J., Goovaerts, M. and Kaas, R. (2005). *Actuarial Theory for Dependent Risks – Measures, Orders and Models*. Wiley, pp. 440.

[7] Dhaene, J., Denuit, M., Goovaerts, M.J., Kaas, R. and Vyncke, D. (2002a). The concept of comonotonicity in actuarial science and finance: theory. *Insurance: Mathematics & Economics*, vol. 31(1), 3-33.

[8] Dhaene, J., Denuit, M., Goovaerts, M.J., Kaas, R. and Vyncke, D. (2002b). The concept of comonotonicity in actuarial science and finance: applications. *Insurance: Mathematics & Economics*, vol. 31(2), 133-161.

[9] Dhaene, J., Wang, S., Young, V. and Goovaerts, M.J. (2000). Comonotonicity and maximal stop-loss premiums. *Bulletin of the Swiss Association of Actuaries*, vol. 2, 99-113.

[10] Duffie, D. and Kan, R. (1996). On the term structure of interest rates. *Mathematical Finance*, vol. 6(4), 379-406.

[11] El Karoui, N. and Quenez, M.C. (1995). Programmation dynamique et évaluation des actifs contingents en marché incomplet. *Siam Journal of Control and Optimization*, vol. 33, 29-66.

[12] Goovaerts, M., Dhaene, J. and De Schepper, A. (2000). Stochastic upper bounds for present value functions. *The Journal of Risk and Insurance*, vol. 67(1), 1-14.

[13] Hobson, D., Laurence, P. and Wang, T.H. (2005). Static-arbitrage upper bounds for the prices of basket options. *Quantitative Finance*, vol. 5(4), 329-342.

[14] Jamshidian, F. (1989). An exact bond option formula. *Journal of Finance*, vol. XLIV(1), 205-209.

[15] Kaas, R., Dhaene, J. and Goovaerts, M. (2000). Upper and lower bounds for sums of random variables. *Insurance: Mathematics & Economics*, vol. 27(2), 151-168.

[16] Kaas, R., Goovaerts, M., Dhaene, J., Denuit, M. (2001). *Modern Actuarial Risk Theory*. Kluwer Academic Publishers, pp. 309.

[17] Kaas, R., Dhaene, J., Vyncke, D., Goovaerts, M. and Denuit, M. (2002). A simple geometric proof that comonotonic risks have the convex-largest sum. *Astin Bulletin*, vol. 32(1), 71-80.

[18] Laurence, P. and Wang, T.H. (2004). What's a basket worth?. *Risk Magazine*, vol. 17, 73-77.

[19] Meilijson, I. and Nadas, A. (1979). Convex majorization with an application to the length of critical paths. *Journal of Applied Probability*, vol. 16, 671-676.

[20] Müller, A. (1997). Stop-loss order for portfolios of dependent risks. *Insurance: Mathematics & Economics*, vol. 21, 219-223.

[21] Nielsen, J.A. and Sandmann, K. (2003). Pricing bounds on Asian options. *Journal of Financial and Quantitative Analysis*, vol. 38(2), 449-473.

[22] Reynaerts, H., Vanmaele, M., Dhaene, J. and Deelstra, G. (2006). Bounds for the price of a European-style Asian option in a binary tree model. *European Journal of Operational Research*, vol. 168(2), 322-332.

[23] Rüschendorf, L. (1983). Solution of statistical optimization problem by rearrangement methods. *Metrika*, vol. 30, 55-61.

[24] Shaked, M. and Shanthikumar, J.G. (1994). *Stochastic Orders and their Applications*. Academic press, pp. 545.

[25] Simon, S., Goovaerts, M. and Dhaene, J. (2000). An easy computable upper bound for the price of an arithmetic Asian option. *Insurance: Mathematics & Economics*, vol. 26(2-3), 175-184.

[26] Su, X. (2005). Hedging basket options by using a subset of underlying assets. *working paper*.

[27] Vanmaele, M., Deelstra, G., Liinev, J., Dhaene, J. and Goovaerts, M. (2006). Bounds for the price of discrete Asian options. *Journal of Computational and Applied Mathematics*, vol. 185(1), 51-90.

[28] Vasicek, O. (1977). An equilibrium characterization of the term structure. *Journal of Financial Economics*, vol. 5(November), 177-188.

[29] Wang, S. and Dhaene, J. (1998). Comonotonicity, correlation order and stop-loss premiums. *Insurance: Mathematics & Economics*, vol. 22(3), 235-243.

Appendix

An algorithm to compute $F_{\bar{\mathbb{S}}^c}(K)$ in the case that $K \leq \sum_{i=1}^n w_i K_{i,m_i}$:

From (18) we find that $F_{\bar{\mathbb{S}}^c}(K)$ can be determined from

$$F_{\bar{\mathbb{S}}^c}(K) = \sup\left\{p \in [0,1] \mid \sum_{i=1}^n w_i F_{\overline{X}_i}^{-1}(p) \leq K\right\}, \tag{112}$$

and also notice that $F_{\bar{\mathbb{S}}^c}(K)$ is equal to one of the $F_{\overline{X}_i}(K_{i,j})$ with $j = 0, 1, \ldots, m_i$ and $i = 1, \ldots, n$. And in this algorithm and only in this algorithm, $\sum_{i=1}^n w_i = 1$ is made as an assumption, so accordingly, after a rescaling, the strike K and the random sum $\bar{\mathbb{S}}^c$ in (112) are the adjusted ones, but the $F_{\overline{X}_i}$ are the same and the result p does not differ from the original one.

Taking into account (70), we find that for any i and for any p the value of $F_{\overline{X}_i}^{-1}(p)$ is given by an element of the set $\{K_{i,0}, K_{i,1}, \ldots, K_{i,m_i}, K_{i,m_i+1}\}$. Hence

$$\sum_{i=1}^n w_i F_{\overline{X}_i}^{-1}(p) \leq K \qquad \Rightarrow \qquad \sum_i w_i K_{i,j} \leq K.$$

In particular, for each term in the sum it should hold that $w_i K_{i,j} \leq K$ or, equivalently,

$$K_{i,j} \leq \frac{K}{w_i}.$$

This observation might limit the number of available strikes that we have to take into account in order to find the supremum in (112).

We have that

$$\min_i F_{\overline{X}_i}(K) \leq F_{\bar{\mathbb{S}}^c}(K). \tag{113}$$

To see this, assume that

$$F_{\bar{\mathbb{S}}^c}(K) < \min_i F_{\overline{X}_i}(K). \tag{114}$$

Under this assumption, we find from the definition of $F_{\overline{X}_i}^{-1}$ that

$$\sum_i w_i F_{\overline{X}_i}^{-1}(F_{\bar{\mathbb{S}}^c}(K)) \leq \sum_i w_i F_{\overline{X}_i}^{-1}(\min_k F_{\overline{X}_k}(K)) \leq K \tag{115}$$

where in the last inequality, we used the fact that $F_{\overline{X}_i}^{-1}(\min_k F_{\overline{X}_k}(K)) \leq K$ holds for all i, and the assumption $\sum_{i=1}^n w_i = 1$. The relations (114)-(115) are in contradiction with definition (112) that states that $F_{\bar{\mathbb{S}}^c}(K)$ is a supremum. We can conclude that the assumption (114) is wrong and that (113) holds.

Obviously, when $K \geq K_{i,m_i+1}$ for some i, we have that $\max_i F_{\overline{X}_i}(K)$ will equal one and we can conclude that

$$\min_i F_{\overline{X}_i}(K) \leq F_{\bar{\mathbb{S}}^c}(K) \leq 1. \tag{116}$$

On the other hand, when $K < K_{i,m_i+1}$ for all i, we have that $\max_i F_{\overline{X}_i}(K)$ is strictly smaller than one and $F_{\bar{\mathbb{S}}^c}(K)$ satisfies

$$\min_i F_{\overline{X}_i}(K) \leq F_{\bar{\mathbb{S}}^c}(K) \leq \max_i F_{\overline{X}_i}(K) < 1. \tag{117}$$

In order to prove (117), assume that

$$\max_i F_{\overline{X}_i}(K) < F_{\bar{\mathbb{S}}^c}(K). \tag{118}$$

Then we have that

$$\sum_i w_i F_{\overline{X}_i}^{-1}(F_{\bar{\mathbb{S}}^c}(K)) > K,$$

since $F_{\overline{X}_i}^{-1}(F_{\bar{\mathbb{S}}^c}(K)) > K$ for all i and $\sum_{i=1}^n w_i = 1$. This conclusion contradicts

$$\sum_i w_i F_{\overline{X}_i}^{-1}(F_{\bar{\mathbb{S}}^c}(K)) \leq \sum_i w_i F_{\overline{X}_i}^{-1(\alpha)}(F_{\bar{\mathbb{S}}^c}(K)) = K.$$

We can conclude that the assumption (118) is wrong, which proves that $F_{\bar{\mathbb{S}}^c}(K)$ lies in the interval (117).

The observations above imply that it is sufficient to compute and rank in ascending order in one list the values $F_{\overline{X}_i}(K_{i,j})$ for all i and, given i, for those $K_{i,j}$ belonging to the interval

$$\Big[F_{\overline{X}_i}^{-1}(\min_k F_{\overline{X}_k}(K)), F_{\overline{X}_i}^{-1}(\max_k F_{\overline{X}_k}(K))\Big] \cap [K_{i,0}, \frac{K}{w_i}] \qquad \text{if } \exists \ell : \max_k F_{\overline{X}_k}(K) = F_{\overline{X}_i}(K_{i,\ell})$$

$$\Big[F_{\overline{X}_i}^{-1}(\min_k F_{\overline{X}_k}(K)), F_{\overline{X}_i}^{-1}(\max_k F_{\overline{X}_k}(K))\Big[\cap [K_{i,0}, \frac{K}{w_i}] \qquad \text{if } \forall \ell : \max_k F_{\overline{X}_k}(K) \neq F_{\overline{X}_i}(K_{i,\ell})$$

In contrast to the suggested algorithm in Hobson et al. (2005), for most of the i we will not have to evaluate $F_{\overline{X}_i}$ in all available strikes.

We propose to search the interval $[\min_i F_{\overline{X}_i}(K), \max_i F_{\overline{X}_i}(K)]$ to find $F_{\bar{\mathbb{S}}^c}(K)$ as follows: We start by computing $\sum_i w_i F_{\overline{X}_i}^{-1}(\max_i F_{\overline{X}_i}(K))$. When this sum is equal to K, the algorithm stops and $F_{\bar{\mathbb{S}}^c}(K) = \max_i F_{\overline{X}_i}(K)$. Otherwise, we suggest to work upwards and downwards in the list at the same time in order to narrow the interval. When moving upwards, the consecutive values $F_{\overline{X}_u}(K_{u,j(u)})$ and $F_{\overline{X}_v}(K_{v,j(v)})$ in the list result in

$$\sum_i w_i F_{\overline{X}_i}^{-1}(F_{\overline{X}_u}(K_{u,j(u)})) \leq K < \sum_i w_i F_{\overline{X}_i}^{-1}(F_{\overline{X}_v}(K_{v,j(v)})),$$

or, while moving downwards, the consecutive values $F_{\overline{X}_c}(K_{c,j(c)})$ and $F_{\overline{X}_d}(K_{d,j(d)})$ in the list lead to

$$\sum_i w_i F_{\overline{X}_i}^{-1}(F_{\overline{X}_d}(K_{d,j(d)})) \leq K < \sum_i w_i F_{\overline{X}_i}^{-1}(F_{\overline{X}_c}(K_{c,j(c)})),$$

the algorithm stops. In the first case $F_{\bar{\mathbb{S}}^c}(K) = F_{\overline{X}_u}(K_{u,j(u)})$, while in the latter case $F_{\bar{\mathbb{S}}^c}(K) = F_{\overline{X}_d}(K_{d,j(d)})$.

When moving up or down in the list from value $F_{\overline{X}_k}(K_{k,j(k)})$ to value $F_{\overline{X}_\ell}(K_{\ell,j(\ell)})$ and when there are no other assets for which $F_{\overline{X}_i}(K_{i,j(i)})$ coincides with one of these two values, the sums $\sum_i w_i F^{-1}_{\overline{X}_i}(F_{\overline{X}_k}(K_{k,j(k)}))$ and $\sum_i w_i F^{-1}_{\overline{X}_i}(F_{\overline{X}_\ell}(K_{\ell,j(\ell)}))$ will differ by only one term. Indeed, $w_k F^{-1}_{\overline{X}_k}(F_{\overline{X}_k}(K_{k,j(k)})) = w_k K_{k,j(k)}$ will be replaced by $w_k K_{k,j(k)+1}$ when going up, while $w_\ell F^{-1}_{\overline{X}_\ell}(F_{\overline{X}_k}(K_{k,j(k)})) = w_\ell K_{\ell,j(\ell)+1}$ is substituted by $w_\ell K_{\ell,j(\ell)}$ when going down. This observation allows us to further optimize the computations.