

Risk management of a bond portfolio using options

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Abstract

In this paper, we elaborate a formula for determining the optimal strike price for a bond put option, used to hedge a position in a bond. This strike price is optimal in the sense that it minimizes, for a given budget, either Value-at-Risk or Tail Value-at-Risk. Formulas are derived for both zero-coupon and coupon bonds, which can also be understood as a portfolio of bonds. These formulas are valid for any short rate model that implies an affine term structure model and in particular that implies a lognormal distribution of future zero-coupon bond prices. As an application, we focus on the Hull-White one-factor model, which is calibrated to a set of cap prices. We illustrate our procedure by hedging a Belgian government bond, and take into account the possibility of divergence between theoretical option prices and real option prices. This paper can be seen as an extension of the work of Ahn et al. (1999), who consider the same problem for an investment in a share.

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1 Introduction

Several studies document risk management practices in a corporate setting, see for example Bodnar et al. (1998), Bartram et al. (2004), Prevost et al. (2000). Survey techniques are often employed to get insights into why and how firms implement hedging strategies. In the vast majority of studies, the widespread usage of these hedging policies is confirmed. In each of the above mentioned surveys, at least 50% of the firms reported that they make use of some kind of derivatives.

The most popular derivatives are forwards, options and swaps. These instruments can be used to hedge exposures due to currency, interest rate and other market risks. Swaps are most frequently used to tackle interest rate risks, followed by forwards and options. Using these kind of derivatives is surely a first step in successful risk management.

However, a second step is formed by using these derivatives in an optimal way. Although tools like swaps and options are basic building blocks for all sorts of other, more complicated derivatives, they should be used prudently and a firm knowledge of their properties is needed. These derivatives have a multitude of decision parameters, which necessitates thoroughly investigating the influence of these parameters on the aims of the hedging policies and the possibility to achieve these goals.

The literature on risk management is much more silent on how to optimally decide on these parameters. The present study partly fills this gap. We consider the problem of determining the optimal strike price for a bond put option, which is used to hedge the interest rate risk of an investment in a bond. In order to measure risk, we focus on both Value-at-Risk (VaR) and Tail Value-at-Risk (TVaR). Our optimization is constrained by a maximum hedging budget. Alternatively, our approach can also be used to determine the minimal budget a firm needs to spend in order to achieve a predetermined absolute risk level.

The setup of our paper is similar in spirit as Ahn et al. (1999). However, we emphasize that our paper contributes in several aspects. First of all, our analysis is carried out for another asset class. Whereas Ahn et al. (1999) consider stocks, our focus lies on bonds. The importance of bonds as an investment tool can hardly be underestimated. As reported in the European institutional market place overview 2006 of Mercer Investment Consulting (see MercerAssetAlloc (2006)), pension funds in continental Europe invest more than half of their resources in bonds. This makes fixed income securities an asset class that should not be neglected. Secondly, Ahn et al. (1999) assume that stock prices are driven by a geometric Brownian motion. Our analysis generalises their results since we only assume that the price of the asset we consider is driven by a one factor model with an affine structure. This encompasses the Brownian motion process which is often used for stocks, but also allows for mean reverting processes, which are crucial in interest rate modelling and the pricing of fixed income securities. Concrete examples of the term structure models that are captured by our approach are: Vasicek, one-factor Hull-White and one-factor Heath-Jarrow-Morton with deterministic volatility. Furthermore, we develop formulas for not only a zero-coupon bond, but also for a coupon-bearing bond. Finally, as risk measure, we consider both VaR and TVaR. As stated below, VaR is a very popular risk measure

but it is not free of criticism. An important drawback of VaR is that it is a risk measure which ignores what really happens in the tail. Furthermore, it is not a coherent measure, as precised by Artzner et al. (1999). These two problems are tackled when TVaR is used as risk measure.

Taking into account the advent of new capital regulations in both the bank (Basel II) and the insurance industry (Solvency II), our insights can play a role in implementing a sound risk management system.

In the next section we introduce the loss function as well as the risk measures that will be used. In Section 3 we formulate the bond hedging problem, first for a zero-coupon bond and next for a coupon-bearing bond. We assume a short rate model for the instantaneous interest rate with an affine term structure. Not only the VaR of the loss function but also its TVaR is minimized under the budget constraint. We pay special attention to the case that the zero-coupon bond price is lognormally distributed. In Heyman et al. (2006) we treat this problem theoretically in a more general framework by only assuming that the cumulative distribution function of the zero-coupon bond price at a later time instance before maturity is known.

In Section 4 we illustrate the procedure by hedging a Belgian government bond, and take into account the possibility of divergence between theoretical option prices and real option prices.

Section 5 concludes the paper.

2 Loss function and risk measures

Consider a portfolio with value W_t at time t . W_0 is then the value or price at which we buy the portfolio at time zero. W_T is the value of the portfolio at time T . The loss L we make by buying at time zero and selling at time T is then given by $L = W_0 - W_T$. The Value-at-Risk of this portfolio is defined as the $(1 - \alpha)$ -quantile of the loss distribution depending on a time interval with length T . A formal definition for the $\text{VaR}_{\alpha,T}$ is

$$\Pr[L \geq \text{VaR}_{\alpha,T}] = \alpha. \quad (1)$$

In other words $\text{VaR}_{\alpha,T}$ is the loss of the worst case scenario on the investment at a $(1 - \alpha)$ confidence level at time T . It is also possible to define the $\text{VaR}_{\alpha,T}$ in a more general way

$$\text{VaR}_{\alpha,T}(L) = \inf \{ \ell \in \mathbb{R} \mid \Pr(L > \ell) \leq \alpha \}. \quad (2)$$

Although frequently used, VaR has attracted some criticisms. First of all, a drawback of the traditional Value-at-Risk measure is that it does not care about the tail behaviour of the losses. In other words, by focusing on the VaR at, let's say a 5% level, we ignore the potential severity of the losses below that 5% threshold. This means that we have no information on how bad things can become in a real stress situation. Therefore, the

important question of ‘how bad is bad’ is left unanswered. Secondly, it is not a coherent risk measure, as suggested by Artzner et al. (1999). More specifically, it fails to fulfil the subadditivity requirement which states that a risk measure should always reflect the advantages of diversifying, that is, a portfolio will risk an amount no more than, and in some cases less than, the sum of the risks of the constituent positions. It is possible to provide examples that show that VaR is sometimes in contradiction with this subadditivity requirement.

Artzner et al. (1999) suggested the use of CVaR (Conditional Value-at-Risk) as risk measure, which they describe as a coherent risk measure. CVaR is also known as TVaR, or Tail Value-at-Risk and is defined as follows:

$$\text{TVaR}_{\alpha,T}(L) = \frac{1}{\alpha} \int_{1-\alpha}^1 \text{VaR}_{1-\beta,T}(L) d\beta. \quad (3)$$

This formula boils down to taking the arithmetic average of the quantiles of our loss, from $1 - \alpha$ to 1 on, where we recall that $\text{VaR}_{1-\beta,T}(L)$ stands for the quantile at the confidence level β , see (1).

A closely related risk measure concerns Expected Shortfall (ESF). It is defined as:

$$\text{ESF}_{\alpha,T}(L) = E[(L - \text{VaR}_{\alpha,T}(L))_+]. \quad (4)$$

In order to determine $\text{TVaR}_{\alpha,T}(L)$, we can also make use of the following equality:

$$\text{TVaR}_{\alpha,T}(L) = \text{VaR}_{\alpha,T}(L) + \frac{1}{\alpha} \text{ESF}_{\alpha,T}(L) \quad (5)$$

$$= \text{VaR}_{\alpha,T}(L) + \frac{1}{\alpha} E[(L - \text{VaR}_{\alpha,T}(L))_+]. \quad (6)$$

This formula already makes clear that $\text{TVaR}_{\alpha,T}(L)$ will always be larger than $\text{VaR}_{\alpha,T}(L)$. If moreover the cumulative distribution function of the loss is continuous, TVaR is also equal to the Conditional Tail Expectation (CTE) which for the loss L is calculated as:

$$\text{CTE}_{\alpha,T}(L) = E[L \mid L > \text{VaR}_{\alpha,T}(L)].$$

This is for example the case in the bond hedging problem that we consider in the subsequent sections, when bond prices are lognormally distributed.

3 The bond hedging problem

Analogously to Ahn et al. (1999), we assume that we have, at time zero, one zero-coupon bond with maturity S and we will sell this bond at time T , which is prior to S . In case of an increase in interest rates, not hedging can lead to severe losses. Therefore, the company decides to spend an amount C on hedging. This amount will be used to buy one or part of

a bond put option with the bond as underlying, so that, in case of a substantial decrease in the bond price, the put option can be exercised in order to prevent large losses. The remaining question now is how to choose the strike price. We will find the optimal strike prices which minimize VaR and TVaR respectively for a given hedging cost. An alternative interpretation of our setup is that it can be used to calculate the minimal hedging budget the firm has to spend in order to achieve a specified VaR or TVar level, a setup which was followed in the paper by Miyazaki (2001) in another setting.

3.1 Zero-coupon bond

Let us assume that the institution has at date zero an exposure to a bond, $P(0, S)$, with principal $N = 1$, which matures at time S , and that the company has decided to hedge the bond value by using a percentage h ($0 < h < 1$) of one put option $ZBP(0, T, S, X)$ with strike price X and exercise date T (with $T \leq S$).

Further, we assume a short rate model for $r(T)$ with an *affine* term structure such that the zero-coupon bond price $P(T, S)$ can be written in the form

$$P(T, S) = A(T, S)e^{-B(T, S)r(T)}, \quad (7)$$

with parameters $A(T, S)(> 0)$ and $B(T, S)(> 0)$ independent of $r(T)$.

This assumption covers a range of commonly used interest rate models such as Vasicek, one-factor Hull-White and one-factor Heath-Jarrow-Morton with deterministic volatility, see e.g. Brigo and Mercurio (2001).

In Heyman et al. (2006) we treat this problem theoretically in a more general framework. We make no assumption on $r(T)$, we only assume that the cumulative distribution function of $P(T, S)$ is known.

Analogously as in the paper of Ahn et al. (1999), we can look at the future value of the hedged portfolio that is composed of the bond P and the put option $ZBP(0, T, S, X)$ at time T as a function of the form

$$H_T = \max(hX + (1 - h)P(T, S), P(T, S)).$$

In a worst case scenario — a case which is of interest to us — the put option finishes in-the-money. Then the future value of the portfolio equals

$$H_T = (1 - h)P(T, S) + hX.$$

Taking into account the cost of setting up our hedged portfolio, which is given by the sum of the bond price $P(0, S)$ and the cost C of the position in the put option, we get for the value of the loss:

$$L = P(0, S) + C - ((1 - h)P(T, S) + hX),$$

and this under the assumption that the put option finishes in-the-money.

In view of the assumption on the form of $P(T, S)$, this loss of the portfolio equals a strictly increasing and continuous function f of the random variable $r(T)$:

$$f(r(T)) := L = P(0, S) + C - ((1 - h)A(T, S)e^{-B(T, S)r(T)} + hX). \quad (8)$$

VaR minimization

We first look at the case of determining the optimal strike X when minimizing the VaR under a constraint on the hedging cost.

Lemma 1 *Under the assumption of an affine term structure such that the zero-coupon bond price $P(T, S)$ is given by (7), the Value-at-Risk at an α percent level of a position $H = \{P, h, ZBP\}$ consisting of the bond $P(T, S)$ and h put options ZBP on this zero-coupon bond (which are assumed to be in-the-money at expiration) with a strike price X and an expiry date T is equal to¹*

$$\text{VaR}_{\alpha, T}(L) = P(0, S) + C - ((1 - h)A(T, S)e^{-B(T, S)F_{r(T)}^{-1}(1-\alpha)} + hX), \quad (9)$$

where $F_{r(T)}$ denotes the cumulative distribution function (cdf) of $r(T)$ and $F_{r(T)}^{-1}$ stands for the inverse of this cdf and is defined in the usual way:

$$F_{r(T)}^{-1}(p) = \inf \{x \in \mathbb{R} \mid F_{r(T)}(x) \geq p\}, \quad p \in [0, 1]. \quad (10)$$

PROOF. We start from the general definition (2) of VaR, use definition (8) of the function f , the fact that f is strictly increasing and the definition (10) of the inverse cdf to obtain consecutively:

$$\begin{aligned} \text{VaR}_{\alpha, T}(L) &= \inf \{\ell \in \mathbb{R} \mid \Pr(L > \ell) \leq \alpha\} \\ &= \inf \{\ell \in \mathbb{R} \mid \Pr(f(r(T)) > \ell) \leq \alpha\} \\ &= \inf \{\ell \in \mathbb{R} \mid \Pr(r(T) > f^{-1}(\ell)) \leq \alpha\} \\ &= \inf \{\ell \in \mathbb{R} \mid \Pr(r(T) \leq f^{-1}(\ell)) \geq 1 - \alpha\} \\ &= \inf \{\ell \in \mathbb{R} \mid F_{r(T)}(f^{-1}(\ell)) \geq 1 - \alpha\} \\ &= f(F_{r(T)}^{-1}(1 - \alpha)). \end{aligned}$$

Finally, invoking again definition (8) of the function f we arrive at (9). \square

Similar to the Ahn et al. problem, we would like to minimize the risk of the future value of the hedged bond H_T , given a maximum hedging expenditure C . More precisely, we

¹ In case of an unhedged portfolio, take $C = h = 0$ in (8) and in (9) to obtain the loss function L with corresponding $\text{VaR}_{\alpha, T}(L)$.

consider the minimization problem

$$\min_{X,h} P(0, S) + C - ((1 - h)A(T, S)e^{-B(T,S)F_{r(T)}^{-1}(1-\alpha)} + hX)$$

subject to the restrictions $C = h\text{ZBP}(0, T, S, X)$ and $h \in (0, 1)$.

This is a constrained optimization problem with Lagrange function

$$\mathcal{L}(X, h, \lambda) = \text{VaR}_{\alpha, T}(L) - \lambda(C - h\text{ZBP}(0, T, S, X)),$$

containing one multiplier λ . Note that the multipliers to include the inequalities $0 < h$ and $h < 1$ are zero since these constraints are not binding. Taking into account that the optimal strike X^* will differ from zero, we find from the Kuhn-Tucker conditions

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial X} = -h + h\lambda \frac{\partial \text{ZBP}}{\partial X}(0, T, S, X) = 0 \\ \frac{\partial \mathcal{L}}{\partial h} = -(X - A(T, S)e^{-B(T,S)F_{r(T)}^{-1}(1-\alpha)}) + \lambda \text{ZBP}(0, T, S, X) = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} = C - h\text{ZBP}(0, T, S, X) = 0 \\ 0 < h < 1 \quad \text{and} \quad \lambda > 0 \end{cases}$$

that this optimal strike X^* should satisfy the following equation

$$\text{ZBP}(0, T, S, X) - (X - A(T, S)e^{-B(T,S)F_{r(T)}^{-1}(1-\alpha)}) \frac{\partial \text{ZBP}}{\partial X}(0, T, S, X) = 0. \quad (11)$$

By a change of numeraire, it is well known that the put option price equals the discounted expectation under the T -forward measure of the payoff:

$$\text{ZBP}(0, T, S, X) = P(0, T)E^T[(X - P(T, S))_+].$$

When the cumulative distribution function $F_{P(T,S)}^T$ of $P(T, S)$ under this T -forward measure has bounded variation and the expectation $E^T[P(T, S)]$ is finite, then by partial integration we find

$$\text{ZBP}(0, T, S, X) = P(0, T) \int_{-\infty}^X (F_{P(T,S)}^T(p) - 1) dp.$$

Its first order derivative with respect to the strike X leads immediately to

$$\frac{\partial \text{ZBP}}{\partial X}(0, T, S, X) = P(0, T)F_{P(T,S)}^T(X). \quad (12)$$

This relation between the cdf and the price of the put option is analogous to a result derived in a Black&Scholes framework in Breeden and Litzenberger (1978). Since the randomness of $P(T, S)$ is completely due to the randomness of $r(T)$, relation (7) implies

the following connection between their cdfs under the T -forward measure (indicated by the subscript T):

$$F_{P(T,S)}^T(X) = 1 - F_{r(T)}^T \left(\frac{\ln A(T, S) - \ln X}{B(T, S)} \right). \quad (13)$$

Hence, (11) is equivalent to

$$\begin{aligned} & \text{ZBP}(0, T, S, X) \\ &= (X - A(T, S))e^{-B(T,S)F_{r(T)}^{-1}(1-\alpha)}P(0, T) \left[1 - F_{r(T)}^T \left(\frac{\ln A(T, S) - \ln X}{B(T, S)} \right) \right]. \end{aligned}$$

Important remarks

- (1) We note that the optimal strike price is independent of the hedging cost C . This independence implies that for the optimal strike X^* , VaR in (9) is a linear function of h (or C):

$$\begin{aligned} \text{VaR}_{\alpha,T}(L) &= P(0, S) - A(T, S)e^{-B(T,S)F_{r(T)}^{-1}(1-\alpha)} \\ &\quad + h(\text{ZBP}(0, T, S, X^*) + A(T, S)e^{-B(T,S)F_{r(T)}^{-1}(1-\alpha)} - X^*). \end{aligned}$$

So, there is a linear trade-off between the hedging expenditure and the VaR level, see Figure 1 in the application of Section 4. It is a decreasing function since in view of (12) $\frac{\partial \text{ZBP}}{\partial X}(0, T, S, X^*) < 1$ and thus according to (11)

$$X^* - A(T, S)e^{-B(T,S)F_{r(T)}^{-1}(1-\alpha)} > \text{ZBP}(0, T, S, X^*). \quad (14)$$

Although the setup of the paper is determining the strike price which minimizes a certain risk criterion, given a predetermined hedging budget, this trade-off shows that the analysis and the resulting optimal strike price can evidently also be used in the case where a firm is fixing a nominal value for the risk criterion and seeks the minimal hedging expenditure needed to achieve this risk level. It is clear that, once the optimal strike price is known, we can determine, in both approaches, the remaining unknown variable (either VaR, either C).

- (2) We also note that the optimal strike price X^* is higher than the bond VaR level

$$A(T, S)e^{-B(T,S)F_{r(T)}^{-1}(1-\alpha)}.$$

This has to be the case since inequality (14) holds with $\text{ZBP}(0, T, S, X)$ being positive. This result is also quite intuitive since there is no point in taking a strike price which is situated below the bond price you expect in a worst case scenario.

When moreover the optimal strike is smaller than the forward price of the bond, i.e.

$$X^* < \frac{P(0, S)}{P(0, T)},$$

then the time zero price of the put option to buy will be small.

TVaR minimization

In this section, we demonstrate the ease of extending our analysis to the alternative risk measure TVaR (3) by integrating $\text{VaR}_{1-\beta,T}(L)$, given by (9) with $\alpha = 1 - \beta$, with respect to β :

$$\text{TVaR}_{\alpha,T}(L) = P(0, S) + C - hX - \frac{1}{\alpha}(1 - h)A(T, S) \int_{1-\alpha}^1 e^{-B(T,S)F_{r(T)}^{-1}(\beta)} d\beta. \quad (15)$$

We again seek to minimize this risk measure, in order to minimize potential losses. The procedure for minimizing this TVaR is analogous to the VaR minimization procedure. The resulting optimal strike price X^* can thus be determined from the implicit equation below:

$$\text{ZBP}(0, T, S, X) - \left(X - \frac{1}{\alpha}A(T, S) \int_{1-\alpha}^1 e^{-B(T,S)F_{r(T)}^{-1}(\beta)} d\beta\right) \frac{\partial \text{ZBP}}{\partial X}(0, T, S, X) = 0 \quad (16)$$

which is in view of (12)-(13) equivalent to

$$\begin{aligned} \text{ZBP}(0, T, S, X) = P(0, T) \left[X - \frac{A(T, S)}{\alpha} \int_{1-\alpha}^1 e^{-B(T,S)F_{r(T)}^{-1}(\beta)} d\beta \right] \times \\ \times \left[1 - F_{r(T)}^T \left(\frac{\ln A(T, S) - \ln X}{B(T, S)} \right) \right]. \quad (17) \end{aligned}$$

As for the VaR-case the optimal strike X^* is independent of the hedging cost C and TVaR can be plotted as a linear function of C (or h) representing a trade-off between the cost and the level of protection.

For the same reason as in the VaR-case, the optimal strike X^* has to be higher than the bond TVaR level $\frac{1}{\alpha}A(T, S) \int_{1-\alpha}^1 e^{-B(T,S)F_{r(T)}^{-1}(\beta)} d\beta$.

Expected shortfall

Substitution of the expressions (9) and (15) for the VaR and the TVaR in (5) or (6) provides immediately the value of the expected shortfall of the loss L :

$$\begin{aligned} \text{ESF}_{\alpha,T}(L) &= \alpha[\text{TVaR}_{\alpha,T}(L) - \text{VaR}_{\alpha,T}(L)] \\ &= (1 - h)A(T, S) \left[\alpha e^{-B(T,S)F_{r(T)}^{-1}(1-\alpha)} - \int_{1-\alpha}^1 e^{-B(T,S)F_{r(T)}^{-1}(\beta)} d\beta \right]. \quad (18) \end{aligned}$$

Summary

The implicit equations (11) and (16) to solve for the optimal strike price X^* in the VaR-case respectively the TVaR-case, have the same structure and only differ by the risk measure level. Hence, we can treat these as one problem when we introduce the notation RM for the risk measures VaR and TVaR. Further we put for the bond risk measure level:

$$\text{RMlevel} = \begin{cases} A(T, S)e^{-B(T, S)F_{r(T)}^{-1}(1-\alpha)} & \text{if VaR} \\ \frac{1}{\alpha}A(T, S) \int_{1-\alpha}^1 e^{-B(T, S)F_{r(T)}^{-1}(\beta)} d\beta & \text{if TVaR.} \end{cases} \quad (19)$$

Hence, the results that we derived above can be summarized as follows:

Theorem 2 *Under the assumption of an affine term structure such that the zero-coupon bond price $P(T, S)$ is given by (7), the constrained minimization problem:*

$$\min_{X, h} \text{RM}_{\alpha, T}(L) \quad (20)$$

$$\text{s.t. } C = h\text{ZBP}(0, T, S, X) \text{ and } h \in (0, 1) \quad (21)$$

with $\text{RM}_{\alpha, T}(L)$ given by (9) or (15), has an optimal solution X^* implicitly given by

$$\text{ZBP}(0, T, S, X) = (X - \text{RMlevel}) \frac{\partial \text{ZBP}}{\partial X}(0, T, S, X). \quad (22)$$

When moreover the cdf of $P(T, S)$ under the T -forward measure has bounded variation and $E^T[P(T, S)]$ is finite, the optimal strike X^* solves:

$$\text{ZBP}(0, T, S, X) = (X - \text{RMlevel})P(0, T) \left[1 - F_{r(T)}^T \left(\frac{\ln A(T, S) - \ln X}{B(T, S)} \right) \right]. \quad (23)$$

The corresponding expected shortfall of the loss is given by

$$\text{ESF}_{\alpha, T}(L) = (1 - h)\alpha(\text{VaRlevel} - \text{TVaRlevel}).$$

RMlevel, VaRlevel and TVaRlevel are defined by respectively (19), (19)(a), (19)(b).

VaR and TVaR minimization and ESF: lognormal case

When the short rate $r(T)$ is a normal random variable, then $P(T, S)$ is lognormally distributed and we can further elaborate the relations of Theorem 2 noting that the assumptions are satisfied.

Theorem 3 *Assume that under the risk neutral measure — in which we also express our risk measures — the short rate $r(T)$ is normally distributed with mean m and variance s^2 . Then $P(T, S)$ in (7) is lognormally distributed with parameters $\Pi(T, S)$ and $\Sigma(T, S)^2$*

given by

$$\Pi(T, S) = \ln A(T, S) - B(T, S)m, \quad \Sigma(T, S)^2 = B(T, S)^2 s^2, \quad (24)$$

and the optimal solution X^* to the constrained minimization problem (20)-(21) satisfies

$$G(\Phi^{-1}(\alpha)) = \frac{P(0, S)\Phi(-d_1(X))}{P(0, T)\Phi(-d_2(X))}, \quad (25)$$

with

$$G(\Phi^{-1}(\alpha)) = \begin{cases} e^{\Pi(T, S) + \Sigma(T, S)\Phi^{-1}(\alpha)} & \text{if VaR} \\ e^{\Pi(T, S) + \frac{1}{2}\Sigma(T, S)^2\Phi(\Phi^{-1}(\alpha) - \Sigma(T, S))} & \text{if TVaR,} \end{cases} \quad (26)$$

where $\Phi(\cdot)$ stands for the cumulative standard normal distribution, and with

$$d_1(X) = \frac{1}{\Sigma(T, S)} \log\left(\frac{P(0, S)}{XP(0, T)}\right) + \frac{\Sigma(T, S)}{2}, \quad d_2(X) = d_1(X) - \Sigma(T, S). \quad (27)$$

The corresponding shortfall of the loss equals:

$$\text{ESF}_{\alpha, T}(L) = (1 - h)e^{\Pi(T, S)}[\alpha e^{\Sigma(T, S)\Phi^{-1}(\alpha)} - e^{\frac{1}{2}\Sigma(T, S)^2}\Phi(\Phi^{-1}(\alpha) - \Sigma(T, S))]. \quad (28)$$

PROOF. When the short rate $r(T)$ is normally distributed with mean m and variance s^2 then the parameters Π and Σ^2 of the lognormally distributed $P(T, S)$ follow immediately from (7) while for the inverse cdf of $r(T)$ we find

$$F_{r(T)}^{-1}(p) = m + s\Phi^{-1}(p), \quad p \in [0, 1]. \quad (29)$$

Since $P(T, S)$ is lognormally distributed, the price at date zero of a European put option with the zero-coupon bond as the underlying security and with strike price X and exercise date T ($T \leq S$), see for example Brigo and Mercurio (2001), is explicitly known:

$$\text{ZBP}(0, T, S, X) = -P(0, S)\Phi(-d_1(X)) + XP(0, T)\Phi(-d_2(X)), \quad (30)$$

where $d_1(X)$ and $d_2(X)$ are defined in (27).

Its first order derivative with respect to X is:

$$\frac{\partial \text{ZBP}}{\partial X}(0, T, S, X) = P(0, T)\Phi(-d_2(X)). \quad (31)$$

Combining (30) and (31) in (22) will provide the required result (25)-(26) when we have an expression for the RMlevel which is in this lognormal case denoted by $G(\Phi^{-1}(\alpha))$ to express the dependence on $\Phi^{-1}(\alpha)$. For the VaR case we substitute (29) in (19)(a) and use the property $\Phi^{-1}(1 - \alpha) = -\Phi^{-1}(\alpha)$ to end up with (26)(a).

For the TVaR-expression of $G(\Phi^{-1}(\alpha))$ we start from the integral in (15) combined with (29), apply a change of variable, use the properties of the cdf $\Phi(\cdot)$ and invoke the well-known result

$$\int_{-\infty}^b e^{\lambda z} \varphi(z) dz = e^{\frac{1}{2}\lambda^2} \Phi(b - \lambda), \quad (32)$$

with $\varphi(\cdot)$ the probability density function of a standard normal random variable:

$$\begin{aligned} G_{\text{TVaR}}(\Phi^{-1}(\alpha)) &= \frac{A(T, S)e^{-B(T, S)m}}{\alpha} \int_{1-\alpha}^1 e^{-B(T, S)s\Phi^{-1}(\beta)} d\beta \\ &= \frac{A(T, S)e^{-B(T, S)m}}{\alpha} \int_{-\infty}^{\Phi^{-1}(\alpha)} e^{B(T, S)sz} \varphi(z) dz \\ &= \frac{1}{\alpha} A(T, S) e^{-B(T, S)m + \frac{1}{2}B(T, S)^2 s^2} \Phi(\Phi^{-1}(\alpha) - B(T, S)s). \end{aligned}$$

Finally we express the right hand-side in terms of the parameters (24) of $P(T, S)$, leading to (26)(b). The expected shortfall of the loss equals in this lognormal case:

$$\begin{aligned} \text{ESF}_{\alpha, T}(L) &= (1-h)\alpha [G_{\text{VaR}}(\Phi^{-1}(\alpha)) - G_{\text{TVaR}}(\Phi^{-1}(\alpha))] \\ &= (1-h)A(T, S)e^{-B(T, S)m} \times \\ &\quad \times [\alpha e^{B(T, S)s\Phi^{-1}(\alpha)} - e^{\frac{1}{2}B(T, S)^2 s^2} \Phi(\Phi^{-1}(\alpha) - B(T, S)s)] \\ &= (1-h)e^{\Pi(T, S)} [\alpha e^{\Sigma(T, S)\Phi^{-1}(\alpha)} - e^{\frac{1}{2}\Sigma(T, S)^2} \Phi(\Phi^{-1}(\alpha) - \Sigma(T, S))]. \end{aligned}$$

The expected short fall $\text{ESF}_{\alpha, T}(L)$ could also be computed directly by combining (8) and (9) in (4). Hereto, we recall (29) and note that

$$r(T) \stackrel{d}{=} m + sZ, \quad Z \sim N(0, 1),$$

such that we may write

$$\text{ESF}_{\alpha, T}(L) = (1-h)A(T, S)e^{-B(T, S)m} E[(e^{B(T, S)s\Phi^{-1}(\alpha)} - e^{-B(T, S)sZ})_+].$$

Then, apply a Black&Scholes like formula and (24) to arrive at (28). \square

Remark 1 It is easily noticed that the case considered in Ahn et al. (1999) is of the same form as formula (25) when using a Brownian motion process.

Remark 2 Two factor models like two-factor additive Gaussian model G2++, two-factor Hull-White, two-factor Heath-Jarrow-Morton with deterministic volatilities which result in lognormally distributed bond prices (see Brigo and Mercurio (2001)), are also applicable in our framework.

3.2 Coupon-bearing bond

We consider now the case of a coupon-bearing bond paying deterministic cash flows $\mathcal{C} = [c_1, \dots, c_n]$ at maturities $\mathcal{S} = [S_1, \dots, S_n]$. Let $T \leq S_1$. The price of this coupon-bearing bond in T is expressed as a linear combination (or a portfolio) of zero-coupon bonds:

$$\text{CB}(T, \mathcal{S}, \mathcal{C}) = \sum_{i=1}^n c_i P(T, S_i). \quad (33)$$

As in the previous section, the company wants to hedge its position in this bond by buying a percentage of a put option on this bond with strike X and maturity T . In order to determine the strike X , the VaR or the TVaR of the hedged portfolio at time T is minimized under a budget constraint. As in the previous section we will be able to treat the VaR-case and the TVaR-case together.

We first have a look at the value of a put option on a coupon-bearing bond as well as at the structure of the loss function.

The prices of the zero-coupon bonds $P(T, S_i)$, given by (7), all depend on the same short rate $r(T)$. Each $P(T, S_i)$ equals a strictly decreasing and continuous function of one and the same random variable $r(T)$, i.e. for all i

$$P(T, S_i) = A(T, S_i)e^{-B(T, S_i)r(T)} := g_i(r(T)). \quad (34)$$

Hence the vector $(P(T, S_1), \dots, P(T, S_n))$ is comonotonic, see Kaas et al. (2000), and a European option on a coupon-bearing bond can be explicitly priced by means of Jamshidian's decomposition, which was originally derived in Jamshidian (1989) in case of a Vasicek interest rate model. In fact a European option on a coupon-bearing bond decomposes into a portfolio of options on the individual zero-coupon bonds in the portfolio, which gives in case of a put with maturity T and strike X :

$$\text{CBP}(0, T, \mathcal{S}, \mathcal{C}, X) = \sum_{i=1}^n c_i \text{ZBP}(0, T, S_i, X_i), \quad (35)$$

$$\text{with } X_i = g_i(r_X) \text{ satisfying } \sum_{i=1}^n c_i X_i = X. \quad (36)$$

Thus r_X is the value of the short rate at time T for which the coupon-bearing bond price equals the strike.

Repeating the reasoning of Section 3.1 we may conclude that in a worst case scenario the loss of the hedged portfolio at time T composed of the coupon-bearing bond (33) and the put option (35) equals a strictly increasing function f of the random variable $r(T)$:

$$L = \text{CB}(0, \mathcal{S}, \mathcal{C}) + C - ((1 - h) \sum_{i=1}^n c_i g_i(r(T)) + hX) := f(r(T)), \quad (37)$$

with $g_i(r(T))$ defined in (34).

VaR and TVaR minimization

The VaR of this loss that we want to minimize under the constraints $0 < h < 1$ and $C = h\text{CBP}(0, T, \mathcal{S}, \mathcal{C}, X)$, is analogously to (9) given by

$$\text{VaR}_{\alpha, T}(L) = \text{CB}(0, \mathcal{S}, \mathcal{C}) + C - hX - (1 - h) \sum_{i=1}^n c_i g_i(F_{r(T)}^{-1})(1 - \alpha). \quad (38)$$

By integrating this relation (38), after replacing α by $1 - \beta$, with respect to β between the integration bounds $1 - \alpha$ and 1, we find for the TVaR of the loss:

$$\text{TVaR}_{\alpha,T}(L) = \text{CB}(0, \mathcal{S}, \mathcal{C}) + C - hX - \frac{1}{\alpha}((1 - h) \sum_{i=1}^n c_i \int_{1-\alpha}^1 g_i(F_{r(T)}^{-1}(\beta))d\beta). \quad (39)$$

Also here we note the similarity in the expressions for the risk measures (RM) VaR and TVaR which could be collected in one expression:

$$\text{RM}_{\alpha,T}(L) = \text{CB}(0, \mathcal{S}, \mathcal{C}) + C - hX - (1 - h) \sum_{i=1}^n c_i G_i(F_{r(T)}^{-1}(1 - \alpha)) \quad (40)$$

with

$$G_i(F_{r(T)}^{-1}(1 - \alpha)) = \begin{cases} g_i(F_{r(T)}^{-1}(1 - \alpha)) = A(T, S_i)e^{-B(T, S_i)F_{r(T)}^{-1}(1 - \alpha)} & \text{if VaR} \\ \frac{1}{\alpha} \int_{1-\alpha}^1 g_i(F_{r(T)}^{-1}(\beta))d\beta = \frac{A(T, S_i)}{\alpha} \int_{1-\alpha}^1 e^{-B(T, S_i)F_{r(T)}^{-1}(\beta)}d\beta & \text{if TVaR.} \end{cases} \quad (41)$$

We now want to solve the constrained optimization problem

$$\min_{X, h} \text{RM}_{\alpha,T}(L) \quad \text{subjected to} \quad C = h\text{CBP}(0, T, \mathcal{S}, \mathcal{C}, X), \quad 0 < h < 1.$$

From the Kuhn-Tucker conditions we find that the optimal strike price X^* satisfies the following equation:

$$\text{CBP}(0, T, \mathcal{S}, \mathcal{C}, X) - [X - \sum_{i=1}^n c_i G_i(F_{r(T)}^{-1}(1 - \alpha))] \frac{\partial \text{CBP}}{\partial X}(0, T, \mathcal{S}, \mathcal{C}, X) = 0. \quad (42)$$

Rewriting this equation in terms of the put options on the individual zero-coupon bonds cfr. (35) leads to the following equivalent set of equations:

$$\sum_{i=1}^n c_i \text{ZBP}(0, T, S_i, X_i) = [X - \sum_{i=1}^n c_i G_i(F_{r(T)}^{-1}(1 - \alpha))] \sum_{i=1}^n c_i \frac{\partial \text{ZBP}}{\partial X_i}(0, T, S_i, X_i) \frac{\partial X_i}{\partial X} \quad (43)$$

$$\sum_{i=1}^n c_i X_i = X \quad (44)$$

$$\sum_{i=1}^n c_i \frac{\partial X_i}{\partial X} = 1. \quad (45)$$

The first equation simplifies by noting that $\frac{\partial \text{ZBP}}{\partial X_i}(0, T, S_i, X_i)$ is independent of i and by

using (45). Indeed, plug (36) in (12)-(13) while recalling (34):

$$\begin{aligned}\frac{\partial \text{ZBP}}{\partial X_i}(0, T, S_i, X_i) &= P(0, T) \left[1 - F_{r(T)}^T \left(\frac{\ln A(T, S_i) - \ln X_i}{B(T, S_i)} \right) \right] \\ &= P(0, T) [1 - F_{r(T)}^T(r_X)]\end{aligned}\quad (46)$$

$$\begin{aligned}\Rightarrow \sum_{i=1}^n c_i \frac{\partial \text{ZBP}}{\partial X_i}(0, T, S_i, X_i) \frac{\partial X_i}{\partial X} &= P(0, T) [1 - F_{r(T)}^T(r_X)] \sum_{i=1}^n c_i \frac{\partial X_i}{\partial X} \\ &\stackrel{(45)}{=} P(0, T) [1 - F_{r(T)}^T(r_X)].\end{aligned}$$

Thus in order to find the optimal strike X^* we proceed as follows:

Step 1 Solve the following equation, which is equivalent to (43), for r_X :

$$\begin{aligned}\sum_{i=1}^n c_i \text{ZBP}(0, T, S_i, g_i(r_X)) \\ = P(0, T) [1 - F_{r(T)}^T(r_X)] \sum_{i=1}^n c_i [g_i(r_X) - G_i(F_{r(T)}^{-1}(1 - \alpha))].\end{aligned}\quad (47)$$

Step 2 Substitute the solution r_X^* in (44):

$$X^* = \sum_{i=1}^n c_i g_i(r_X^*) = \sum_{i=1}^n c_i A(T, S_i) e^{-B(T, S_i) r_X^*}.\quad (48)$$

Remark In all cases, the optimal strike price is independent of the hedging cost and one can look at the trade-off between the hedging expenditure and the RM level, cfr. Section 3.1.

We summarize these results in the following theorem.

Theorem 4 *Under the assumption of an affine term structure model so that for all i the zero-coupon bond price $P(T, S_i)$ is given by (34) and assuming for all i that the cdf of $P(T, S_i)$ under the T -forward measure has bounded variation and that $E^T[P(T, S_i)]$ is finite, the hedging problem for a coupon bond (33):*

$$\min_{X, h} \text{RM}_{\alpha, T}(L) \quad (49)$$

$$\text{s.t. } C = h \text{CBP}(0, T, S, X) \text{ and } h \in (0, 1) \quad (50)$$

with $\text{RM}_{\alpha, T}(L)$ defined by (40)-(41), has an optimal solution X^* given by (47)-(48).

VaR and TVaR minimization and ESF: lognormal case

We consider the special case that $r(T)$ is a normal random variable cfr. (29) such that the zero-coupon bond prices $P(T, S_i)$ are lognormally distributed with parameters $\Pi(T, S_i)$

and $\Sigma(T, S_i)^2$ given by (24) for $S = S_i$. Then the put option prices $ZBP(0, T, S_i, g_i(r_X))$ in relation (47) are given by (30)-(27), while $G_i(F_{r(T)}^{-1}(1 - \alpha))$ is defined by (26) for $S = S_i$ and will be denoted $G_i(\Phi^{-1}(\alpha))$. The factor $P(0, T)[1 - F_{r(T)}^T(r_X)]$ in (47) equals according to (46) the first order derivative of the put option prices, which is in the lognormal case given by (31):

$$P(0, T)[1 - F_{r(T)}^T(r_X)] = P(0, T)\Phi(-d_2(g_i(r_X))), \quad \text{for any } i.$$

This implies that $d_2(g_i(r_X))$ is independent of i .

Thus the optimal strike X^* can be found as follows:

Step 1 Solve the following equation for r_X :

$$\begin{aligned} & \sum_{i=1}^n c_i [-P(0, S_i)\Phi(-d_1(g_i(r_X))) + P(0, T)g_i(r_X)\Phi(-d_2(g_i(r_X)))] \\ & - P(0, T)\Phi(-d_2(g_1(r_X))) \sum_{i=1}^n c_i [g_i(r_X) - G_i(\Phi^{-1}(\alpha))] = 0. \end{aligned} \quad (51)$$

Step 2 Substitute the solution r_X^* in (48).

The expected shortfall in case of a coupon bearing bond is derived in a similar way as for the zero-coupon bond:

$$\begin{aligned} & \text{ESF}_{\alpha, T}(L) \\ & = \alpha[\text{TVaR}_{\alpha, T} - \text{VaR}_{\alpha, T}] \\ & = \sum_{i=1}^n c_i A(T, S_i) [\alpha e^{-B(T, S_i)F_{r(T)}^{-1}(1-\alpha)} - \int_{1-\alpha}^1 e^{-B(T, S_i)F_{r(T)}^{-1}(\beta)} \beta] \\ & = (1-h) \sum_{i=1}^n c_i A(T, S_i) e^{-B(T, S_i)m} \times \\ & \quad \times [\alpha e^{B(T, S_i)s\Phi^{-1}(\alpha)} - e^{\frac{1}{2}B(T, S_i)^2 s^2} \Phi(\Phi^{-1}(\alpha) - B(T, S_i)s)] \\ & = (1-h) \sum_{i=1}^n c_i e^{\Pi(T, S_i)} [\alpha e^{\Sigma(T, S_i)\Phi^{-1}(\alpha)} - e^{\frac{1}{2}\Sigma(T, S_i)^2} \Phi(\Phi^{-1}(\alpha) - \Sigma(T, S_i))]. \end{aligned}$$

The expected short fall $\text{ESF}_{\alpha, T}(L)$ can also be computed directly by combining (37) and (38) in relation (4) and by invoking comonotonicity properties (see Kaas et al. (2000)) for calculating a stop-loss premium of a comonotonic sum. This implies that $\text{ESF}_{\alpha, T}(L)$ is in fact a linear combination of the expressions in the right hand side of (18) with S replaced by S_i for $i = 1, \dots, n$.

We derived formula (11), (16) and formula (47) combined with (48) to calculate the optimal strike price for the hedging problems under consideration. In all cases, the specification of an interest rate model is necessary. Until now, the optimization has been achieved with the most important modelling assumption that the bond price $P(T, S)$ has the form

(7) such that the term structure is affine. We also looked at a special case that the bond price $P(T, S)$ is lognormally distributed. We did not yet form concrete beliefs on how the (instantaneous) interest rate will move. By forming these beliefs, or in other words, by specifying a model for the evolution of the interest rate, we also get explicit expressions for the bond and bond option prices, which then enables us to determine the (theoretically) optimal strike price.

In the next section, we will define and explain the specification of the model for the evolution of the instantaneous interest rate.

4 Application

4.1 The Hull-White model

There exists a whole literature concerning interest rate models. For a comprehensive overview we refer for example to Brigo and Mercurio (2001). For our analysis, we focus on the Hull-White one-factor model, first discussed in Hull and White (1990). We choose this model because it is still an often used model in financial institutions for risk management purposes, (see Brigo and Mercurio (2001)). Two main reasons explain this popularity. First of all, it is a model that allows closed form solutions for bond and plain vanilla European option pricing. So, since there are exact pricing formulas, there is no need to run time consuming simulations. But of course, if the model lacks credibility, fast but wrong price computations do not offer any benefit. But that is where the second big advantage of the Hull-White model comes from since it succeeds in fitting a given term structure by having (at least) one time-dependent parameter. Therefore, today's bond prices can be perfectly matched. It belongs to the class of so called no-arbitrage interest rate models. This means that, in contrast to equilibrium models (such as Vasicek, Cox-Ingersoll-Ross), no-arbitrage models succeed in fitting a given term structure, and thus can match today's bond prices perfectly.

An often cited critique is that applying the model sometimes results in a negative interest rate, but with up-to-date calibrated parameters which are used for a rather short period, it can be proved that the probability of obtaining negative interest rates is very small.

Hull and White (1990) assume that the instantaneous interest rate follows a mean reverting process also known as an Ornstein-Uhlenbeck process:

$$dr(t) = (\theta(t) - \gamma(t)r(t))dt + \sigma(t)dZ(t) \quad (52)$$

for a standard Brownian motion $Z(t)$ under the risk-neutral measure Q , and with time dependent parameters $\theta(t)$, $\gamma(t)$, and $\sigma(t)$. The parameter $\theta(t)/\gamma(t)$ is the time dependent long-term average level of the spot interest rate around which $r(t)$ moves, $\gamma(t)$ controls the mean-reversion speed and $\sigma(t)$ is the volatility function. By making the mean reversion

level θ time dependent, a perfect fit with a given term structure can be achieved, and in this way arbitrage can be avoided. In our analysis, we will keep γ and σ constant, and thus time-independent. According to Brigo and Mercurio (2001), this is desirable when an exact calibration to an initial term structure is wanted. This perfect fit then occurs when $\theta(t)$ satisfies the following condition:

$$\theta(t) = F_t^M(0, t) + \gamma F^M(0, t) + \frac{\sigma^2}{2\gamma}(1 - e^{-2\gamma t}),$$

where, $F^M(0, t)$ denotes the instantaneous forward rate observed in the market on time zero with maturity t .

It can be shown (see Hull and White (1990)) that the expectation and variance of the stochastic variable $r(t)$ are:

$$E[r(t)] = m(t) = r(0)e^{-\gamma t} + a(t) - a(0)e^{-\gamma t} \quad (53)$$

$$\text{Var}[r(t)] = s^2(t) = \frac{\sigma^2}{2\gamma}(1 - e^{-2\gamma t}), \quad (54)$$

with the expression $a(t)$ calculated as follows:

$$a(t) = F^M(0, t) + \frac{\sigma^2}{2} \left(\frac{1 - e^{-\gamma t}}{\gamma} \right)^2.$$

Based on these results, Hull and White developed an analytical expression for the price of a zero-coupon bond with maturity date S

$$P(t, S) = A(t, S)e^{-B(t, S)r(t)}, \quad (55)$$

where

$$B(t, S) = \frac{1 - e^{-\gamma(S-t)}}{\gamma}, \quad (56)$$

$$A(t, S) = \frac{P^M(0, S)}{P^M(0, t)} e^{B(t, S)F^M(0, t) - \frac{\sigma^2}{4\gamma}(1 - e^{-2\gamma t})B^2(t, S)}, \quad (57)$$

with P^M the bond price observed in the market. Since $A(t, S)$ and $B(t, S)$ are independent of $r(t)$, the distribution of a bond price at any given time must be lognormal with parameters Π and Σ^2 :

$$\Pi(t, S) = \ln A(t, S) - B(t, S)m(t), \quad \Sigma(t, S)^2 = B(t, S)^2 s^2(t), \quad (58)$$

with $m(t)$ and $s^2(t)$ given by (53) and (54).

4.2 Calibration of the Hull-White model

Until now, we theoretically discussed the issue of minimizing the VaR and TVaR of our investment. If the firm wants to pursue this minimization into practice, it needs credible parameters for the interest rate model it uses. Focusing in particular on the Hull-White model that we discussed above, we need to have parameter values for γ and σ . The process to obtain these parameters is calibration. The most common way to calibrate the Hull-White model is by using interest rate options, such as swaptions or caps. The goal of the calibration is to find the model parameters that minimize the relative difference between the market prices of these interest rate options and the prices obtained by applying our model.

Suppose we have M market prices of swaptions or caps, then we search the γ and σ such that the sum of squared errors between the market and model prices are minimized. Formally,

$$\min_{\gamma, \sigma} \sqrt{\sum_{i=1}^M \left(\frac{\text{model}_i - \text{market}_i}{\text{market}_i} \right)^2}.$$

Interest rate caps are instruments that provide the holder of it protection against a specified interest rate (e.g. the three month EURIBOR, R_L) rising above a specified level (the cap rate, R_C). Suppose a company issued a floating rate note with as reference rate the three month EURIBOR. When EURIBOR rises above the cap rate, a payoff is generated such that the net payment of the holder only equals the cap rate. One cap consists of a series of caplets. These caplets can be seen as call options on the reference rate. The maturity of the underlying floating interest rate of these call options equals the tenor, which is the time period between two resets of the reference rate. In our case, this is three months, or 0.25 year.

If in our case, at time t_k , the three month EURIBOR rises above the cap rate, the call will be exercised, which leads to a payoff at time t_{k+1} (0.25 year later) that can be used to compensate the increased interest payment on the floating rate note. Formally, the payoff at time t_{k+1} equals (see Hull (2003)):

$$\max(0.25(R_L - R_C), 0).$$

This is equivalent to a payoff at time t_k of

$$\frac{\max(0.25(R_L - R_C), 0)}{1 + 0.25R_L}.$$

This can be restated as:

$$\max\left(1 - \frac{1 + 0.25R_C}{1 + 0.25R_L}, 0\right).$$

This is the payoff of a put option with strike 1, expiring at t_k , on a zero-coupon bond with principal $1 + 0.25R_C$, maturing at t_{k+1} . This means that each individual caplet corresponds

here comes Table 1

Table 1: Overview cap data

to a put option on a zero-coupon bond. Thus, a cap can be valued as a sum of zero-coupon bond put options. Since these put options can be valued using the Hull-White model, this offers us a way to fit our model to the market data. The market data we have used are to be found in Table 1 where cap maturities are listed, along with the volatility quotes of these caps and the cap rate. The data are obtained on 11 April 2005 and have as reference rate EURIBOR. Note that the volatility quotes have the traditional humped relation with respect to the maturity of the cap: the volatility reaches its peak at the 2 year cap and then decreases steadily as the maturity increases. Although the cap rate can be freely determined, it is most common to put it equal to the swap rate for a swap having the same payment dates as the cap. The volatility quotes that are provided are based on Black's model. This means that we first have to use Black's formula for valuing bond options in order to arrive at the prices of the caps. These prices are shown in the fourth column. Now we still have to calculate the model prices. Therefore, we use, for each caplet, the following formula:

$$\text{ZBP}(0, T, S, X, N) = -NP(0, S)\Phi(-d_1(X)) + XP(0, T)\Phi(-d_2(X)), \quad (59)$$

As strike price X we take 1, and as principal N we take $1 + 0.25R_C$. $P(0, T)$ and $P(0, S)$ can be read from the term structure.

Taking the sum of all the caplets in a given cap, we get an expression for which we need to seek the parameters that, globally, make the best fit. The calibration procedure results in the following parameter values:

$$\gamma = 0.31621 \quad \sigma = 0.011631.$$

4.3 VaR and TVaR minimization

Supposing we have gone through this calibration procedure, the next step in our hedging programme would then be to provide this protection to our portfolio. This can basically be achieved in two ways: first of all, by buying a put option, or secondly, by replicating this option. In the first approach, we are also facing two possibilities: either we buy the put option at a regulated exchange market, either we buy it over the counter (OTC). If options are bought as protection against interest rate risk, it is most common to buy them OTC. Genuine bond options are only available at a restricted number of exchanges. Furthermore, at these exchanges, trading in bond options is usually very thin. The second approach, replicating the option synthetically, involves quite some follow up and adjustment in positions, and can entail a considerable amount of transaction costs. Therefore, it is not unreasonable to consider the OTC market as the only viable possibility for a firm

to buy protection. A major advantage of buying over the counter is that we can completely tailor the option to our needs. What is of utter importance to the firm is that the option can be bought at any desired strike. This opposes to buying options on an exchange market, where options can only be bought at predetermined strike prices. A source of uncertainty is the discrepancy between the theoretical option prices that were calculated and the option price that has to be paid over the counter. Therefore, the firm could perform the optimization procedure using the prices of the financial institution. However, this restricts the possible calculation methods to using formula (42). The combination of (51) and (48) cannot be used since this requires the knowledge of d_1 and d_2 , which we clearly not have, since we only have the price of the option. So, it is necessary to use formula (42). The difference in optimal strike price in both approaches is an empirical question and will be dealt with in this part.

For our numerical illustration, we suppose the firm has an OLO 35. OLO (which stands for Obligation linear/lineaire Obligatie) are debt instruments issued by the Belgian government, and as such, believed to be risk-free. OLOs have a fixed coupon. The OLO we consider was issued on 28 Sept 2000 and will mature on 28 Sept 2010, so the maturity is 10 years, i.e. $S = 10$. It pays a yearly coupon of 5.75 %, on 28 Sept of each year, i.e. $c_i = 0.0575$ for all i . As there are no traded options for this kind of bond, we have to protect by buying OTC options. Therefore, we got OTC prices from a financial institution. The date on which these data were delivered, is 30 Sept 2005. This means that the bond then has a remaining maturity of 4.99 years, and coupons will be paid out at $S_1 = 0.99$, $S_2 = 1.99$, $S_3 = 2.99$, $S_4 = 3.99$ and $S_5 = 4.99$. At that particular date, 30 Sept 2005, the bond had a market price of 1.1393. We received the option prices for a wide range of strikes: going from a strike price of 1.05 to a strike of 1.199, with steps of 0.001. The option maturity is exactly one year, i.e. $T = 1$.

This means that the maturity of the option lies between the first and second coupon payment, whereas when deriving optimal strike price, we supposed that the option matured before the first coupon payment. This problem can easily be solved by reducing our coupon payment vector to the last four observations.

We now have **three** methods of computing the optimal strike price.

- (1) The first method is solving equation (51) and substituting in (48).
- (2) The second method still uses the theoretical option prices, but solves equation (42) and approximates the first derivative of the option price with respect to the strike price by the difference quotient of the changes in the option prices to the changes in the strike price.
- (3) The third is equivalent to the second approach, but uses the option prices received from the financial institution.

Using a 5% confidence level, the bond VaR level for a holding period of one year (in other words, a worst case expectation of the evolution of the bond price) is 1.0716. Using this number, we can calculate the optimal strike price in the **three** different methods. Note that VaR has to be calculated under the true probability measure. Since we have calibrated our interest rate model using option prices, the parameters we obtained are under the risk-

neutral measure. So, in order to know the parameters under the true probability measure, we would need to estimate the market price of risk. However, as quite often done (see Stanton (1997)), we assumed the market price of risk to be zero.

- (1) The first method results in an optimal strike price of 1.0833.
- (2) The second method yields an optimum which is very close to this: 1.084.
- (3) The last method finds as an optimum a slightly higher strike price: 1.087.

In all three cases, the optimum is situated above the VaR level of the bond as predicted by the theory. The close correspondence between the first two methods is evident, since the difference can only be attributed to approximation errors in the second method. Although not dramatic, we observe a difference between the first two and the last method. The third method is resulting in an option that is a bit more in the money.

Using a 1% confidence level, a comparable picture emerges. The bond VaR level of course is lower this time, namely 1.0561. This results in lower optimal strike prices: in the first method, we obtain an optimal strike price of 1.0649. The second method shows an optimum of 1.065. The third method again shows a higher optimum, this time the strike price amounts to 1.068.

For both confidence levels the results are summarized in Table 2:

here comes Table 2

Table 2: Optimal strike prices for one and five percent confidence levels, for different calculation methods.

As stated earlier, the firm that wishes to hedge its exposure is now facing a linear trade-off between VaR and hedging expenditure. This is illustrated in Figure 1. On this graph, the firm can clearly see the consequence of choosing a particular hedging cost. Alternatively, it can read the hedging cost required to obtain a certain protection, expressed in VaR terms. Note that the hedging cost is restricted to the range $[0, 0.003171]$, with the left hand side of the range corresponding to no hedging, and the right hand side corresponding to buying an entire put option (at the OTC price) at the optimal strike price (so, $h = 1$). No hedging leads to a VaR of 0.0677. Buying an entire option at the optimal strike price reduces the VaR to 0.0557. It is clear that the exact position a firm takes, is determined by both the budget and the risk aversion or appetite of the firm, which we cannot judge. Furthermore, it makes economic sense to execute the hedge since we observe that the hedging cost is smaller than the reduction in VaR you get by hedging.

Conclusions are comparable when performing a TVaR minimization. Of course, the bond TVaR level lies below the VaR level. For the 5% level, it is situated at 1.0621. The first method results in an optimum of 1.0717. The second method finds 1.072 as optimal strike price, and the third method (taking into account the OTC prices) produces an optimum of 1.075. Again we observe the difference between the first two methods and the third method.

here comes figure 1

Fig. 1: VaR in function of the hedging expenditure C

For the 1% level, the bond TVaR is 1.0485. The optimum in the first method is now at 1.0537. The second method results in an optimum of 1.056. Using the OTC prices, an optimal strike price is reached at a level of 1.059.

We can thus conclude that, based on OTC prices, the optimum is situated slightly higher than the optimum reached under theoretical prices. This conclusion is robust for different risk criteria and different confidence levels.

5 Conclusions

We provided a method for minimizing the risk of a position in a bond (zero-coupon or coupon-bearing) by buying (a percentage of) a bond put option. Taking into account a budget constraint, we determine the optimal strike price, which minimizes a Value-at-Risk or Tail-Value-at-Risk criterion. Alternatively, our approach can be used when a nominal risk level is fixed, and the minimal hedging budget to fulfil this criterion is desired. From the class of short rate models which result in lognormally distributed future bond prices, we have selected the Hull-White one-factor model for an illustration of our optimization. This Hull-White model is calibrated to a set of cap prices, in order to obtain credible parameters for the process. We illustrated our strategy using as investment asset a Belgian government bond, on which we want to buy protection. We calculated the optimal strike price of the bond option that we use, both with theoretical Hull-White prices, and with real market prices. The results are comforting in the sense that the optimal strike prices in both approaches show a close correspondence. The strike price based on real prices is only slightly higher than the one based on theoretical prices.

Further research possibilities are mainly situated in two directions. First of all, we can consider other instruments to hedge our investment. The use of a swaption to hedge a swap is very widespread in the financial industry. It should be possible to determine the optimal swap rate to hedge the swap. The second direction concerns the interest rate models that can be used in our analysis. It is often stated that two-factor models are better suited to capture interest rate behaviour. Such a model cannot be used here to hedge an investment in a coupon-bearing bond. The reason is that the Jamshidian decomposition cannot be applied. An alternative could be the comonotonicity approach of Dhaene et al. (2002a) and Dhaene et al. (2002b), which results in a lower and upper bound for the bond put option. As an alternative for a two-factor model, a model with a jump component can be considered. Johannes (2004) finds evidence for the importance of adding a jump term to interest rate models. The use of jump models, however, raises new pricing and hedging issues.

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