Remarks on 'Boundary Crossing Result for Brownian Motion.'

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Introduction

In "Boundary Crossing Result for Brownian Motion" (1993), Teunen and Goovaerts are interested in the evaluation of the first-passage density of the surplus to a certain boundary $h(t)$. Since the management of the company can decide to pay a dividend once the surplus has crossed the boundary, it is interesting to know the crossing probability $I\!P\left(\sup_{t_o \le t < t_1}(x(t) - h(t)) \ge 0\right)$, where the surplus $x(t)$ is described by e.g. a Gaussian process.

Using path integrals, Teunen and Goovaerts obtain by the methodology of Kac some explicit expressions for the probability of Brownian motion crossing a piecewise linear boundary. Spitefully, this methodology results in complicated differential equations and involves difficult calculations. In Scheike (1990), the results were obtained using scaling properties and time inversion for Brownian motion and there is no need to solve complicated equations.

In a previous paper "Remarks on the methodology introduced by Goovaerts et al." (1992), we have shown how the path integral models are related to stochastic differential equations. Explicit expressions for the crossing probability follow immediately from the theory of Wiener processes and from Girsanov's theorem.

In the theory of Brownian motion, see for example Revuz-Yor(1991), the calculations of such crossing probabilities are standard. In case of a horizontal line boundary, the problem is known as the "ruin problem". The formula which states that the density of the first-passage time of Brownian motion over the boundary $\psi(t) = \Lambda + bt$ is given by

$$p(t) = \frac{\Lambda}{t^{3/2}}\Phi\left(\frac{\psi(t)}{\sqrt{t}}\right) \qquad \text{with} \quad \Phi(y) = \frac{1}{\sqrt{2\pi}}e^{-y^2/2t},$$

is called the Bachelier-Levy formula, see e.g. Lerche (1986). Indeed, Levy (1948) refers in "Processus stochastiques et mouvement Brownien." to Bachelier who has already treated first-passage densities in 1900 in his "Théorie de la Spéculation."

Meanwhile, the list of authors who have worked on first-passage times is too long to mention them all. Therefore, I will name only some of them.

Keilson (1963) has shown that for one class of processes $X(t)$ the distribution of the first passage time has an especially simple expression in terms of the

distribution of $X(t)$. Borovkov (1965) observes one class of processes with independent increments but the expression remains intractable analytically.
In Robbins and Siegmund (1970) and Lerche (1986), the crossing probability is obtained for a class of functions $h(t)$ which are the solutions to

$$f(x,t) \equiv \int_0^\infty \exp(\vartheta x - \frac{1}{2}\vartheta^2 t) F(d\vartheta) = a,$$

where $F(\vartheta)$ is a positive σ-finite measure and $a > 0$. Remark that this class contains no piecewise linear boundaries.
Under mild conditions, Durbin (1971, 1985) obtained an explicit expression for the first-passage density of a continuous Gaussian process to a general boundary. However, this expression is too hard to compute so that we have to resort to numerical methods of solution.
A lot of work has been done on asymptotic estimates, e.g. by Daniels (1974), Cuzick (1981), Jennen and Lerche (1981).

In the rest of the paper, we will show how to obtain the same results as Teunen and Goovaerts, but without solving differential equations. By repeated applications of Girsanov's theorem, the Markov property and the reflection principle of Brownian motions - see for example Revuz and Yor (1991) p.105 - the case of a piecewise linear boundary can be transformed to the known result of the ruin problem.

In the first section, we will give the reasoning for a line boundary $a+bt$ for finite and for infinite time. Then, we will extend the result for a boundary consisting of two lines for infinite and for finite time. Further generalizations proceed in the same way.

1. A line boundary

We will use a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, I\!\!P)$, where Ω denotes the Wiener space and $I\!\!P$ denotes the Wiener measure such that the coordinate process $(X_t)_{t\geq 0}$ is a Brownian Motion with $I\!\!P[X_o = 0] = 1$. Let $I\!\!P_a$ denote the measure such that $I\!\!P_a[X_0 = a] = 1$ and $(X_t)_{t\geq 0}$ is a Brownian motion starting in a. We suppose that the natural filtration satisfies the usual assumptions. This means that $\mathcal{F}_o$ contains all null sets of $\mathcal{F}$ and that the filtration is right continuous.

In this section, we are interested in finding an expression for

$$I\!\!P\left(\sup_{0\leq t\leq T}(B_t - bt) \geq a\right)$$

where $(B_t)_{t\geq 0}$ denotes a Brownian motion.
Let us define the stopping time $\tau = \inf\{t \mid B_t = a + bt\}$. Then,

$$\mathbb{P}\left(\sup_{0\leq t\leq T}(B_t - bt) \geq a\right) = \mathbb{P}(\tau < T)$$

Since $\left(e^{2bB_t - 2b^2 t}\right)_{t\geq 0}$ is a martingale, we obtain the equality

$$\int_{\{\tau<T\}} e^{2bB_T - 2b^2 T} d\mathbb{P} = \int_{\{\tau<T\}} e^{2bB_\tau - 2b^2\tau} d\mathbb{P}$$

By the definition of the stopping time τ, the right hand side equals $e^{2ba}\mathbb{P}(\tau < T)$.
Consequently,

$$\mathbb{P}(\tau < T) = e^{-2ba}\int_{\{\tau<T\}} e^{2bB_T - 2b^2 T} d\mathbb{P} \tag{1}$$

By Girsanov's theorem, $(B_t - bt)_{t\geq 0}$ is a Brownian motion under the measure $\mathbb{Q}$ defined by $\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{bB_T - b^2T/2}$. We rewrite the crossing probability and apply Girsanov's theorem:

$$\begin{aligned}
\mathbb{P}(\tau < T) &= e^{-2ba}\int_{\{\sup_{0\leq t\leq T}(B_t - bt)\geq a\}} e^{bB_T - b^2T/2} e^{b(B_T - bT)} e^{-b^2T/2} d\mathbb{P} \\
&= e^{-2ba}\int_{\{S_T = \sup_{0\leq t\leq T} B_t \geq a\}} e^{bB_T} e^{-b^2T/2} d\mathbb{P}
\end{aligned}$$

Thanks to the reflection principle, we know the density of the pair (B_t, S_t) where $S_t = \sup_{u\leq t} B_u$. For $a > 0$, we have to consider two possibilities (see Revuz-Yor, p.105):

$$\begin{cases}
\mathbb{P}[S_t > a, B_t < x] = \mathbb{P}_{2a}[B_t < x] = \mathbb{P}[B_t < x - 2a] & \text{for } x \leq a \\
\mathbb{P}[S_t > a, B_t > x] = \mathbb{P}[B_t > x] & \text{for } x > a
\end{cases}$$

Thus, we have to divide our integral into two parts:

$$\mathbb{P}(\tau < T) = e^{-2ba}\int_{-\infty}^{a} e^{bx - b^2T/2} f_{B_T}(x - 2a)dx + e^{-2ba}\int_{a}^{\infty} e^{bx - b^2T/2} f_{B_T}(x)dx$$

where $f_{B_T}(x)$ denotes the density of B_T, namely $f_{B_T}(x) = \frac{1}{\sqrt{2\pi T}} e^{-\frac{x^2}{2T}}$. We will rewrite the equality as a combination of the normal cumulative distribution function $\Phi(u) = \int_{-\infty}^{u} \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy$.

$$\mathbb{P}(\tau < T) = e^{-2ba}\left(\int_{-\infty}^{a} e^{bx - b^2T/2}\frac{e^{-\frac{1}{2T}(x-2a)^2}}{\sqrt{2\pi T}}dx + \int_{a}^{\infty} e^{bx - b^2T/2}\frac{e^{-\frac{x^2}{2T}}}{\sqrt{2\pi T}}dx\right)$$

$$\begin{aligned}
&= e^{-2ba}\int_{-\infty}^{a}\frac{1}{\sqrt{2\pi T}}e^{-\frac{1}{2T}(x-2a-bT)^2}e^{2ba}dx + e^{-2ba}\int_{a}^{\infty}\frac{1}{\sqrt{2\pi T}}e^{-\frac{1}{2T}(x-bT)^2}dx \\
&= \int_{-\infty}^{\frac{-a-bT}{\sqrt{T}}}\frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy + e^{-2ba}\int_{\frac{a-bT}{\sqrt{T}}}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy \\
&= \Phi(\frac{-a-bT}{\sqrt{T}}) + e^{-2ba}\Phi(-\frac{a-bT}{\sqrt{T}}) \\
&= 1-\Phi(\frac{a}{\sqrt{T}}+b\sqrt{T}) + e^{-2ba}\Phi(b\sqrt{T}-\frac{a}{\sqrt{T}})
\end{aligned}$$

Thus, for $a > 0$, the crossing probability for the boundary $a + bt$ is given by:

$$I\!P\left(\sup_{0\le t\le T}(B_t - bt) \ge a\right) = 1-\Phi(\frac{a}{\sqrt{T}}+b\sqrt{T}) + e^{-2ba}\Phi(b\sqrt{T}-\frac{a}{\sqrt{T}})$$

For $a \le 0$, this probability equals 1.
For $T \to \infty$, we immediately obtain the following result:

$$I\!P\left(\sup_{0\le t\le\infty}(B_t - bt) \ge a\right) = e^{-2ba}$$

for $a, b > 0$. If $a \le 0$ or $b \le 0$ the probability is 1.

2. Boundary consisting of two lines in infinite time

Let $h(t)$ be the boundary of the form

$$\begin{cases} h(t) = a_1 + b_1 t & \text{for } t < T \\ h(t) = a_2 + b_2 t & \text{for } T \le t \end{cases}$$

We want to derive an explicit expression for the crossing probability:

$$I\!P\left(\sup_{0\le t<\infty}(B_t - h(t)) \ge 0\right) \tag{2}$$

Again, let us define a stopping time

$$\tau = \inf\{t \mid t < T \;\; X_t = a_1 + b_1 t, t \ge T \;\; X_t = a_2 + b_2 t\}$$

so that

$$I\!P\left(\sup_{0\le t<\infty}(B_t - h(t)) \ge 0\right) = I\!P(\tau < \infty)$$

Trivially,

$$I\!P(\tau < \infty) = I\!P(\tau \le T) + I\!P(\tau > T, \tau < \infty)$$

Let us first have a look at the first probability on the right hand side.

$$\mathbb{P}(\tau \leq T) = \mathbb{P}\left((\sup_{0\leq t\leq T}(B_t - b_1 t) \geq a_1) \text{ or } B_T \geq \min(a_1 + b_1 T, a_2 + b_2 T)\right)$$

Let us denote the minimum of $a_1 + b_1 T$ and $a_2 + b_2 T$ by k, then

$$\mathbb{P}(\tau \leq T) = \mathbb{P}(B_T \geq k) + \mathbb{P}\left(\sup_{0\leq t\leq T}(B_t - b_1 t) \geq a_1 \text{ and } B_T \leq k\right)$$

Since $\mathbb{P}(B_T \geq k) = 1 - \Phi(\frac{k}{\sqrt{T}})$, it remains to have a look at the second probability:

$$\begin{aligned}&\mathbb{P}\left((\sup_{0\leq t\leq T}(B_t - b_1 t) \geq a_1) \text{ and } B_T \leq k\right)\\ &= \int_{\{\sup_{0\leq t\leq T}(B_t - b_1 t)\geq a_1, B_T - b_1 T \leq k - b_1 T\}} e^{b_1 B_T - b_1^2 T/2} e^{-b_1(B_T - b_1 T)} e^{-b_1^2 T/2} d\mathbb{P}\end{aligned}$$

By Girsanov's theorem, $(B_t - b_1 t)_{t\geq 0}$ is a Brownian motion under the measure $\mathbb{Q}$ defined by $\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{b_1 B_T - b_1^2 T/2}$. If we apply Girsanov's theorem, we find:

$$\int_{\{\sup_{0\leq t\leq T} B_t \geq a_1, B_T \leq k - b_1 T\}} e^{-b_1 B_T} e^{-b_1^2 T/2} d\mathbb{P}$$

Noticing that $k - b_1 T \leq a_1$, we can use the reflection principle for $a_1 > 0$ and repeat the reasoning of the first section:

$$\begin{aligned}&\int_{-\infty}^{k-b_1 T} e^{-b_1 x - b_1^2 T/2} f_{B_T}(x - 2a_1) dx\\ &= \int_{-\infty}^{k-b_1 T} e^{-b_1 x - b_1^2 T/2} \frac{1}{\sqrt{2\pi T}} e^{-\frac{1}{2T}(x-2a_1)^2} dx\\ &= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{k-b_1 T} e^{-\frac{1}{2T}(x+b_1 T - 2a_1)^2} e^{-2a_1 b_1} dx\\ &= \frac{e^{-2a_1 b_1}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{k-2a_1}{\sqrt{T}}} e^{-y^2/2} dy\\ &= e^{-2a_1 b_1} \Phi\left(\frac{k-2a_1}{\sqrt{T}}\right)\end{aligned}$$

Thus, summarizing, we have shown that:

$$\mathbb{P}(\tau \leq T) = 1 - \Phi(\frac{k}{\sqrt{T}}) + e^{-2a_1 b_1} \Phi\left(\frac{k-2a_1}{\sqrt{T}}\right)$$

It remains to calculate the probability $I\!P(\tau > T, \tau < \infty)$. First, we remark that this equals $I\!E\left[I\!P(\tau > T, \tau < \infty \mid \mathcal{F}_T)\right]$. We notice that for $\tau > T$, τ can be rewritten as $\tau = T + \sigma \circ \vartheta_T$ with ϑ the shiftoperator and σ a stopping time, namely $\sigma = \inf\{s \mid X_s \geq a_2 + b_2(s+T)\}$. We now apply the strong Markov property:

$$I\!P(\tau > T, \tau < \infty) = \int_{\{\tau > T\}} I\!P_{B_T}[\sigma < \infty]$$

If we define a new stopping time $\sigma' = \inf\{s \mid X_s \geq a_2 + b_2(s+T) - X_T\}$, the process starts at 0:

$$I\!P(\tau > T, \tau < \infty) = \int_{\{\tau > T\}} I\!P_o[\sigma' < \infty]$$

As we have shown in section 1, there exists an explicit expression for $I\!P_o[\sigma' < \infty]$, namely for $b_2 > 0$: $I\!P_o[\sigma' < \infty] = e^{-2(a_2 + b_2 T - B_T)b_2}$. We substitute this expression in the last equation:

$$I\!P(\tau > T, \tau < \infty) = e^{-2a_2b_2} \int_{\{\tau > T\}} e^{2b_2(B_T - b_2T)}$$

$$= \quad e^{-2a_2b_2} \int_{\{\sup_{0\leq t\leq T}(B_t - b_1 t) \leq a_1, B_T - 2b_2T \leq k - 2b_2T\}} e^{2b_2(B_T - b_2T)}$$

By Girsanov's theorem, $(B_t - 2b_2t)_{t\geq 0}$ is a Brownian motion under the measure $Q\!\!\!\!Q$ defined by $\frac{dQ\!\!\!\!Q}{dI\!P} = e^{2b_2B_T - 2b_2^2T}$. If we apply Girsanov's theorem, we find:

$$I\!P(\tau > T, \tau < \infty) = e^{-2a_2b_2} I\!P\left[\sup_{0\leq t\leq T}(B_t - b_1t + 2b_2t) \leq a_1, B_T \leq k - 2b_2T\right]$$

$$= \quad e^{-2a_2b_2}\left(1 - I\!P[B_T \geq k - 2b_2T]\right)$$

$$- e^{-2a_2b_2} I\!P\left[\sup_{0\leq t\leq T}(B_t + (2b_2 - b_1)t) \geq a_1, B_T \leq k - 2b_2T\right]$$

$$= \quad e^{-2a_2b_2}\Phi\left(\frac{k}{\sqrt{T}} - 2b_2\sqrt{T}\right)$$

$$- e^{-2a_2b_2} I\!P\left[\sup_{0\leq t\leq T}(B_t + (2b_2 - b_1)t) \geq a_1, B_T + (2b_2 - b_1)T \leq k - b_1T\right]$$

Let us have a look at the last term:

$$e^{-2a_2b_2} I\!P\left[\sup_{0\leq t\leq T}(B_t + (2b_2 - b_1)t) \geq a, B_T + (2b_2 - b_1)T \leq k - b_1T\right]$$

$$= \quad e^{-2a_2b_2} \int_A e^{-(2b_2 - b_1)B_T - (2b_2 - b_1)^2T/2} e^{(2b_2 - b_1)(B_T + (2b_2 - b_1)T) - (2b_2 - b_1)^2T/2}$$

where A denotes $\{\sup_{0\leq t\leq T}(B_t + (2b_2 - b_1)t) \geq a_1, B_T + (2b_2 - b_1)T \leq k - b_1T\}$. By Girsanov's theorem, $(B_t + (2b_2 - b_1)t)_{t\geq 0}$ is a Brownian motion under the measure $\mathbb{Q}$ defined by $\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-(2b_2-b_1)B_T-(2b_2-b_1)^2T/2}$. If we apply Girsanov's theorem, we find that the probability equals:

$$e^{-2a_2b_2}\int_{\{\sup_{0\le t\le T} B_t\ge a_1, B_T\le k-b_1T\}} e^{(2b_2-b_1)B_T-(2b_2-b_1)^2T/2}$$

By an application of the reflection principle, we find for $a_1 > 0$:

$$\begin{aligned}
&e^{-2a_2b_2}\int_{-\infty}^{k-b_1T} e^{(2b_2-b_1)x-(2b_2-b_1)^2T/2} f_{B_T}(x-2a_1)dx \\
&= e^{-2a_2b_2}\int_{-\infty}^{k-b_1T} e^{(2b_2-b_1)x-(2b_2-b_1)^2T/2}\frac{1}{\sqrt{2\pi T}}e^{-\frac{1}{2T}(x-2a_1)^2}dx \\
&= e^{-2a_2b_2-2a_1b_1+4a_1b_2}\int_{-\infty}^{k-b_1T}\frac{1}{\sqrt{2\pi T}}e^{-\frac{1}{2T}(x-(2b_2-b_1)T-2a_1)^2}dx \\
&= e^{-2a_2b_2-2a_1b_1+4a_1b_2}\Phi\left(\frac{k}{\sqrt{T}}-2b_2\sqrt{T}-\frac{2a_1}{\sqrt{T}}\right)
\end{aligned}$$

For $a_1 > 0$ and $b_2 > 0$, the crossing probability (2) equals:

$$\begin{aligned}
&\mathbb{P}\left(\sup_{0\le t<\infty}(B_t-h(t))\ge 0\right) \\
&= 1-\Phi(\frac{k}{\sqrt{T}})+e^{-2a_1b_1}\Phi(\frac{k-2a_1}{\sqrt{T}})+e^{-2a_2b_2}\Phi(\frac{k}{\sqrt{T}}-2b_2\sqrt{T}) \\
&\quad - e^{-2a_2b_2-2a_1b_1+4a_1b_2}\Phi\left(\frac{k}{\sqrt{T}}-2b_2\sqrt{T}-\frac{2a_1}{\sqrt{T}}\right)
\end{aligned}$$

If $a_1 \le 0$ or $b_2 \le 0$, then the probability is 1.

3. Boundary consisting of two lines in finite time

In finite time the boundary is of the form:

$$\begin{cases} h(t) = a_1 + b_1 t & \text{for } t < T \\ h(t) = a_2 + b_2 t & \text{for } T \le t \le T_s \end{cases}$$

If we define the stopping time

$$\tau = \inf\{t \mid t < T \;\; B_t = a_1 + b_1 t,\; T \le t \le T_s \;\; B_t = a_2 + b_2 t\}$$

the crossing probability can be rewritten:

$$\mathbb{P}\left(\sup_{0\le t<T_s}(B_t-h(t))\ge 0\right) = \mathbb{P}(\tau < T_s)$$

Trivially,

$$\begin{aligned} I\!\!P(\tau < T_s) &= 1 - I\!\!P(\tau > T_s) \\ &= 1 - I\!\!P(\tau > T) I\!\!P(\tau > T_s \mid \tau > T) \end{aligned}$$

By the strong Markov property and the results in section 1 and 2, we find that for $a_1, a_2 > 0$

$$\begin{aligned} & I\!\!P\left(\sup_{0 \le t < \infty} (B_t - h(t)) \ge 0\right) \\ &= 1 - \int_{-\infty}^{k} \frac{e^{-y^2/2}}{\sqrt{2\pi}} \left(1 - e^{-\frac{2a_1}{T}(a_1 + b_1 T - y)}\right) \\ & \quad . \left\{\Phi\left(\frac{a_2 + b_2(T_s - T) - y}{\sqrt{T_s - T}}\right) - e^{-2a_2 b_2} \Phi\left(\frac{a_2 + b_2(T_s - T) - y}{\sqrt{T_s - T}} - \frac{2a_2}{\sqrt{T_s - T}}\right)\right\} \end{aligned}$$

If $a_1 \le 0$ or $a_2 \le 0$, then the probability is 1.

Conclusion

In the present contribution, we show that an expression for the first-passage density of the surplus to a linear upper boundary can be found by repeated applications of Girsanov's theorem, the Markov property and the reflection property of Brownian motion. We obtained the same results as Teunen and Goovaerts, but we do not need to solve complicated differential equations. Remark that also in Scheike (1990) these boundary crossing result for the Brownian motion are obtained by straightforward calculations, namely by using scaling properties and time inversion for Brownian motion.

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Zusammenfassung

Einige Bemerkungen zu 'Boundary Crossing Result for the Brownian Motion.'

In Scheike (1990) wurde für die Brownsche Bewegung ein allgemeines Ergebnis über die Randüberschreitung erzielt. M. Teunen und M. Goovaerts erreichen dieses Resultats mit hilfe von Pad-Integralen. Für den Überschuss eines Versicherungsportfolios wird ein Brownscher Prozess betracht, welcher eine vorgegebene obere Schranke nicht überschreiten soll. Anderfalls müss z.B. eine Dividende gezahlt werden. Diese Schranke wird stückweise linear angenommen. In der vorliegenden Arbeit wird eine direkte Herleitung dieses Resultats angegeben. Wir müssen keine schwierigen Differentialgleichungen auflösen. Wir wenden standardisierte Methoden an: den Satz von Girsanov, die Eigenschaft von Markov und das Reflexion-Prinzip; siehe z.B. Revuz-Yor (1991).

Summary

In Scheike (1990) a general boundary crossing result for the Brownian motion is obtained. Using path integrals, M. Teunen and M. Goovaerts obtained this result and some generalisations by the methodology of Kac. A Brownian motion process for the surplus of an insurance portfolio is considered which may not cross a given upper boundary. This boundary can be a piecewise linear one consisting of one or more lines.
In the present contribution, we show that these results can be found by a straightforward derivation. We do not need to solve differential equations but use several applications of Girsanov's theorem, the Markov property and the reflection property of Brownian motion. The exercise consists in recognizing the Bachelier-Levy formula, transformed by Girsanov's formula. Remark that these methods are standard; see for example Revuz-Yor (1991).