Pricing and Hedging Asian Basket Spread Options in a Nutshell

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In this paper we study the pricing and hedging of arithmetic Asian basket spread options of the European type and present the main results of Deelstra et al. (2008). Asian basket spread options are written on a multivariate underlying. Thus we first need to specify a financial market model containing multiple stocks. We choose to use the famous Black and Scholes model.

The set up and the problem

More formally we assume a financial market composed of m risky assets such that the dynamics of the price of the jth asset under the historical probability measure P are

$$dS_j(t) = \mu_j S_j(t)dt + \sigma_j S_j(t)d\widetilde{B}_j(t)$$

where μ_j is the constant instantaneous return, σ_j the return's volatility and $\widetilde{B}_j(t)$ is a standard Brownian motion under P. Furthermore we assume that asset returns are correlated according to

$$\text{cov}(B_{jt_v}, B_{it_s}) = \rho_{ji}\min(t_v, t_s)$$

The final pay off at time T of an Asian basket spread option is of the form

$$(\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m}\varepsilon_j a_j S_j(t_i) - K)_+$$

with $(x)_+ = \max(x,0)$ and where a_j is the weight given to asset j, ε_j its sign in the spread and K is the strike price. We

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assume that $\varepsilon_j = 1$ for j=1,...p, $\varepsilon_j = -1$ for $j=p+1,...,m$, where p is an integer such that $1 \le p \le m-1$ and $t_0 < t_1 < t_2 < ... < t_n = T$. Such Asian basket spread options are frequently encountered in the energy markets where they are used by energy producers to cover their profit margins.

From the fundamental theorem of arbitrage free pricing we know that the arbitrage free price of an Asian basket spread option can be obtained by evaluating

$$e^{-rT}E_Q(\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m}\varepsilon_j a_j S_j(t_i) - K)_+$$

where E_Q denotes the expectation with respect to the risk neutral probability measure Q and r is the risk-free interest rate. The risk neutral probability measure Q is a probability measure equivalent to P such that the discounted asset prices are martingales under Q.

Unfortunately things are not as simple as they seem. Since the underlying is a linear combination of the asset prices, formula (1) does not have a closed form expression. This means that for evaluation we need to resort to numerical methods. One of the most popular methods is the use of Monte Carlo simulations. Unfortunately, Monte Carlo methods can be time consuming especially in the case of path dependency. This poses some serious problems to the implementation of arbitrage free pricing by traders. Furthermore, financial institutions are not only interested in the evaluation of the option price. In order to control the risk of their position they also want to evaluate the Greeks: the derivatives of the option price with respect to the parameters of the stock prices. This additional task can considerably increase the computational time making the use of Monte Carlo methods even more complicated.

Two Solutions

In this paper we consider two ways of approximating the price of an Asian basket spread option: the comonotonic approximations and the moment matching methods. The logic behind both techniques is similar. It consists in replacing the original underlying, denote it by S, by a new one whose structure is simpler.

First: the comonotonic bounds

As said above a first way of approximating (1) is to use the so called comonotonic approximations. The idea behind the comonotonic approximations is to replace the underlying S in (1) by a new underlying Z of the form

$$Z = \sum_{i=1}^{n} \sum_{j=1}^{m} \varepsilon_j Z_{ji},$$

where the Z_{ji} are chosen such that Z_{ji} and $a_j S_j(t_i)$ have the same marginal distribution. However the dependence structure between the components of Z will be different from the dependence within S. Indeed the dependence structure between the components of Z shall be maximal, it shall be comonotone.

Two reasons explain the success of the theory of comonotonic random vectors in the option pricing literature. First, it can be shown that if X is a n-dimensional comonotonic random vector with marginal components X_i then

$$E(\sum_{i=1}^{n} X_i - K)_+ = \sum_{i=1}^{n} E(X_i - \widehat{K_i})_+,$$

where $\widehat{K_i}$ can be evaluated using the marginal cumulative distribution functions of the vector of $a_j S_j(t_i)$'s. Thus the stop-loss premium of the sum of the components of a comonotonic random vector can be written as the sum of the marginal stop-loss premia. Something of interest since the expectation of the marginal stop-loss premium is simply the Black and Scholes price. Second, there are different ways of building comonotonic variables. And they do not lead to the same comonotonic random sum. Furthermore depending on which comonotonic sum is used we can tell whether our approximation will be an upper or a lower bound.

Using the theory of comonotonic random vectors, we derived, in the first part of Deelstra et al. (2008), four different comonotonic approximations of the real price. In finance comonotonicity is used for pricing and hedging of Asian, basket or Asian basket options (see Simon et al. (2000), Vanmaele et al. (2006), Deelstra et al. (2004), Chen et al. (2008)). To our knowledge, this is the first time that this approach has been used in order to approximate basket spread or Asian basket spread options.

Second: moment matching

The basic intuition behind the moment matching methods goes as follows: consider the original problem of evaluating (1). Since the distribution of a correlated sum of log-normal random variables is not known, we cannot derive a closed form expression for this expectation. However, we can derive an approximation to the price by replacing the original underlying of (1) by a new random variable with a treatable distribution with p parameters whose stop-loss premium has a closed form expression. The p parameters are fixed such that the p first moments of the new random variable are equal to those of the original underlying. This is what is called moment matching.

Deelstra et al. (2008) contributes to the literature on moment matching approximations in two ways. First, we study and improve the hybrid moment matching method that was introduced for basket spread options by Castellacci and Siclari (2003). Their original approach is the following: start by noticing that the original underlying of (1) can be split in two parts, one containing the stock prices with a positive sign (S_1) and another containing the stock prices with a negative sign (S_2):

$$E_Q(\underbrace{\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{p} a_j S_j(t_i)}_{S_1} - \underbrace{\frac{1}{n}\sum_{i=1}^{n}\sum_{j=p+1}^{m} a_j S_j(t_i)}_{S_2} - K)_+$$

Castellacci and Siclari propose to replace S_1 and S_2 by two log-normal random variables X_1 and X_2, where X_i has the same mean and variance as S_i, for $i=1,2$. Doing this, they transform the problem of evaluating the price of a basket spread option into the problem of evaluating the price of a spread option. The evaluation of the price of a spread option is a well studied problem in the literature and many approximations are available.

We extend the approach of Castellacci and Siclari in two ways. First, we improve their method by choosing a better approximation technique for the spread option. Originally Castellacci and Siclari used the Kirk method to approximate the spread. We use two new approximations for the spread, one is based on a recent approximation due to Li et al. (2008), the second approximation is based on the improved comonotonic upper bound belonging to the comonotonic bounds that we proposed as a first set of approximations. Second, we also study the performances of this so-called hybrid moment matching on Asian basket spread options extending so the results obtained by Castellacci and Siclari who only considered basket spread options.

Second, we mix and extend the approach of Borokova et al. (2007) and Zhou and Wang (2008). Borokova et al. (2007) studied the problem of approximating the price of basket spread options using a shifted log-normal random variable. While Zhou and Wang studied the problem of pricing Asian and basket spread options using a log-skew normal extended distribution (see Azzalini (1985)). More

exactly we replace the underlying in (1) by a random variable of the form

$$e^{\mu+\sigma X} + \eta,$$

where X has a skew extended distribution and where μ, σ and η are respectively location, scale and shift parameters. This is an improvement over the original paper of Borokova and al. (2007). By using a skew extended distribution for X instead of a normal distribution we gain two additional parameters to match, this should enhance the quality of our approximation. It is an extension of the approach of Zhou and Wang (2008) since by introducing a shift parameter we can control for the fact that our underlying can take negative values and can be applied to basket spreads.

Numerical Results

We compare our approximations using extensive numerical simulations. The simulations are split in three parts. First, we compare the performances of the approximations on spread option prices. We find out that the improved comonotonic upper bound is an extremely accurate approximation to the price of a spread option. The quality of the approximation was even superior to the one obtained with the method of Li et al. (2008) which is one of the best approximations that can be found in the literature. This result justified the use of the improved comonotonic upper bound to approximate the spread in the hybrid moment matching method.

Second, we study the approximation of the price of a basket spread option. Unfortunately it was not optimal to use a single approximation technique when dealing with basket spread options. The best approximation technique seems to be based on a combination of hybrid moment matching with the improved comonotonic upper bound and the shifted log-normal approximation.

Finally, we study the approximation of Asian basket spread options. In this case we find that hybrid moment matching combined with the improved comonotonic upper bound (HybMMICUB) is the best approximation. Thus we recommend its use in this case.

The performances of our approximations are briefly illustrated in table 1 and 2. The second column of table 1 gives the approximated price of a spread option when the price is approximated using the improved comonotonic upper bound (ICUB). The second column of table 2 contains the approximated HybrMMICUB price of an Asian basket spread option. The third column reports in both tables the "real" price which is computed through Monte Carlo simulations (MC).

Greeks and Options Written in a Foreign Currency

In the last two sections of Deelstra et al. (2008), we derive the Greeks of the approximate option prices and explain how the approximation techniques we develop could be applied to price options written in a foreign currency. To compute the approximations of the Greeks, we derive the approximation obtained by hybrid moment matching when the spread is approximated by an improved comonotonic upper bound. It should be emphasized that two things make this computation feasible. First, the improved comonotonic upper bound we use to approximate the spread has a nice closed form expression that can easily be derived. Second, since we used log-normal random variables to approximate S_1 and S_2, we only had to solve a linear system to compute the matching distribution. This linearity introduces considerable simplifications in our problem and allows us to use a chain rule to compute the approximation of the Greeks.

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